Stable and efficient resource allocation with contracts

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Abstract

Consider a model of indivisible-object allocation with contracts, such as college admissions in which contracts specify majors. Does including contracts in a market allow designers to guarantee a stable and (student) efficient matching? I find that it is difficult to ensure both properties, as adding contracts can often put stability and efficiency at odds. Theorem 1 shows that a necessary condition to secure these properties is student-lexicographic priorities—colleges must rank all contracts from “second-tier” students consecutively. I develop a new framework to analyze a broad class of mechanisms with contracts. The main result characterizes the restriction which guarantees efficiency, stability and group strategy-proofness for this class of mechanisms. I use this framework to apply the main result to two famous mechanisms, deferred acceptance and top trading cycles. I conclude by extending the main result to many-to-many matching and substitutable college priorities.

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1 Introduction

I consider a model of allocating indivisible and scarce resources to agents in which contracts are allowed. Introduced to economists by Hatfield and Milgrom

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contracts are an attractive way to model many real-world allocation situations in which agents have preferences not only over their partners, but also the terms by which they are assigned. Some interesting interpretations are the allocation of workers to jobs based on wage (Kelso and Crawford, 1982) and the matching of U.S. Military Academy graduates to military branches in which contracts differ by term length (Sönmez (2013), Sönmez and Switzer (2013), Kominers and Sönmez (2013)). In countries with centralized college admissions processes such as Turkey\textsuperscript{1} and China, contracts allow, as an example, for a school to give high priority to a physics prodigy if she applies to study physics, but low priority if she wishes to instead study poetry.

Following this convention in the literature, as an extended analogy, I phrase the model as a \textit{school choice problem}: there are finite sets of students, schools and contracts. Each student can be matched to at most one school under the terms of an available contract. Each school is endowed with a supply of seats which it cannot exceed. Students have strict preferences over contracts, and schools have strict priority rankings over the same contracts. Priorities are fixed exogenously, often by law.

A first-best solution would be to create a mechanism that always results in a matching between students and schools that is \textit{stable} (a student is denied a contract only if the school in question has filled all seats with higher priority contracts), \textit{Pareto efficient for students} (there is no other matching that makes all students no worse off, and at least one student strictly better off) and \textit{group strategy-proof} (no group of students can be better off by jointly misreporting preferences). These desireata emphasize the notion that schools are resources that exist to serve the students and are not considered in welfare notions, nor are they strategic entities.

Putting aside contracts for a moment, the best candidate for such a mechanism is the \textit{(student proposing) deferred acceptance algorithm} of Gale and Shapley (1962), because it always delivers the student optimal stable matching. Therefore, if a stable and efficient allocation exists, it can be found via this mechanism. Unfortunately, there does not always exist a stable and efficient matching, as is illustrated in Example 1 in Section 2.1.

Due to this negative finding, a second-best result is sought in the literature. Because priorities are frequently fixed by law and schools are not permitted to be strategic, the next candidate solution is finding the minimal restriction on (equivalently the maximal domain of) priorities over which efficiency, stability and group strategy-proofness are guaranteed for any preference profile of the students. Without contracts, Ergin (2002) shows that stability and efficiency are irreconcilable whenever school priorities are sufficiently heterogeneous, as these differences can cause cycles between students which result in one student blocking a “trade” between two others. It is precisely the domain of acyclic priorities that guarantee stability, efficiency, and group strategy-proofness for deferred acceptance (Ergin, 2002). Unfortunately, this acyclicity condition is quite restrictive, as cycles may frequently be present in real-world priority structures.

\textsuperscript{1}Balinski and Sönmez (1999) discuss in depth the Turkish admissions process.
This paper asks: Under what conditions can a market designer ensure efficiency and stability when contracts are added to the model?

As matching without contracts is a special case of a matching with contracts model, it is clear that we require acyclicity à la Ergin as a necessary condition for the desired properties. However, Example 2 in Section 2.2 demonstrates that the addition of contracts can result in a new trade-off between stability and efficiency—within a single school’s priority ranking, whereas without contracts, they can only occur across the rankings of several schools. I introduce a class of priority structures, student-lexicographic priorities, that is necessary for stability, efficiency and group strategy-proofness. In contrast to acyclicity which requires homogeneity of different schools’ priorities, student-lexicographic priorities restrict the rankings within each school; every school must sort students in two ways: 1) “top tier” students who are guaranteed admission to a particular school must have all contracts naming them at the top of the rankings of that school, and 2) “second tier” students who are not guaranteed admission to a particular school must have all of their contracts ranked consecutively. Intuitively, this means that a school cannot view contracts involving the same student very differently, and for example, cannot “strongly” favor a physics prodigy studying physics over the same physics prodigy studying poetry.

Although this may seem impossibly restrictive at first glance, real world markets that exhibit this kind of priority structure are the college admissions processes in the United States, Canada and Scotland. Students often declare an intended major in their applications, only to have the option to choose any desired major once enrolled. In this sense, a college cannot favor a student for a particular major while denying her the option of studying a different major.

Theorem 1 in Section 2.3 demonstrates that acyclic and student-lexicographic priorities are together necessary and sufficient for the existence of a stable and efficient matching with contracts. In view of the second theorem in Ergin (2002), this is a negative result.

The main contribution of this paper, however, is twofold. First, this paper offers a systematic framework in which to view matching with contracts. I give a clear definition of what it means for a mechanism with contracts to be “the same as” a mechanism without contracts, and give a recipe on how to extend mechanisms to a world with contracts. Until now, such comparisons between mechanisms have been ad-hoc and based on properties of the resulting matchings. Second, Theorem 2, the main result, proves that student-lexicographic priorities and (a mechanism specific) acyclicity together are necessary and sufficient for any mechanism satisfying independence of irrelevant contracts to guarantee stability, efficiency and group strategy-proofness when contracts are added to the model. Therefore, this paper gives a general cook book to extend the insight of previous “maximal domain” inquiry to many mechanisms, not just a particular one.

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2This is a class of mechanisms that gives bargaining power to the student. It is defined formally in the following section.
As hinted above, many authors have studied maximal domain questions in settings without contracts. Kesten (2006) shows that a more restricted domain of priority structures (a stronger acyclicity condition than the one proposed in Ergin (2002)) is needed for Gale’s Top Trading Cycles algorithm (introduced in Shapley and Scarf (1974) and adapted to the school choice environment by Abdulkadiroğlu and Sönmez (2003)) to be stable, efficient and group strategy-proof. Haeringer and Klijn (2009) show that acyclicity is necessary and sufficient for both deferred acceptance and top trading cycles to result in stable and efficient equilibrium outcomes when students cannot list all schools in their submitted preferences. Ehlers and Erdil (2009) find that a stronger acyclicity condition is necessary and sufficient for the existence of a stable and efficient matching when schools have indifferences over classes of students. Kumano (2009) generalizes the non-contract model to allow for acceptant substitutable priorities, which allow schools to have rankings over sets of students, and shows that acyclicity is necessary and sufficient for the existence of a stable and efficient matching. Kumano and Watabe (2010) take an axiomatic approach to a school choice problem and show that acyclicity is necessary for an equilibrium outcome that is stable and efficient. Kojima (2011) finds that acyclicity is necessary and sufficient for the deferred acceptance algorithm to satisfy robust stability, a stronger truth-telling condition. Kojima (2013) extends Ergin’s analysis to the case of multi-unit demand on the part of students, and shows that an extremely strong acyclicity condition is necessary and sufficient to guarantee stability, efficiency and group strategy-proofness. A recurring theme throughout these papers is that, before introducing contracts, acyclicity conditions are restrictive.

To give practical applications of the main result of this paper to well-known mechanisms, I generalize the main results of several papers listed above in Sections 2, 4, 5 and 6. I show that contractual deferred acceptance (proposed in Hatfield and Milgrom (2005)) fits into the framework of this paper. I also propose a generalized top trading cycles mechanism which, to the best of my knowledge, is the first instance of top trading cycles which can accommodate contracts. Therefore, I am able to generalize the results of Ergin (2002) and Kesten (2006) to the contractual setting, giving the maximal domain of priorities over which deferred acceptance and top trading cycles are stable, efficient and group strategy-proof. I also discuss how this model can be applied to a many-to-many matching framework in which students have multi-unit demand, and show that the main result carries over to this more general setting. This allows me to generalize the main result of Kojima (2013) by giving the exact priority restriction to ensure that the multi-unit deferred acceptance algorithm satisfies desired properties when contracts are added. Before concluding, I also extend the main theorem to allow for acceptant substitutable priorities, allowing schools to rank sets of contracts to allow for diversity considerations; I generalize the result of Kumano (2009), showing that acyclic and lexicographic priorities are equivalent to the efficiency and group strategy-proofness of deferred acceptance when contracts are added.
2 Model

2.1 Setup

Let $I$ denote a set of students, $S$ a set of schools, and $T$ a set of terms under which students and schools can be matched. These sets are finite and disjoint. A feasible contract is an element $x \in \mathcal{X}$ where $\mathcal{X} \subset I \times S \times T$. I write $X_i \equiv \{x|x_1 = i\} \cup \emptyset$ and $X_s \equiv \{x|x_2 = s\}$ to denote the set of contracts within $\mathcal{X}$ involving student $i$ and school $s$, respectively, where $\emptyset$ is each student’s outside option (such as attending a foreign college). Let $X$ be a subset of feasible contracts and similarly define $X_i$ and $X_s$. I assume that $\emptyset \in X_i$ for all $i \in I$ and all $X \subset \mathcal{X}$. Throughout, I assume that a contract is available to a student if and only if it is also available to the school named in the contract, that is.

