Worst-Case Capacity of Gaussian Vector Channels

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Abstract

We study the (Shannon) capacity of Gaussian multiple input multiple output (MIMO) systems when the exact distribution of the interference and/or noise is unknown, but the family the distribution belongs to is known (assumed Gaussian). This capacity can be written as a min-max optimization problem over convex sets with matrix variables. We employ Lagrange duality from convex optimization theory to convert the min-max optimization into a single convex minimization, which is easily solved. Our technique is illustrated for several example problems involving multiuser channels and multiple antenna channels.

I. Introduction

The capacity of a scalar (single antenna) Gaussian channel is characterized in closed form by an average power constraint on the transmitter and on the interference [3, 4]. Note that the distribution of a zero mean scalar Gaussian random variable is determined by its second moment alone. However, a solution for vector channels is more complicated since the distribution of a Gaussian random vector is based on a covariance matrix. Thus, to calculate the channel capacity of a vector channel, we require knowledge of the exact distribution of the interference, i.e., knowledge of both the type of distribution (which is assumed to be Gaussian in this paper) and the covariance matrix of this interference distribution. In most practical scenarios, however, the actual covariance of the interference is unknown, but is known to be limited to a (finite or infinite) convex set $\mathcal{S}$. In this scenario, a "worst-case capacity" for this system is defined [1, 2] as the minimum capacity over the set $\mathcal{S}$ of possible interference covariances.

In this paper we study three capacity problems that are of interest to the research community. These are:

1. Worst-case capacity with a second order constraint on the interference [2];
2. Worst-case capacity when there are N other users in the system, each with an average power constraint; and
3. Sum capacity of a vector broadcast channel.

A mathematical formulation of these problems and their solution can be found in the next two sections of this paper. A closed form solution for high SNR for the first problem has been obtained in [2]. Also, a clever game theoretic understanding of such problems has also been studied [2, 11]. More generally, however, the three worst-case capacity problems defined above are difficult to solve directly. We therefore employ Lagrangian duality on these optimization problems to simplify their solution. This is termed the "dual-problem method" in optimization literature. This method converts the min-max optimization of the worst-capacity capacity to a single convex minimization. There are many standard techniques for solving such convex minimization problems, and simple software implementations of these techniques are readily available [9, 16].

We have limited our study to the cases when the type of distribution of the interference is Gaussian. The capacity of channels without knowledge of the type of distribution or with a different type of distribution have also been studied ([1, 4, 7, 8] and the references therein). In many cases, particularly when there is a power constraint on the interference, the worst case interference distribution has been shown to be Gaussian [1]. Thus, Gaussianity of the interference is not a restrictive assumption.

In the next section, we present the system model and provide a mathematical formulation for these min-max problems. In Section III we present the dual-problem method. We conclude with Section V.

II. System Model and Problem Formulation

We use boldface to denote matrices and vectors, $\dagger$ to denote transpose conjugate, $|.|$ to denote the determinant and $\mathrm{Tr}(.)$ to denote the trace of a matrix.

The channel for all the cases mentioned in Section I can be modeled as

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{z} \tag{1}$$

where $\mathbf{x}$ is the transmitted signal, $\mathbf{H}$ the channel matrix, $\mathbf{z}$ the combined interference and noise and $\mathbf{y}$ the received signal. We assume that $\mathbf{z}$ is Gaussian, and that its covariance is restricted to some compact set. The optimum input distribution can be easily shown to be Gaussian, and hence the worst-case capacity for such a system may be expressed as

$$\min_{\mathbf{A}\in\mathcal{S}} \max_{\mathbf{Q}\in\mathcal{T}} \log|\mathbf{I} + \mathbf{A}^{-1}\mathbf{H}^\dagger\mathbf{Q}\mathbf{H}|, \tag{2}$$

where $\mathbf{A}$ is the covariance matrix of the interference and noise and belongs to some bounded convex set $\mathcal{S}$ and $\mathbf{Q}$ is the covariance of the input $\mathbf{x}$ which is restricted to the set $\mathcal{T} = \{\mathbf{Q} : \mathrm{Tr}(\mathbf{Q}) \le Q\}$ for some scalar $Q > 0$.