A preference profile is a vector of linear orders $R^X = (R_i^X)_{i \in I} \in R^{2X}$ over subsets of the set of feasible contracts where $R_i^X$ represents student $i$’s complete, transitive and antisymmetric preference relation over contracts in $X_i$. I restrict that the relative ordering between contracts does not vary as the subset of contracts being considered changes. Formally, this means that for all $i \in I$ and any $\mathcal{X}$, if there exist $x, y \in \mathcal{X}_i$ such that $xR_i^Xy$, then $xR_i^Xy$ for any $X$ containing $x$ and $y$. A contract $x \in X_i$ is acceptable to student $i$ if $xR_i^X\emptyset$. I define $P_i^X$ as the asymmetric subset of $R_i^X$, that is, for each $i \in I$ and $x, y \in X_i$, $xP_i^Xy$ if $xR_i^Xy$ and $yR_i^Xx$.

A priority structure is a pair $(\succeq^X, q) = (\succeq^X_s, q_s)_{s \in S} \in \mathcal{P}^{2X}$ in which $\succeq^X_s$ is a linear order representing school $s$’s complete, transitive and antisymmetric ranking of contracts in $X_s$, while $q_s \geq 1$ denotes the number of seats available at school $s$.\footnote{Note that this description of priorities implies that all contracts naming a particular school are acceptable to that school. Furthermore, the priority structures studied here are responsive, meaning that a school’s ranking of a contract is not affected by the set of other contracts matched to the school in question (Roth, 1985).} Again, I restrict that for all $s \in S$ and any $\mathcal{X}$, if there exist $x, y \in \mathcal{X}_s$ such that $x \succeq^X_s y$, then $x \succeq^X_s y$ for any $X$ containing $x$ and $y$. For ease of exposition, I often drop the capacity vector when referring to the priority structure. I write “contract $x$ has (weakly) higher priority at school $s$ than contract $y$” as $x \succeq^X_s y$. Let $\succ^X_s$ represent the asymmetric subset of $\succeq^X_s$. When there is no ambiguity with respect to the set of contracts, I drop the superscripts in the “$\succeq^X$” and “$\succ^X$” relations, and write, for example, $(R_i, \succeq, X)$ instead of $(R_i^X, \succeq^X, X)$. For ease of exposition, I often drop the capacity vector when referring to the priority structure.

A matching (or assignment) is a correspondence $\mu : I \cup S \rightarrow \mathcal{X}$ satisfying:

\begin{itemize}
  \item $\forall i \in I, \mu(i) \in X_i$,
  \item $\forall s \in S, \mu(s) \subset X_s$ and $|\mu(s)| \leq q_s$, and
  \item $\forall i \in I$ and $\forall s \in S$, if $x \in X_i \cap X_s$ then $\mu(i) = x$ if and only if $x \in \mu(s)$.
\end{itemize}

A matching $\mu$ is blocked by $(i, s) \in I \times S$ via contract $x \in X_i \cap X_s$ if $xP_i\mu(i)$ and either
• $|\mu(s)| < q_s$ and $i$ is not matched to $s$,

• $|\mu(s)| = q_s$ and $i$ is not matched to $s$ and $x \geq y$ for some $y \in \mu(s)$, or

• $|\mu(s)| \leq q_s$ and $i$ is matched to $s$ under some contract $x'$ and $x \geq x'$.

$\mu$ is individually rational if $\mu(i)R_i\emptyset$ for all $i \in I$. A matching is stable if it is unblocked and individually rational. A matching $\mu$ is (Pareto) efficient if there does not exist another matching $\mu'$ such that $\mu'(i)R_i\mu(i)$ for all $i \in I$ and $\mu'(j)P_i\mu(j)$ for some $j \in I$. As schools are viewed as resources to be consumed by students, they are not considered in the efficiency criterion.

For what follows, it will be useful to think of mappings taking subsets of the set of all feasible contracts into final matchings. Formally, I define a mechanism as a function $\varphi$ from $R^2 \times \mathcal{P}^2 \times 2^X$ to the set of matchings. Therefore, a mechanism takes as inputs $X \subset X$ and preferences and priorities over contracts in $X$, and delivers a matching.

A mechanism $\varphi$ is efficient if $\varphi(R, \succeq, X)$ is efficient for any $(R, \succeq, X)$, and it is stable if $\varphi(R, \succeq, X)$ is stable for any $(R, \succeq, X)$. A mechanism $\varphi$ is group strategy-proof if no collection of students can improve their assignments by jointly misreporting preferences; that is, for any $(R, \succeq, X)$ there does not exist some non-empty subset $I' \subset I$, and $R' \in R$ such that $\varphi_i((R_{I \setminus I'}, R'_{I'}), \succeq, X) R_i\varphi_i(R, \succeq, X)$ for all $i \in I'$ and $\varphi_j((R_{I \setminus I'}, R'_{I'}), \succeq, X) R_j\varphi_j(R, \succeq, X)$ for some $j \in I'$. A rule is strategy proof if truth-telling is a weakly dominant strategy for all students.

In this setting, efficiency is often viewed as the first-order criterion for judging the quality of a matching, but stability is an important fairness property. Empirically, Roth (2002) finds that stable mechanisms are more likely to remain in use in market design settings that non-stable ones. Unfortunately, it is often impossible to generate a matching that is both stable and fair; as the following example demonstrates, there does not always exist a stable and efficient matching, even without contracts.

**Example 1** : There are two schools, each with one seat, and three students. Consider the following priorities and preferences over the set of feasible contracts. (This notation indicates that student $i$ prefers school $s$ to school $r$ and school $r$ to being unmatched, and so on.)

<table>
<thead>
<tr>
<th>$R_i$</th>
<th>$R_j$</th>
<th>$R_k$</th>
<th>$\geq_r$</th>
<th>$\geq_s$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s$</td>
<td>$r$</td>
<td>$r$</td>
<td>$i$</td>
<td>$k$</td>
</tr>
<tr>
<td>$r$</td>
<td>$s$</td>
<td>$s$</td>
<td>$j$</td>
<td>$i$</td>
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<tr>
<td>$k$</td>
<td>$j$</td>
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4 This single-contract blocking definition is without loss of any generality because the priorities are substitutable (Hatfield and Kominers, 2013).

5 Stability is viewed as a fairness property because it seems quite plausible that a student who is rejected from a school which has filled its seats with lower priority students has a legitimate cause for complaint, and indeed, the school may be interested in listening.
The unique stable matching is $\mu(i) = r$, $\mu(j) = \emptyset$, and $\mu(k) = s$. However, this matching is not efficient, as it is Pareto dominated by $\mu'$ with $\mu'(i) = s$, $\mu'(j) = \emptyset$, and $\mu'(k) = r$. In this example, student $k$ is ranked very differently at the two schools, and she is only a competitive applicant at school $s$. This is problematic because a cycle has formed: students $i$ and $k$ would like to trade their assignments, but the resulting allocation would admit student $j$ and school $r$ as a blocking pair.

Since school priorities are fixed, a second-best solution is finding the minimal restriction on the priority structure that guarantees the existence of a stable and efficient matching for any preference profile of the students. Ergin (2002) proves that Ergin acyclicity of the priority structure is both necessary and sufficient for this guarantee. Ergin acyclicity puts a limit on how different the rankings of schools can be from one another. Below is the acyclicity condition found in Ergin (2002), which I have adapted to the framework of contracts.

**Definition 1**: Let $(\succeq, q)$ be a priority structure. An Ergin cycle is constituted of distinct $r, s \in S$ and distinct $i, j, k \in I$ and possibly distinct $t, u, v, w, p \in T$ such that the following are satisfied:

(C) Cycle condition: $(i, r, t) \succeq_r (j, r, u)$ and $(k, r, v) \succeq_r (k, s, w)$ and $(i, s, p) \succeq_s (i, s, p)$,

(S) Scarcity condition: There exist (possibly empty) disjoint sets of contracts $X_r, X_s \subset X \setminus \{X_i \cup X_j \cup X_k\}$ such that:

1. $|X_r| = q_r - 1$ and $|X_s| = q_s - 1$,
2. $x \succeq_r (j, r, u)$ for every $x \in X_r$ and $y \succeq_s (i, s, p)$ for every $y \in X_s$, and
3. $|X_r \cap X_\ell| + |X_s \cap X_\ell| \leq 1$ for all $\ell \in I$.