This problem can be shown to be convex in $\mathbf{A}$ and concave in $\mathbf{Q}$ [1]. This is sufficient to show that [15]

$$\min_{\mathbf{A}\in\mathcal{S}} \max_{\mathbf{Q}\in\mathcal{T}} \log|\mathbf{I} + \mathbf{A}^{-1}\mathbf{H}^\dagger\mathbf{Q}\mathbf{H}| = \max_{\mathbf{Q}\in\mathcal{T}} \min_{\mathbf{A}\in\mathcal{S}} \log|\mathbf{I} + \mathbf{A}^{-1}\mathbf{H}^\dagger\mathbf{Q}\mathbf{H}| \tag{3}$$

i.e., the min-max problem can be written as a max-min problem.

As explained in Section I, many practical problems can be written in the form (2). In the definition of (2) we have specified the set $\mathcal{T}$ to be all $\mathbf{Q}$ with a trace constraint. However, we have not placed any constraint on $\mathcal{S}$ except that it consists of compact sets. For concreteness, we shall focus of solving three common worst-case problems for which the set $\mathcal{S}$ can be given explicitly. Note, however, that our dual-problem method applies to any worst-case capacity problem defined by (2) and is not limited to just these three example problems.

Problem 1: Computing the worst-case capacity of a multi-antenna system with an average power constraint on the interference and noise. This can be written mathematically as:

$$\begin{aligned} &\min_{\mathbf{A}\in\mathcal{S}}\max_{\mathbf{Q}\in\mathcal{T}} \log|\mathbf{I}+\mathbf{A}^{-1}\mathbf{H}^\dagger\mathbf{Q}\mathbf{H}| \\ \text{such that}\quad & \\ &\mathcal{S}=\{\mathbf{A}:\mathrm{Tr}(\mathbf{A})\le P,\mathbf{A}>0\}, \\ &\mathcal{T}=\{\mathbf{Q}:\mathrm{Tr}(\mathbf{Q})\le Q\}. \end{aligned} \tag{4}$$

Problem 2: Next, we consider a system where the interference arises from N other users with channels $\mathbf{G}_1,\dots,\mathbf{G}_N$ sharing the same bandwidth as the desired user with channel $\mathbf{H}$. Assuming that $\mathbf{G}_1,\dots,\mathbf{G}_N$ are known, the interferers have average power constraintsP_i and the noise is white, we have the worst-case capacity of such a system given by the optimization problem

$$\begin{aligned} &\min_{\mathbf{A}\in\mathcal{S}}\max_{\mathbf{Q}\in\mathcal{T}} \log|\mathbf{I}+\mathbf{A}^{-1}\mathbf{H}^\dagger\mathbf{Q}\mathbf{H}| \\ \text{such that}\quad & \\ &\mathcal{S}=\{\mathbf{A}:\mathbf{A}=\mathbf{I}+\sum_{i=1}^{N}\mathbf{G}_i^\dagger\Delta_i\mathbf{G}_i,\mathrm{Tr}(\mathbf{\Delta}_i)\le P_i,\mathbf{\Delta}_i\ge 0\}, \\ &\mathcal{T}=\{\mathbf{Q}:\mathrm{Tr}(\mathbf{Q})\le Q\}. \end{aligned} \tag{5}$$

Problem 3: The worst-case noise correlation problem for calculating the sum-rate capacity of the vector broadcast channel [10, 12, 13] can be written as

$$\begin{aligned} &\min_{\mathbf{A}\in\mathcal{S}}\max_{\mathbf{Q}\in\mathcal{T}} \log|\mathbf{I}+\mathbf{A}^{-1}\mathbf{H}^\dagger\mathbf{Q}\mathbf{H}| \\ \text{such that}\quad & \\ &\mathcal{S}=\left\{\mathbf{A}:\mathbf{A}=\begin{bmatrix}\mathbf{I} & \mathbf{X}^\dagger\\ \mathbf{X} & \mathbf{I}\end{bmatrix},\mathbf{A}>0\right\}, \\ &\mathcal{T}=\{\mathbf{Q}:\mathrm{Tr}(\mathbf{Q})\le Q\}. \end{aligned} \tag{6}$$

Next, we present the dual problem method for solving the min-max problem.

III. The Dual-Problem Method

The optimization problem

$$\max_{\mathbf{Q}\in\mathcal{T}} \log|\mathbf{I}+\mathbf{A}^{-1}\mathbf{H}^\dagger\mathbf{Q}\mathbf{H}| \tag{7}$$

over the convex set $\mathcal{T}=\{\mathrm{Tr}(\mathbf{A})\le P\}$ for some $P>0$ is a convex optimization problem [9, 14], and a dual optimization problem can be derived whose optimum value equals the optimum value of the primal problem (7). The dual problem to (7) has been derived in [14]. We shall repeat that derivation here for completeness. Note that the primal problem given in (7) is the capacity of a multi-antenna system with channel given by $\mathbf{A}^{-1/2}\mathbf{H}$ and additive white Gaussian noise [5]. The capacity of a system with the conjugate transpose channel $\mathbf{H}^\dagger\mathbf{A}^{-1/2}$ can be shown to be exactly equal to that in (7) (See [5, 13] for proof). The capacity of $\mathbf{H}^\dagger\mathbf{A}^{-1/2}$ is given by the maximization problem

$$\max_{\mathbf{Q}\in\mathcal{T}} \log|\mathbf{I}+\mathbf{H}\mathbf{A}^{-1/2}\mathbf{Q}\mathbf{A}^{-1/2}\mathbf{H}^\dagger| \tag{8}$$

The problem given by (8) is a convex optimization problem, i.e., it has a concave objective function and a convex constraint set. Hence a convex Lagrangian dual minimization problem can be obtained that achieves the same optimum value at (8). For this, we rewrite (8) as

$$\begin{aligned} &\min -\log|\mathbf{X}| \\ \text{such that}\quad &\mathbf{X}=\mathbf{I}+\mathbf{H}\mathbf{A}^{-1/2}\mathbf{Q}\mathbf{A}^{-1/2}\mathbf{H}^\dagger \\ &\mathrm{Tr}(\mathbf{Q})=P,\quad \mathbf{Q}\ge 0 \end{aligned}$$

Note that matrix inequalities are associated with dual variables that are matrices, while scalar inequalities are associated with scalar dual variables. Using dual variables $\mathbf{K}, \lambda$, the Lagrangian for this problem can be written as:

$$\mathcal{L}(\mathbf{X}, \mathbf{Q}, \mathbf{K}, \mathbf{S}, \lambda) = -\log|\mathbf{X}| + \mathrm{Tr}[\mathbf{K}(\mathbf{X} - \mathbf{I} - \mathbf{H}\mathbf{A}^{-1/2}\mathbf{Q}\mathbf{A}^{-1/2}\mathbf{H}^{\dagger})] + \lambda(\mathrm{Tr}(\mathbf{Q}) - \mathbf{P}) + \mathrm{Tr}(\mathbf{V}\mathbf{Q}) \tag{9}$$

The saddle point of this Lagrangian (minimizing with respect to $\mathbf{X}, \mathbf{Q}$ and maximizing with respect to $\mathbf{K}, \mathbf{V}, \lambda$) equals the optimal value of the primal problem. We obtain the optimality conditions by differentiating (9) to get

$$\begin{aligned} \lambda\mathbf{I} &= \mathbf{A}^{-1/2}\mathbf{H}\mathbf{K}\mathbf{H}^{\dagger}\mathbf{A}^{-1/2} + \mathbf{V} \\ \mathbf{X}^{-1} &= \mathbf{K} \end{aligned}$$

Substituting these into the Lagrangian, we obtain

$$\mathcal{L}(\mathbf{K}, \mathbf{V}, \lambda) = \log|\mathbf{K}| - \mathrm{Tr}(\mathbf{K}) - \lambda Q + M \tag{10}$$

The dual problem is obtained by maximizing (10)with respect to the dual variables as

$$\begin{aligned} &\max_{\mathbf{K},\lambda} \log|\mathbf{K}| - \mathrm{Tr}(\mathbf{K}) - \lambda Q + M \\ \text{such that} \quad &\mathbf{K} > 0, \lambda > 0 \\ &\lambda\mathbf{A} \geq \mathbf{H}\mathbf{K}\mathbf{H}^{\dagger} \end{aligned} \tag{11}$$