(C) gives a heterogeneity restriction on the relative ordering of contracts across schools. (S) describes a situation where the students in question are actually competing over seats at the schools: (1.) there exist sets of contracts that are smaller than each schools’ respective capacity by one seat that have (2.) higher priority at the respective schools than those held by students $j$ and $i$ and (3.) that no student is named in more than one of these contracts. A priority structure is Ergin acyclic if it does not contain any Ergin cycles.

The preceding definition is stated generally with no restriction on the number of available contracts between each school and each student. This causes the most noticeable difference between Definition 1 and the original definition of an Ergin cycle (Ergin, 2002); point (3.) is necessitated by the addition of contracts. By restricting the number of possible contracts between each school and student to be less than or equal to one, it is possible to state the main finding of Ergin (2002):
**Proposition 1** : Let $|X_i \cap X_s| \leq 1$ for all $i \in I$ and $s \in S$. For any priority structure $(\succeq, q)$, the following are equivalent:

- There exists a (unique) stable and efficient matching for any $R$,
- $(\succeq, q)$ is Ergin acyclic, and
- Gale and Shapley’s deferred acceptance algorithm is group strategy-proof, stable and efficient.

Therefore, there is a stable and efficient matching precisely when the priority structure is Ergin acyclic and there is a trivial contract structure. However, the following section shows that when multiple contracts are available between schools and students, acyclicity is insufficient for the guaranteed existence of a stable and efficient matching.

### 2.2 The case with contracts

This section introduces contracts to the model. The following example highlights a new possible conflict between stability and efficiency in the presence of contracts.

**Example 2** : There is one school with a single seat, and two students. Consider the priority order and preferences over the set of feasible contracts:

<table>
<thead>
<tr>
<th>$\succeq_s$</th>
<th>$R_i$</th>
<th>$R_j$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(i, s, t)$</td>
<td>$(i, s, u)$</td>
<td>$(j, s, t)$</td>
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<tr>
<td>$(j, s, t)$</td>
<td>$(i, s, t)$</td>
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<tr>
<td>$(i, s, u)$</td>
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</table>

This priority structure satisfies Definition 1, as there is only one school in the market (recall that acyclicity places restrictions on how different the priority rankings of two schools can be). The unique stable matching is $(\mu(i), \mu(j)) = ((i, s, t), \emptyset)$ and the efficient allocations are $(\mu'(i), \mu'(j)) = ((i, s, u), \emptyset)$ and $(\mu''(i), \mu''(j)) = (\emptyset, (j, s, t))$. Therefore, in contrast to Proposition 1, there is no stable matching that is also efficient; student $i$ can only ensure herself a seat by giving up her most preferred terms. Intuitively, when contracts are introduced, it is possible for priority cycles to exist within schools as well as between schools.

It is now apparent that acyclicity is not enough to guarantee stability and efficiency once contracts are introduced. The analysis proceeds in the direction of ruling out cycles that can occur within the ranking of a single school. The
following definition introduces the concept of two contracts being *indistinguishable* to a school; a school is unable to reject one contract but not the other if two contracts are indistinguishable.

**Definition 2**: Two contracts \( x, y \in X_i \cap X_s \) with \( x \succ_s y \) are *indistinguishable* to school \( s \) with respect to \( X \) if either

- \( |\{ j \in I \setminus \{ i \} : \exists z \in X_j \cap X_s \text{ such that } z \succ_s y \}| \leq q_s - 1 \), or
- there is no \( z \in X \setminus X_i \) such that \( x \succ_s z \succ_s y \).

In words, two contracts from the same individual are indistinguishable to a school if the contracts are sufficiently high on its priority list relative to its quota so that the students named in the contract are guaranteed seats at the school if they choose (the first bullet), or there is no contract from another student which interrupts the two contracts in question on its priority list (the second bullet). Note that the school is able to distinguish between all contracts in Example 2.

One possibility is that all of the contracts involving the same student are indistinguishable to a school. This corresponds to the notion that a student is ultimately able to select her most preferred contract at a school once she gets her foot in the door, meaning that a school cannot choose to admit a student and simultaneously deny her any contract. This practice is common in college choice in the United States, Canada and Scotland; while a student may claim to be applying to a school to specialize in physics, once the student is admitted she is free to change her major. This restriction, which I call *student-lexicographic priorities*, is an important counterpart to acyclicity. Whereas acyclicity restricts how the priority structure can rank contracts across different schools, student-lexicographic priorities restrict the ranking of contracts within each school.

**Definition 3**: A priority structure \( \succeq \) is *student-lexicographic* if for all \( i \in I, s \in S \), and distinct \( x, y \in X_i \cap X_s \), \( x \) and \( y \) are indistinguishable to school \( s \) with respect to \( X \).

Throughout, I will denote the set of all priority structures which are student-lexicographic by \( \mathcal{L} \). The following figure gives an illustration of student-lexicographic priorities for a single school.

**Example 3**: There is one school with \( q \) seats. The following priority structure is student-lexicographic. The dotted line separates the contracts from the top \( q \) students from the remaining students.
In Example 3 the school ranks all contracts (except those belonging to the top \( q \) students) involving the same student consecutively in its priority order. The top \( q \) students’ contracts are ranked without restriction, other than being at the top of the ranking. In fact, any student-lexicographic priority structure requires every school’s rankings to look like the illustration in Example 3, although there is no restriction that a student whose contracts are found near the top of one school’s rankings must be found near the top of another school’s rankings. Therefore, unlike acyclicity, it is straightforward to identify a student-lexicographic priority structure. The following proposition formalizes this point.

**Proposition 2**: A priority structure is student-lexicographic if and only if every school \( s \) ranks every contract from the top \( q \) students above the other students and each student not in the top \( q \) has her contracts ranked consecutively.

In practice, the restriction of student-lexicographic priorities requires that every school creates two tiers of students. The first tier (those above the dotted line) are those who are guaranteed admission to a particular school, and the contracts for these students can be ranked arbitrarily. The second tier of students (those below the dotted line) are only admitted if enough students more favored by the school choose to study elsewhere, and the contracts from each of these students

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6Ergin (2002) gives a characterization of acyclic priority structures. This condition is relatively difficult to verify when schools have more than 1 seat, as it requires checking all schools’ rankings simultaneously.
must be ranked consecutively.

2.3 Existence of a stable and efficient matching (and how to find it)

Preventing cycles across school priorities is important in guaranteeing stability and efficiency. Additionally, Example 2 suggests that it may be important to also limit cycles within the priority rankings of a school (which is achieved with student-lexicographic priorities). This section will show that Ergin acyclicity and student-lexicographic priorities are together necessary and sufficient to guarantee a stable and efficient matching.

Of course, the existence of a stable and efficient matching is not very useful without a means of finding it. To this end, I define (student proposing) deferred acceptance, which is an extension of the original deferred acceptance mechanism of Gale and Shapley (1962) to markets with contracts. Fleiner (2003) and Hatfield and Milgrom (2005) show that this mechanism always delivers the student optimal stable matching. The description below is based on Kominers and Sönmez (2013).

Definition 4: The student proposing deferred acceptance mechanism (denoted $\varphi^{DA}$) involves school choice sets such that for all $s \in S$ and for all $Y \subset X_s$,

$$C_s(Y) = \max_{Z \supseteq Y} \{Z \subset Y : \forall i \in I, \forall x, y \in Z, x \in X_i \Rightarrow y \notin X_i \} \land \{|Z| \le q_s\}$$

In words, the school holds the highest priority contracts available with the restriction that only one contract is held for each student and the number of contracts held does not exceed the supply of the school in question. Students propose contracts to schools in a sequence of steps $t = 1, 2, ...$

Step 1: An arbitrary student $i^1 \in I$ proposes his most preferred contract $x^1 \in X_{i^1}$. This contract names some school $s^1 \in S$. School $s^1$ holds $x^1$ if $x^1 \in C_{s^1}({\{x^1\}})$, and rejects $x^1$ otherwise. Set $A^2_{s^1} = C_{s^1}({\{x^1\}})$ and set $A^2_s = \emptyset$ for all $s \neq s^1$. These sets denote the set of available contracts for each school at the beginning of Step 2.