Thus, the dual problem to the worst-case capacity problem (2) can be written as:

$$\begin{aligned} &\min_{\mathbf{A}\in\mathcal{S}} \min_{\mathbf{K},\lambda} -\log|\mathbf{K}| + \mathrm{Tr}(\mathbf{K}) + \lambda Q - M \\ \text{such that} \quad & \\ &\mathbf{K} \geq 0, \lambda \geq 0, \\ &\lambda\mathbf{A} \geq \mathbf{H}\mathbf{K}\mathbf{H}^{\dagger}. \end{aligned} \tag{12}$$

Note that the optimization problem in (12) is a single minimization problem, and hence can be solved using many traditional optimization techniques. For this paper, we shall focus on the three problems considered in Section II.For these problems, we can rewrite (12) as a jointly convex minimization problem with linear inequality constraints, and hence employ the methodology given in [9] to obtain the optimum solution.

For Problem (4), the dual convex minimization problem with linear constraints is:

$$\begin{aligned} &\min_{\hat{\mathbf{A}},\mathbf{K},\lambda} -\log|\mathbf{K}| + \mathrm{Tr}(\mathbf{K}) + \lambda Q - M \\ \text{such that} \quad & \\ &\mathbf{K} \geq 0, \hat{\mathbf{A}} \geq 0, \lambda \geq 0, \\ &\hat{\mathbf{A}} \geq \mathbf{H}\mathbf{K}\mathbf{H}^{\dagger}, \\ &\mathrm{Tr}(\hat{\mathbf{A}}) \leq \lambda P. \end{aligned}$$

For Problem (5), the dual convex minimization problem with linear constraints is:

$$\begin{aligned} &\min_{\mathbf{\Gamma}_i,\mathbf{K},\lambda} -\log|\mathbf{K}| + \mathrm{Tr}(\mathbf{K}) + \lambda Q - M \\ \text{such that} \quad & \\ &\mathbf{K} \geq 0, \mathbf{\Gamma}_i \geq 0, \lambda \geq 0, \\ &\lambda\mathbf{I} + \sum_{i=1}^{N} \mathbf{G}_i\mathbf{\Gamma}_i\mathbf{G}_i^{\dagger} \geq \mathbf{H}\mathbf{K}\mathbf{H}^{\dagger}, \\ &\mathrm{Tr}(\mathbf{\Gamma}_i) \leq \lambda P_i. \end{aligned}$$

For Problem (6), the dual convex optimization problem with linear constraints is:

$$\begin{aligned} &\min_{\hat{\mathbf{A}},\mathbf{K},\lambda} -\log|\mathbf{K}| + \mathrm{Tr}(\mathbf{K}) + \lambda Q - M \\ \text{such that} \quad & \\ &\mathbf{K} \geq 0, \hat{\mathbf{A}} \geq 0, \lambda \geq 0, \\ &\hat{\mathbf{A}} \geq \mathbf{H}\mathbf{K}\mathbf{H}^{\dagger}, \\ &\hat{\mathbf{A}} = \begin{bmatrix} \lambda\mathbf{I} & \mathbf{Z}^{\dagger} \\ \mathbf{Z} & \lambda\mathbf{I} \end{bmatrix}. \end{aligned}$$

where $\mathbf{K}$ is assumed to be an $M \times M$ matrix. Note that all three problems above are convex minimization problems with linear matrix constraints, and can be solved using the interior-point technique in [9]. We use the software implementation SPDSOL to obtain our numerical results. This software, which implements the methodology in [9] for convex minimization, is available at [16].

Note that there many other techniques in the field of min-max optimization theory that also solve these min-max problems. For a explanation of other techniques, see [20–22].

IV. Examples

To better illustrate the method, we provide some examples. The first three cases deal with the worst-case capacity under multiuser interference described by Problem 2 in Section II. The fourth case is an example of the worst-case noise for vector broadcast sum-rate capacity described by Problem 3 in Section II. Cases 1,2 and 4 are constructed such that the answers are intuitive and can be guessed, which can then be verified with the dual problem method. Case 3 is non-intuitive, and hence the solution can only be obtained by solving the dual problem. In all cases we assume $Q = 1$ and $P_i = 1$ for all i.

Case 1:

$\mathbf{H} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\mathbf{G}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\mathbf{G}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. In this case, it is clear that a malicious interferer will affect both eigenvalues of $\mathbf{H}$ equally. Thus, the optimum is clearly at $\mathbf{Q} = \mathbf{\Delta}_i = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}$. Using SDPSOL [16] on the dual problem for Problem 2, we find the optimum value to be 0.4463 nats/channel use, which is the same capacity as that achieved by using these covariances.

Case 2:

$\mathbf{H} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $\mathbf{G}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\mathbf{G}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Now, the user must put all its power in the only eigenmode of the channel. Thus, the interferers should put all their power along that eigenmode as well, giving us the solution $\mathbf{Q} = \mathbf{\Delta}_i = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Again, SDPSOL gives us the optimum to be 0.2877 nats/channel use, which is the worst-case capacity.

Case 3:

$\mathbf{H} = \begin{bmatrix} -0.4326 & 0.1253 \\ -1.6656 & 0.2877 \end{bmatrix}$ $\mathbf{G}_1 = \begin{bmatrix} -1.146571.1892 & \\ 1.1909 & -0.0376 \end{bmatrix}$ $\mathbf{G}_2 = \begin{bmatrix} 0.3273 & -0.1867 \\ 0.1746 & 0.7258 \end{bmatrix}$. The channels in this case are completely random, and hence guessing the solution is difficult. Using the dual problem and SDPSOL, we get the capacity to be 0.9141 nats/channel use.

Case 4: As the last example, we consider Problem 3. We want the worst-case sum e-rate capacity for a three user broadcast MIMO system with channels given by

$$\begin{aligned} \mathbf{h}_1^\dagger &= [\ 0 \quad 1\ ], \\ \mathbf{h}_2^\dagger &= [\ -\sqrt{3}/2 \quad -1/2\ ], \\ \mathbf{h}_3^\dagger &= [\ \sqrt{3}/2 \quad -1/2\ ] \end{aligned}$$

and the total power constraint $Q = 1$. Moreover, the channels are unit vectors in Euclidean space, and are spaced 120 degrees apart. Note that the matrix $\mathbf{H}$ for the cooperative system in Problem 3 will be given by

$$\mathbf{H} = \begin{bmatrix} 0 & 1 \\ -\sqrt{3}/2 & -1/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix} \tag{13}$$

and it has rank two, i.e. $\mathbf{h}_1 = -(\mathbf{h}_2 + \mathbf{h}_3)$.

Let us now try to obtain the worst case noise for this problem without numerical computation. If we assume the noise to be correlated such that $z_1 = -(z_2 + z_3)$, then, $\mathbf{h}_1 = -(\mathbf{h}_2 + \mathbf{h}_3)$, implies that $y_1 = -(y_2 + y_3)$. This makes the received signal at antenna 1 a linear combination of the received signal at antennas 2 and 3. Therefore, antenna 1 can be eliminated from this system without any loss in capacity. Let us assume, for the moment, that the noise correlation corresponding to $z_1 = -(z_2 + z_3)$ is the worst case noise correlation. Since the diagonal elements of $\mathbf{A}$ are constrained to unity, $\mathbf{A}$ is uniquely given by

$$\mathbf{A} = \begin{bmatrix} 1 & -0.5 & -0.5 \\ -0.5 & 1 & -0.5 \\ -0.5 & -0.5 & 1 \end{bmatrix}, \tag{14}$$

and the capacity achieved (employing waterfilling [5, 6]) is given by 1.1699 bits/channeluse. The dual problem method yields the same value for capacity, which verifies our assumption about worst case noise correlation.

V. Conclusion

In this paper, we study the worst-case capacity of a system, defined as the capacity when the covariance of the Gaussian interference and/or noise in the system is unknown, but it limited to a compact set. We find these worst-case capacity problems to be min-max optimization problems, with no closed form solutions for most cases. hence, we introduce numerical techniques to solve them. We use duality of optimization problems to replace the min-max problem with a jointly convex minimization problem that can be solved using existing techniques and software. Lastly, we apply our method to both multiuser and multi-antenna examples.

VI. Acknowledgment

We sincerely thank Prof. Arkadi Nemirovski for providing us with his unpublished work on worst-case capacity [22].

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