Step $t$: Let $J^t$ be the set of students named in a contract which is held by any school after Step $t - 1$. An arbitrary student $i^t \in I \setminus J^t$ proposes his most preferred contract $x^t \in X_{i^t}$ which he has not proposed in a previous step. This contract names some school $s^t \in S$. School $s^t$ holds all contracts $x \in C_{s^t}(A^t_{s^t} \cup \{x^t\})$, and rejects all others. All other $s \neq s^t$ continue to hold all contracts they held at the end of Step $t - 1$. Set $A^{t+1}_{s^t} = C_{s^t}(A^t_{s^t} \cup \{x^t\})$ and set $A^{t+1}_s = A^t_s$ for all $s \neq s^t$. 

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The algorithm terminates at some step $T$ when no student proposes any new contract. This means each student either has a contract that is held by a school, or that he has already proposed all acceptable contracts and been universally rejected. The set $(A_T^1, A_T^2, ..., A_T^S)$ gives the final matchings.\footnote{In the case in which $|X(i) \cap X(s)| \leq 1$ for all $s \in S$, $i \in I$, the algorithm becomes identical to the non-contractual deferred acceptance mechanism (denoted $\varphi^{DA}$) in which students propose to schools, not contracts. For a full description, see Gale and Shapley (1962).}

**Theorem 1**: The following are all equivalent:

- $\varphi^{DA}(\cdot, \succeq, \cdot)$ is efficient,
- $\varphi^{DA}(\cdot, \succeq, \cdot)$ is group strategy-proof, and
- the priority structure is Ergin acyclic and student-lexicographic.

This is a generalization of Ergin (2002), in that it fully characterizes the domain of priority structures over which (contractual) deferred acceptance guarantees stability, efficiency, and group strategy-proofness.

There is also an interesting negative interpretation of this theorem that stems from the (necessary) restriction of student-lexicographic priorities. As seen in Proposition 2, this means that every school must prioritize all the contracts involving the same student consecutively (excluding the contracts of the top tier students). In the context of labor matching in which contracts are wages (Kelso and Crawford, 1982) this means that if a firm gives higher priority to worker $i$ at wage $w$ to worker $j$ at wage $v$, then it must give higher priority to $i$ at any wage than worker $j$ at every wage. In the context of military matching in which contracts are service length, student-lexicographic priorities mean that if cadet $\ell$ signed for $t$ years is given higher priority than cadet $k$ signed for $u$ years, then cadet $\ell$ always has higher priority than cadet $k$, regardless of term length. Clearly, this restriction prevents the “receiver” from having any discriminatory power over the terms of a matching. Therefore, this theorem tells us that a market designer may have to forgo at least one desired property if she wishes to enrich the environment to include contracts and simultaneously allow both sides of the market to be picky about the contracts selected, not just the agents to whom they are matched.

### 2.4 A general class of mechanisms that recover desired properties

So far, the focus has been on the existence of a stable and efficient matching, and how to find it using deferred acceptance. Nevertheless, some markets may use different mechanisms to allocate students to schools, as market designers may have different objectives (i.e. one designer may require an efficient allocation and
view stability as a bonus, while another may value stability above efficiency). Therefore, it is important to generalize the results of Theorem 1 to a general class of mechanisms. To this end, this section introduces a set of mechanisms which satisfy independence of irrelevant contracts, which, informally speaking, give bargaining power to students. The main result shows that stability and efficiency can be guaranteed for these mechanisms.

Furthermore, this section gives a mathematically rigorous way to construct mechanisms for markets with contracts which are analogues of existing non-contract mechanisms. In the past, such comparisons have been ad-hoc, and based upon properties of delivered matchings. For example, $\varphi^{DA}$ is seen as an extension of the original deferred acceptance mechanism because it always delivers the student optimal stable matching. However, this is not a criterion which can be applied to extend every mechanism to markets with contracts. This section clearly defines what it means for a mechanism with contracts to be “the same as” one without contracts, and gives a recipe for creating an extension of a non-contract mechanism.

Finally, this section presents a general framework to study acyclicity notions. As seen in Kesten (2006), Ehlers and Erdil (2009), Hatfield et al. (2013) and Kojima (2013), different acyclicity notions are necessary to ensure desired properties for different mechanisms. Since a goal of this paper is to generalize maximal domain results for stability and efficiency from settings without contracts for many mechanisms, it is necessary to understand which acyclicity condition is necessary, and indeed, what an acyclicity condition formally means.

A class of well-behaved contract mechanisms

In order to find a sufficient condition for the existence of a stable and efficient matching it is first necessary to specify what “a world with contracts” means and how this differs from “a world without contracts.” In traditional non-contract allocation settings, a match between student $i$ and school $s$ can be denoted $(i, s)$. Note that the same matching can be written as contract $(i, s, t)$ for some term $t$ and no generality is gained or lost in this notation if there is only one possible contract between $i$ and $s$, that is, consider the general model presented in this paper but let $T = \{t\}$. Therefore, markets without contracts are equivalent to markets with contracts in which there is at most one contract available between each school in the context of this exposition. For clarity, the following definition of a contract-free mechanism is offered to differentiate between settings with and without contracts.

**Definition 5**: A mechanism $\varphi$ is contract-free if it is only defined when $|X_i \cap 8$.

---

For example, New Orleans has adopted a top trading cycles mechanism for incoming high school students (http://www.nola.com/education/index.ssf/2012/04/centralized_enrollment_in_reco.html), and Turkey uses a constrained serial dictatorship in college admissions (Balinski and Sönmez (1999)). Both of these mechanisms guarantee efficiency (and group strategy-proofness) but not stability.
$|X_i| \leq 1$ for all $i \in I$ and $s \in S$. A mechanism is *contractual* if it is not contract-free.

Throughout, I generally reserve the notation $\varphi$ for contract-free mechanisms and $\bar{\varphi}$ for contractual mechanisms.

Having now defined how a contractual mechanism differs from a contract-free mechanism, the next step is defining a class of mechanisms to which it is possible to extend the maximal domain results discussed above. The first step in doing so is noting that the analysis thus far in the paper has only focused on school priorities. However, it is possible to say more about certain contracts by taking the preferences of the students into account. Combining the priority structure with the preference profile, I describe contracts that are undesirable to both the school and the student named in the contract. The reason for defining these *irrelevant* contracts is that a “good” mechanism should never be affected by the presence of such a contract.

**Definition 6**: A contract $x \in X_i \cap X_s$ is *irrelevant* with respect to $X$ if:

- there exists $y \in X_i \cap X_s$ with $x$ and $y$ indistinguishable to $s$ (with respect to $X$), and $yP_ix$, or
- there exists $y \in X_i \cap X_s$ with $y \succ_s x$ and $yP_ix$, or
- $\emptyset P_i x$.

A contract is *relevant* if it is not irrelevant.

It should be apparent that any matching which involves an irrelevant contract is suboptimal. Indeed, such an assignment is necessarily inefficient, which provides justification for an irrelevant contract to be ignored in the assignment process, especially as one goal is to ensure a student efficient outcome. A mechanism satisfies *independence of irrelevant contracts* if the omission of an irrelevant contract does not impact the matching generated by the mechanism. The following definition states this condition formally.

**Definition 7**: For any $(R, \succeq, X)$ suppose there exists an irrelevant contract $x$. A mechanism $\bar{\varphi}$ satisfies *independence of irrelevant contracts* (IIC) if $\bar{\varphi}(R, \succeq, X) = \bar{\varphi}(R^{X \setminus \{x\}, \succeq^{X \setminus \{x\}}, X \setminus \{x\})$.

Inherent in the definitions of irrelevant contracts and IIC mechanisms is the notion that students have more bargaining power than schools. A contract cannot be judged irrelevant (and hence thrown out) if a student prefers it to every other contract with higher priority, but a contract can be deemed irrelevant even if a school gives it a higher priority than every other contract which is strictly preferred by the student as long as there is not much difference in the priority
of the two contracts (i.e. they are indistinguishable). This essentially says that any time a student is faced with consecutively ranked contracts in the priority ranking of a school, she can always pick her most preferred of these options. The appendix outlines how this condition is satisfied by well-known mechanisms.

As with other instances of deletion of contracts from preferences (one recent occurrence of this is Example 4 in Kesten (2010)), it is imperative to be cautious regarding the way in which contracts are deleted. Indeed, if the set of irrelevant contracts is dependent on the order of deletion of contracts, then the class of IIC mechanisms may not offer a clean way to ensure efficiency and stability. However, if the priority structure is student-lexicographic then there is no need to worry about the order of deletion the set of irrelevant contracts. The following proposition formalizes this point.

**Proposition 3**: Let \((\succeq, q)\) be student-lexicographic. For any \((R, \succeq, X)\) let \(Q^X\) be the set of all irrelevant contracts, and let \(x \in Q^X\). Then \(Q^X \setminus \{x\} = Q^X \setminus \{x\}\).

As shown in Theorem 1, student-lexicographic priorities are necessary for the guaranteed existence of an efficient and stable matching. In addition, the class of IIC mechanisms is guaranteed to function “correctly” exactly when this necessary condition is met.\(^9\) Therefore, an IIC mechanism never depends on the order of deletion of contracts when efficiency and stability are guaranteed.

I now state the final definition which can be viewed as an anonymity condition for contracts. **Contract neutrality** dictates that a mechanism cannot inherently favor one contract over another, and would produce the same matching if the names of two contracts were switched. This rules out, for example, a mechanism that treats the contract “student \(i\) will study physics at school \(s\)” differently from “student \(i\) will study poetry at school \(s\)” merely on the basis that the mechanism values the term “physics” differently from the term “poetry.”

**Definition 8**: Let \(\varphi\) satisfy IIC. For any \((R, \succeq, X)\) suppose there exist \(x, y \in X_i \cap X_s\) where \(x\) and \(y\) are indistinguishable to school \(s\) and \(xP_i y\). Let \(R_i^*\) reverse the rankings of \(x\) and \(y\), that is, for \(w, z, v \in X_i\):

- \(yP_i^* x\),
- \(yR_i^* w R_i^* x\) if and only if \(xR_i w R_i y\), and

\(^9\)It is possible to derive the main result of this paper with a weaker version of IIC. For any \((R, \succeq, X)\) let \(Q^X\) be the set of all irrelevant contracts. A mechanism \(\varphi\) satisfies weak independence of irrelevant contracts (WIIC) if \(Q^X \setminus \{x\} = \varphi(R, \succeq, X) = \varphi(R^X \setminus \{x\}, \succeq, X \setminus \{x\})\) for any contract \(x \in Q^X\). Nevertheless, as Proposition 3 shows, this distinction is not important to answer the maximal domain questions that are discussed in this paper. Therefore, for simplicity, I proceed with IIC as a regularity condition, not WIIC. Of course, as WIIC is weaker than IIC, all proofs go through without modification if WIIC is used instead.
• \( zR^*_iv \) if and only if \( zR^*_iv \) for all other \( v \in X_i \).

Then \( \varphi \) is contract neutral (CN) if \( \varphi_j ((R_{I\setminus i}, R^*_i), \succeq, X) = \varphi_j (R, \succeq, X) \) for all \( j \neq i \), and

\[
\varphi_i ((R_{I\setminus i}, R^*_i), \succeq, X) = \begin{cases} 
\varphi_i (R, \succeq, X) & \text{if } \varphi_i (R, \succeq, X) \neq x \\
y & \text{if } \varphi_i (R, \succeq, X) = x 
\end{cases}
\]

It is worth noting that contract neutrality is not necessary to guarantee a stable and efficient allocation. As the proof of Theorem 2 (found in the appendix) shows, contract neutrality is only necessary to guarantee group strategy-proofness. Therefore, if a market designer is willing to forgo this property, she need not focus her attention on contract neutral mechanisms (although I do not believe this is a very strong restriction).

### Extending non-contract mechanisms

Having formalized exactly how IIC mechanisms must function, I now formalize what it means for a contractual mechanism to be “the same as” a contract-free mechanism. I use a very weak equivalence notion: if deleting all irrelevant contracts from the priorities and preferences yields at most a single contract between each school and each mechanism, then the contract-free mechanism run on these reduced preferences and priorities must yield the same matching as its extension on the original preferences and priorities. Note that the following definition, while formalizing this point, does not restrict the extended mechanism in the case that there are at least two relevant contracts between a particular school and a particular student. Therefore there are many possible extensions of a given mechanism.

**Definition 9:** Let \( \varphi \) be a contract-free mechanism, and \( \bar{\varphi} \) be a mechanism satisfying IIC. Let \( Q^X \) be the set of all irrelevant contracts associated with \( (R, \succeq, X) \), and suppose \( \varphi (R \setminus Q^X, \succeq \setminus Q^X, X \setminus Q^X) \) is well-defined (there is at most one relevant contract between each \( i \) and \( s \)). Then \( \bar{\varphi} \) is an extension of \( \varphi \) if \( \bar{\varphi} (R, \succeq, X) = \varphi (R \setminus Q^X, \succeq \setminus Q^X, X \setminus Q^X) \).

A natural question is: When does the definition of extension bind? The following proposition answers this question, and gives the exact domain of priorities over which it is possible to reject the claim “\( \bar{\varphi} \) is an extension of \( \varphi \).”

**Proposition 4:** A priority structure is student-lexicographic if and only if there is at most one relevant contract with respect to \( X \) between each school and each student for every preference profile.

Proposition 4 turns out to be extremely useful as it provides insight into how to extend mechanisms to include contracts. Building off of this proposition, Remark 1 gives a cook book on how to construct an extension of a contract-free mechanism.
**Remark 1**: \( \bar{\varphi} \) is an extension of \( \varphi \) if \( \bar{\varphi} \) is equivalent to \( \varphi \) while only considering the students’ most preferred contracts for any \((R, \succeq, X)\). More precisely, for any \((R, \succeq, X)\) define \( Z_X = \{ z \in X : \exists x \text{ s.t. } x \preceq_i z \text{ for some } i \in I \} \). If \( \bar{\varphi}(R, \succeq, X) = \varphi(R \setminus Z_X, \succeq \setminus Z_X, X \setminus Z_X) \), then \( \bar{\varphi} \) is an extension of \( \varphi \).

Clearly, \( \bar{\varphi}_DA \) was not constructed following Remark 1. This emphasizes the point that there are many possible extensions of \( \varphi_{DA} \), and the one presented in the previous section is just a particular one (although other extensions will not always deliver a stable matching).

**General acyclicity conditions**

Throughout, I use the term “acyclicity condition” to denote a generic condition on the relative ordering between contracts in different schools together with a scarcity condition. Definition 1 formally defines Ergin acyclicity. As shown in Proposition 1, Ergin acyclicity is necessary for the guaranteed existence of a stable and efficient matching. Therefore, any mechanism that is able to find a stable and efficient matching must require an acyclicity condition no weaker than Ergin’s. On the other hand, recalling that the intuitive purpose of acyclicity is to limit the differences in school rankings across different schools, there is also a strongest acyclicity condition, namely, a homogeneous priority structure. The following definition formally defines this strongest acyclicity condition, and is a generalization of the virtual homogeneity restriction of Hatfield et al. (2013).

**Definition 10**: A priority structure \( (\succeq, q) \) is **homogeneous** if there exist no distinct \( r, s \in S \) and \( i, j \in I \) and possibly distinct \( t, u, v, w \in T \) such that

\[
\text{(C)} \quad (i, r, t) \succ_r (j, r, u) \quad \text{and} \quad (j, s, v) \succ_s (i, s, w) \quad \text{and}
\]

\[
\text{(S)} \quad \text{There exists a set of contracts } X_s \subset X \setminus \{X_i \cup X_j\} \text{ such that}
\]

1. \( |X_s| = q_s - 1 \),
2. \( x \succeq_s (i, s, w) \) for all \( x \in X_s \), and
3. \( |X_s \cap X_\ell| \leq 1 \) for all \( \ell \in I \).

Unlike Ergin acyclicity which allowed more heterogeneity between the rankings of different schools, a homogeneous priority structure requires the rankings of every student (except those who are guaranteed a spot at every school) to be virtually identical across schools.

With Definition 10 in hand, I can now formally define an acyclicity condition.

**Definition 11**: An **acyclicity condition** is any restriction that is implied by a homogeneous priority structure and in turn implies Ergin acyclicity.

The main result will show that acyclicity and student-lexicographic priorities are together necessary and sufficient to guarantee stability and efficiency. Therefore, the final preliminary result shows that neither of these two restrictions
rules the other one out.

**Proposition 5**: The set of student-lexicographic priority structures that satisfy any acyclicity condition is non-empty.

I have now stated all the definitions and preliminary results necessary for the main result. The next section is dedicated to this aim.

## 3 Main Result

Below is the statement of the main theorem. It states that student-lexicographic priorities and an (mechanism specific) acyclicity condition are necessary and sufficient to ensure stability, efficiency and group strategy-proofness with contracts.

**Theorem 2**: Let $\varphi$ be a contract-free mechanism. Let $\varphi$-acyclicity be the weakest acyclicity restriction on the priority structure such that $\varphi$ is stable, efficient and group strategy-proof. Let $\bar{\varphi}$ be an extension of $\varphi$ satisfying independence of irrelevant contracts and contract neutrality. Then

1. $\bar{\varphi}$ is stable and efficient if and only if the priority structure is student-lexicographic and satisfies $\varphi$-acyclicity, and
2. $\bar{\varphi}$ is stable and group strategy-proof if and only if the priority structure is student-lexicographic and satisfies $\varphi$-acyclicity.

The first part of the theorem states that student-lexicographic priorities and $\varphi$-acyclicity are necessary and sufficient to guarantee that IIC mechanism $\bar{\varphi}$ is always stable and efficient. The second part makes a similar conclusion about stability and group strategy-proofness. Therefore, the set of priority structures that recovers all of the desired properties is precisely the set of $\varphi$-acyclic and student-lexicographic priorities.

This is actually slightly stronger than the informal statements of Theorem 2 in the introduction, as the above domain restriction of student-lexicographic priorities and $\varphi$-acyclic priorities is sufficient, but not necessary, for each of stability, efficiency and group strategy-proofness. For example, the top trading cycles mechanism (described in the following section) is group strategy-proof and efficient for the priority structure in Example 2, but it is not stable (indeed, the mechanism is always group strategy-proof and efficient). Similarly, $\bar{\varphi}^{DA}$ is stable for all priority structures, but not necessarily efficient and group strategy-proof. In general, many mechanisms that are group strategy-proof are also efficient.\(^{10}\)

\(^{10}\)Under the assumption that a mechanism is *onto* (any matching $\mu$ is attainable for some preference vector $R$), it is easy to prove that group strategy-proofness implies efficiency. The proof is as follows: Suppose for contraposition that for some $(R, \succeq, X)$ a mechanism $\bar{\varphi}$
Given that student-lexicographic priorities may not be found in many real-world markets, this result indicates that often times a market designer may have to choose whether she would rather guarantee efficiency or stability. (One exception, as mentioned before, is the approach that US, Canadian and Scottish colleges take to this problem: admit students, and then allow them to pick their contracts ex-post.)

The following section demonstrates examples of the “cook book” provided in Theorem 2 and how it applies to another real-world mechanism, top trading cycles.

4 Top trading cycles

I now consider a different mechanism, an extension of top trading cycles. Unlike $\varphi^{DA}$ which always yields a stable matching, top trading cycles (denoted $\varphi^{TTC}$) is always efficient and group strategy-proof (Abdulkadiroğlu and Sönmez, 2003). I will show that these properties carry over to the extended mechanism $\varphi^{TTC}$. To the best of my knowledge, this is the first top trading cycles mechanism that can accommodate contracts. $\varphi^{TTC}$ is defined as follows, and is based upon Gale’s top trading cycles algorithm.

**Definition 12**: The top trading cycles mechanism proceeds in a sequence of steps $t = 1, 2, ...$:

Step 1: Each $s \in S$ is endowed with a counter, $C^0_s = q_s$. Each $s \in S$ points to the student named in its highest priority contract. Each student similarly points to the school named in his most preferred contract. The null object points to all students, and has $q_s = \vert I \vert$. Due to finiteness, there is at least one cycle. Let $I^1$ and $S^1$ be the sets of students and schools involved in cycles. Each $i \in I^1$ is matched to school $s \in S$ to which student $i$ is pointing via the student’s most preferred contract. For all $s \in S^1$ set $C^1_s = q_s - 1$ and $C^1_s' = q_s'$ for all $s' \notin S^1$. Let $J^1 = I^1$ and let $W^1 = \{s \in S : C = 0\}$. Define $X^1 \equiv X \setminus X_{J^1} \cup X_{W^1}$.

Step $t$: Each $s \in S \setminus W^{t-1}$ points to the student named in its highest priority contract (with respect to priorities $\succeq X^{t-1}$). Each student $i \in I \setminus J^{t-1}$ similarly points to the school named in his most preferred contract (with respect to preferences $R^{X^{t-1}}$). The null object points to all students. There is at least one cycle. Let $I^t$ and $S^t$ be the sets of students and schools involved in cycles. Each $i \in I^t$ is matched to $s \in S^t$ to which student $i$ is pointing. If there is a matching $\mu$ such that $\mu(i) R_i \varphi_i(R, \succeq, X)$ for all $i$ and $\mu(j) P_j \varphi_j(R, \succeq, X)$ for some $j$. But since $\varphi$ is onto by assumption, there is some $R^*$ for which $\mu = \varphi(R^*, \succeq, X)$. Then $\varphi$ is not group strategy-proof.

$\square$
some school \( s \in S \) via the student’s most preferred contract, that is, the \( R^t_{i}-1 \)-maximal element of \( X^t_{i} \cap X^t_{s} \). For all \( s \in S \) set \( C^t_{s} = C^t_{s} - 1 \) and \( C'_{s} = C'_{s} - 1 \) for all \( s' \notin S \). Let \( J^t = J^t \cup I^t \) and let \( W^t = \{ s \in S : C^t_{s} = 0 \} \). Define \( X^t = X^t \setminus X^t_{J^t} \cup X^t_{W^t} \).

The allocation terminates at some step \( M \) when \( W^M = S \) or \( J^M = I \), whichever comes first; either no acceptable contracts remain to unmatched students, or all of the seats are filled at every school. In the case in which \( |X^i \cap X^s| \leq 1 \) for all \( s \in S \), \( i \in I \), the algorithm becomes identical to the contract-free top trading cycles mechanism, denoted \( \varphi^{TTC} \) (see Abdulkadiroğlu and Sönmez (2003) for a formal description).

Since this is a new top trading cycles mechanism, I will first show that this mechanism is always efficient and group strategy-proof for any priority structure, just like \( \varphi^{TTC} \). I will then apply Theorem 2 to find the maximal domain over which stability can also be recovered. Taking the hint from Remark 1, I make the following observation:

**Observation 1:** \( \bar{\varphi}^{TTC} \) yields the same matching as \( \varphi^{TTC} \) while treating the student’s most preferred contract as her highest priority contract at every school for any \((R, \succeq, X)\). More precisely, for any \((R, \succeq, X)\) let \( x^*_{i,s} \) be student \( i \)'s favorite contract at school \( s \), and let \( x'_{i,s} \) be student \( i \)'s highest priority contract at school \( s \). Let \( \tilde{\succeq}^X \) switch the rankings of \( x^*_{i,s} \) and \( x'_{i,s} \) for all \( i \) and \( s \). Let \( Z^X = \{ z \in X \mid z \neq x^*_{i,s} \text{ for any } i \in I \text{ and any } s \in S \} \). Then \( \bar{\varphi}^{TTC}(R, \succeq, X) = \varphi^{TTC}(R \setminus Z^X, \tilde{\succeq}^X \setminus Z^X, X \setminus Z^X) \).

It is well known that \( \varphi^{TTC} \) is efficient and group strategy-proof (Abdulkadiroğlu and Sönmez, 2003). Coupling this result with Observation 1, the following result is evident.

**Proposition 6:** \( \bar{\varphi}^{TTC} \) is efficient and group strategy-proof.

Having now shown that \( \bar{\varphi}^{TTC} \) satisfies the same efficiency and group strategy-proofness characteristics as \( \varphi^{TTC} \), I turn my focus to applying the main result to these mechanisms.

As in Ergin (2002), Kesten (2006) provides an acyclicity condition under which \( \varphi^{TTC} \) is stable, efficient and group strategy-proof. Again, I have adapted the definition to be consistent with the inclusion of contracts.

**Definition 13:** Let \((\succeq, q)\) be a priority structure. A Kesten cycle is constituted of distinct \( r, s \in S \) and distinct \( i, j, k \in I \) and possibly distinct \( t, u, v, w, q \in T \) such that the following are satisfied:

(C) Cycle condition: \( (i, r, t) \succ_r (j, r, u) \succ_r (k, r, v), (k, s, w) \succ_s (i, s, q) \) and \( (k, s, w) \succ_s (j, s, z) \).
(S) Scarcity condition: There exist (possibly empty) disjoint sets of contracts $X_r \subset X_r \setminus \{X_i \cup X_j \cup X_k\}$ such that

1. $|X_r| = q_r - 1$,
2. For every $x \in X_r$ either $x \succ_r (i, r, t)$, OR $x \succ_r (j, r, u)$ and $(k, s, w) \succ_s x$, and
3. $|X_r \cap X_r' \cap X_r''| \leq 1$ for all $\ell \in I$ and for all $s' \in S$.

This scarcity condition is slightly different than the one found in Ergin (2002). (1.) says that there exists a set of contracts that is smaller than school $r$’s capacity by one that has (2.) higher priority at school $r$ than that held by student $i$, OR a higher priority than student $j$’s contract and there is no contract with a higher priority than that held by $k$ at school $s$ and (3.) that no student is named in more than one of these contracts at each school. A priority structure is Kesten acyclic if there are no Kesten cycles. Note that Kesten acyclicity is stronger than Ergin acyclicity. The following proposition states that $\varphi^{TTC}$ requires this stronger acyclicity condition to guarantee the desired properties. It is stated here without proof, but is the main result of Kesten (2006).

**Proposition 7:** Let $|X_i \cap X_s| \leq 1$ for all $i \in I$ and $s \in S$. $\varphi^{TTC}$ is stable if and only if $(\succeq, q)$ is Kesten acyclic.

In order to adopt this proposition into the framework of this paper, it remains to be shown that $\tilde{\varphi}^{TTC}$ satisfies the regularity conditions. Luckily, Observation 1 does the legwork in proving that the regularity conditions are satisfied. The result is stated formally in the following proposition.

**Proposition 8:** $\tilde{\varphi}^{TTC}$ satisfies IIC and CN and is an extension of $\varphi^{TTC}$.

Finally, I provide the following result, which is a corollary of Proposition 7 and Theorem 2, and generalizes the main result of Kesten (2006) to allow for contracts.

**Corollary 1:** $\tilde{\varphi}^{TTC}$ is stable if and only if $(\succeq, q)$ satisfies Kesten acyclicity and is student-lexicographic.

The interpretation is, again, that this is a very small domain over which $\tilde{\varphi}^{TTC}$ is stable. Since Kesten acyclicity is stronger than Ergin acyclicity, this domain is smaller than the domain over which $\tilde{\varphi}^{DA}$ is efficient and group strategy-proof.

This section demonstrates how to use the results of this paper to apply previous maximal domain results to the setting with contracts. Following Remark 1, I construct a mechanism which is an extension of $\varphi^{TTC}$. Kesten (2006) provides the necessary and sufficient priority restriction of which $\varphi^{TTC}$ is always stable and efficient. Because $\tilde{\varphi}^{TTC}$ satisfies IIC and CN, I am able to apply Theorem 2 to $\tilde{\varphi}^{TTC}$ to generalize the main result of Kesten (2006).
5 Multi-unit Demand

I will now briefly demonstrate the applicability of the theory developed in this paper to the context of multi-unit demand (many-to-many matching). Following Kojima (2013), I consider students who can be matched to a number of courses (for similarity of notation, I use the letter $s$ to denote a generic course and $S$ to denote the set of courses). Each student $i \in I$ has a quota $p_i \geq 1$ of courses she can attend, with the restriction that each student and each course can only be matched together under a single contract. For example, a contract could specify whether a student audits a particular course, takes the course on a pass/fail basis, or takes the course for a letter grade. A course allocation problem is $(I, R, p, S, \succeq, q, X)$. The rest of the notation generally carries through from above.\(^{11}\) In the non-contractual setting stable, efficient and strategy-proof mechanisms are very limited under multi-unit demand; multi-unit deferred acceptance\(^{12}\) $\varphi^{MDA}$ is equivalent to a serial dictatorship under the necessary domain restriction to ensure that the mechanism is stable, efficient and strategy-proof (Kojima, 2013). The necessary restriction is that, excluding top students who are guaranteed seats in any course, the priority structure must rank all students in the same position across all courses. The definition, adapted from Kojima (2013), is presented below.

**Definition 14** : A priority structure $(\succeq, q)$ is **essentially homogenous** if there exist no $r, s \in S$ and $i, j \in I$ and possibly distinct $u, v, w, t \in T$ such that:

(C) Cycle condition: $(i, r, u) \succ_r (j, r, v)$ and $(j, s, w) \succ_s (i, s, t)$ and,

(S) Scarcity condition: There exist (possibly empty) disjoint sets of contracts $X_r, X_s \subset X$ such that

1. $|X_r| = q_r - 1$ and $|X_s| = q_s - 1$,
2. $x \succ_r (j, r, v)$ for every $x \in X_r$ and $y \succ_s (i, s, t)$ for every $y \in X_s$, and
3. $|X_r \cap X_\ell| + |X_s \cap X_\ell| \leq 1$ for all $\ell \in I$.

This is clearly the most restrictive of the acyclicity conditions applied to a mechanism seen in this paper. Kojima (2013) proves that essential homogeneity of the priority structure, efficiency of $\varphi^{MDA}$ and strategy-proofness of $\varphi^{MDA}$ are all equivalent.

The proof of Theorem 2 in this paper goes through with minimal modification to include the general case of many-to-many matching. This allows the following result, which generalizes the main result of Kojima (2013) to include contracts.

\(^{11}\)The one exception is that the “no blocking” condition of stability now requires the contract $x$ to be preferred by student $i$ to some $y \in \mu(i)$.

\(^{12}\)Deferred acceptance extends in the natural way to the multi-unit demand case. At each step, every student $s$ has $p_s$ offers out to courses.
Corollary 2: The following are all equivalent:

- \( \bar{\varphi}^{MDA} \) is efficient,
- \( \bar{\varphi}^{MDA} \) is strategy-proof,
- \( \succeq \) is essentially homogenous and student-lexicographic.

Not surprisingly, given the extreme restriction of essential homogeneity, the domain over which \( \bar{\varphi}^{MDA} \) is efficient and strategy proof is quite small. However, this section demonstrates how the results of this paper extend to a more general problem.

### 6 Acceptant substitutable priorities

The analysis thus far has assumed that school priorities are responsive, meaning that a school’s ranking over sets of contracts is uniquely defined by its ranking over individual contracts. For example, if school \( s \) gives highest priority to contract \( x \) and second highest priority to contract \( y \), then of all the sets of contracts of size two, it gives highest priority to the set \( \{x, y\} \). It is well known that allowing for general complementarities in school priorities may lead to non-existence of stable matchings (Kelso and Crawford (1982), Hatfield and Milgrom (2005)). Nevertheless, stability is guaranteed if school priorities are substitutable, meaning that if a contract is accepted by a school from a set of available contracts, it must also be accepted when only a subset of those available contracts are presented to the school. Indeed, substitutable priorities have a natural interpretation in school choice settings as they allow school systems to achieve diversity goals by favoring sets of applicants which include students from underrepresented groups (Abdulkadiroğlu and Sönmez (2003)). In the contract setting, substitutable priorities also allow for schools, as an example, to favor applicant sets listing a wide range of majors to a set of applicants who all wish to study physics, even if each of the physics students individually have higher priority than all others. This is a powerful tool for schools, as there may be internal constraints on the number of students that can study any particular major.

School priorities are also often acceptant in that schools favor filling as many seats as possible. In countries such as China and Turkey with centralized college systems, a likely governmental objective of providing as much education to the best students, subject to capacity constraints, demands that schools have acceptant priorities. Indeed, acceptant priorities are also necessary if a market designer wishes to ensure efficiency, for otherwise an empty seat at a school may be denied to a student who wishes to attend.

A natural question is whether it is possible to guarantee stability, efficiency and group strategy-proofness in the contracts setting when the priority structure is acceptant and substitutable. This section will show that the answer is yes. More specifically, I show that (with a slight generalization of notation) Theorem
2 extends to this more general framework, so acyclic and student-lexicographic priorities are again necessary and sufficient to guarantee stability, efficiency and group strategy-proofness when contracts are added to the model. This allows me to generalize the result of Kumano (2009), and show that $\varphi^{DA}$ is stable, efficient and group strategy-proof under acceptant substitutable priorities if and only if the priority structure is Ergin acyclic and student-lexicographic.

6.1 Changes to the priority structure

Here, I introduce acceptant substitutable priorities and modify the setting introduced in Section 2 to apply to this new context.

A priority order $(\succeq^X, q) = (\succeq_s^X, q_s)_{s \in S}$ is an exogenous vector of linear orders along with school quotas in which $\succeq_s^X$ represents school $s$’s complete, transitive and antisymmetric ranking of sets of contracts $A \subset X_s$.\footnote{To comport with the exposition of the rest of the paper, I abuse notation and write, for example, $\succeq^X$ to mean $\succeq^{2^X}$ throughout this section.} Again, $\succ_s$ is the asymmetric subset of $\succeq_s$. Priorities are complete, transitive and antisymmetric. Note that the priority ranking does not depend on the feasibility of a set of contracts. Nevertheless, it is important to denote which sets of contracts are actually available to which a school can be matched. This is done by adding a choice function $C_s(\cdot)$. The choice function satisfies $\forall A \subset X_s$, $C_s(A) \subset A$, and feasibility, meaning that 1) $|C_s(A)| \leq q_s$ and 2) $\forall i \in I$ and if $\exists x, y \in C_s(A) \cap X_i$ then $x = y$. Feasibility requires that 1) the chosen set does not exceed the size of the school, and 2) that no student is chosen for multiple contracts. The relation of the choice function to the priority order is that $C_s(A) = Y$ if and only if $Y \succeq_s Z$ for all feasible $Z \subset A$. Throughout, I use the notation $I_A$ to denote the set of students who are named in a contract in $A$, that is, $I_A = \{i | \exists x \in A \cap X_i\}$.

Although the choice function defines the highest priority set of contracts from an available set, it does not specify how a school chooses this highest priority set of contracts. More specifically, the choice function does not yet restrict the priority structure to exclude complementarities. The following definition formally describes the acceptant substitute property.

**Definition 15**: A priority structure is acceptant substitutable with respect to $X$ if

\begin{align*}
(\text{Acc}) \forall s \in S, \forall A \subset X_s, |C_s(A)| = \min\{|I_A|, q_s\}, \text{ and} \\
(\text{Sub}) \forall s \in S, \forall A, A' \subset X_s \text{ with } A' \subset A, C_s(A) \cap A' \subset C_s(A').
\end{align*}

This definition states that a school’s priorities must (Acc) accept as many students as possible, and must (Sub) accept any contract from a set $A'$ when the same contract was accepted from a superset of $A'$.\footnote{To comport with the exposition of the rest of the paper, I abuse notation and write, for example, $\succeq^X$ to mean $\succeq^{2^X}$ throughout this section.}
Having now defined acceptant substitutes, I address whether it is possible to guarantee stability, efficiency and group strategy-proofness with contracts. One important point to make is that the set of responsive priorities is a subset of the set of acceptant substitutable priorities (Kumano, 2009). Therefore, some form of acyclic and student-lexicographic priorities will again be necessary for the existence of a stable and efficient matching. These concepts, however, need to be redefined to account for acceptant substitutable priorities. Indeed, the former definition of student-lexicographic priorities is not well-defined with these more general priority structures. Before giving the new definition of student-lexicographic priorities, I provide the following example, which illustrates the intuition of a cycle within the priority order of a single school that conflicts stability and efficiency.

**Example 4** : There is one school with two seats, and there are three students. Consider the priority order (over feasible sets of contracts) and preferences:

School s: \( \{x_i, y_j\} \succ \{x_i, x_k\} \succ \{x_i, x_j\} \succ \{y_j, x_k\} \succ \{x_j, x_k\} \succ \{x_k\} \succ \emptyset \)

Student i: \( x_i \not\in C_s(\emptyset) \)

Student j: \( x_j \in C_s(\emptyset) \)

Student k: \( x_k \not\in C_s(\emptyset) \)

This priority structure is acceptant substitutable. Since all students prefer to be matched and student j most prefers contract \( x_j \), the efficient matchings are \( \{x_i, x_k\} \), \( \{x_i, x_j\} \) and \( \{x_j, x_k\} \). However, the unique stable matching is \( \{x_i, y_j\} \).

Intuitively, the reason that there are no stable and efficient matchings in Example 4 is that the sets of contracts involving students i and j are interrupted in the priority order by \( \{x_i, x_k\} \). Similarly to Example 2, the problem is that the sets of contracts naming the same students are not back-to-back in the priority order, and so the school can distinguish between the contracts that a student selects.

The generalized setting of acceptant substitutable priorities, therefore, is not much different conceptually than responsive preferences. Nevertheless, several

\[^{14}\]The “no blocking pair” definition has to be slightly restated to account for the substitutable priority structure. An allocation is stable if it is individually rational and there are no blocking pairs. Formally, a matching is stable if:

(\textbf{IR}) \( \forall i \in I \) \( \mu(i) \not\in X_i \) and

(\textbf{NB}) \( \exists (i, s) \in I \times S \) with \( x \in X_i \cap X_s \) such that \( x \in C_s(\mu(i)) \) and \( x \not\in C_s(\mu(s) \cup \{x\}) \).

\[^{15}\]It clearly satisfies the acceptant property. To see that it is also substitutable, note that for any set of contracts \( A' \neq \{x_i, y_j\} \) with \( |A'| \leq 2 \), \( C_s(A') = A' \) and so the definition of substitutability has no bite. Similarly, the case in which \( |A'| = 4 \) has no bite since the only superset of \( A' \) is \( A' \) itself. Therefore, it suffices to consider \( A = \{x_i, x_j, y_j, x_k\} \) with \( A' \) being any of the \( \binom{4}{3} = 4 \) subsets of size 3, and \( A' = \{x_j, y_j\} \) with \( A \) being any of the three supersets which contain \( A' \). None of these cases violate Definition 15.
Definitions in Sections 2.2 and 2.4 need to be translated into the more general setting. Again, the root of the following definitions is the notion of indistinguishable (sets of) contracts.

**Definition 16**: \( Y, Z \subset X_s \) with \( Y \succ_s Z \) are indistinguishable sets of contracts to school \( s \) with respect to \( X \) if \( Y \) and \( Z \) are feasible and either:

- \( |Y| \leq q_s \) or
- \( I_Y = I_Z \) and \( \nexists \) a feasible \( W \subset X_s \) with \( I_W \neq I_Y \) such that \( Y \succ_s W \succ_s Z \).

The first point in this definition takes advantage of the acceptant property. If a school can accommodate all interested students, then it is not important how sets of contracts of size smaller than the capacity of the school are ranked. This corresponds to the first bullet point (scarcity) in Definition 2. Similarly, the second bullet point is analogous to the “back-to-back” condition in Definition 2.

As in the case of responsive priorities, when all sets of contracts involving the same students are indistinguishable to the school named in those contracts, the priority structure is set-lexicographic.

**Definition 17**: A priority structure is set-lexicographic if for every school \( s \), any two sets of feasible contracts \( Y, Z \subset X_s \) with \( I_Y = I_Z \) are indistinguishable to school \( s \) with respect to \( X \).

The notions of irrelevant sets of contracts, independence of irrelevant sets of contracts, and set-contract neutrality follow in a straightforward way from Definition 16. For ease of exposition, formal definitions of these terms are relegated to the appendix.

Having now translated the relevant definitions into the setting of acceptant substitutable priorities, I state Theorem 3, which itself is a translation of Theorem 2. It states that set-lexicographic priorities and an (mechanism specific) acyclicity condition are necessary and sufficient to ensure stability, efficiency and group strategy-proofness with acceptant substitutable priorities when contracts are added. The reasoning behind this theorem follows from the logic of Theorem 2.

**Theorem 3**: Let \( \varphi \) be a contract free mechanism. Let \( \varphi \)-acyclicity be the weakest acyclicity restriction on an acceptant substitutable priority structure such that \( \varphi \) is stable, efficient and group strategy-proof. Let \( \bar{\varphi} \) be an extension of \( \varphi \) satisfying independence of irrelevant sets of contracts and set-contract neutrality. Then
1. $\bar{\varphi}$ is stable and efficient if and only if the priority structure is set-lexicographic and satisfies $\varphi$-acyclicity, and

2. $\bar{\varphi}$ is stable and group strategy-proof if and only if the priority structure is set-lexicographic and satisfies $\varphi$-acyclicity.

### 6.2 Application to deferred acceptance

Similarly to the analysis in Sections 4 and 5, I now apply the result of the previous theorem to exactly characterize the set of acceptant substitutable priority structures over which $\varphi^{DA}$ is stable, efficient and group strategy-proof. The punch line of this section is that Ergin acyclicity (which also needs to be translated to accommodate substitutable priorities) and set-lexicographic priorities are necessary and sufficient to guarantee the desired properties.

First, I give a definition of set-Ergin acyclicity. The intuition is identical to Ergin acyclicity presented at the beginning of this paper, but accommodates the more general priority structure.

**Definition 18**: Let $(\succeq, q)$ be an acceptant substitutable priority structure. A *set-Ergin cycle* is constituted of distinct $r, s \in S$, distinct $i, j, k \in I$ and possibly distinct $t, u, v, w, q \in T$ and (possibly empty) disjoint sets of contracts $X_r, X_s \subset X \setminus \{X_i \cup X_j \cup X_k\}$ such that:

(C) **Cycle condition:**
- $(j, r, u) \notin C_r (X_r \cup \{(j, r, u), (i, r, t)\})$,
- $(k, r, v) \notin C_r (X_r \cup \{(j, r, u), (k, r, v)\})$, and
- $(i, s, q) \notin C_s (X_s \cup \{(i, s, t), (k, s, w)\})$.

(S) **Scarcity condition:**
1. $|X_r| = q_r - 1$ and $|X_s| = q_s - 1$, and
2. $|X_r \cap X_\ell| + |X_s \cap X_\ell| \leq 1$ for all $\ell \in I$.

A priority structure is *set-Ergin acyclic* if it does not contain any set-Ergin cycles.

The following proposition states that without contracts, set-Ergin acyclicity, efficiency and group strategy-proofness are all equivalent for $\varphi^{DA}$. It is presented without proof, but is the main result of Kumano (2009).

**Proposition 9**: Let $|X_i \cap X_s| \leq 1$ for all $i \in I$ and $s \in S$. For any acceptant substitutable priority structure $(\succeq, q)$, the following are equivalent:
Corollary 3: Let $(\succeq, q)$ be an acceptant substitutable priority structure. The following are all equivalent:

- $\varphi^{DA}$ is efficient,
- $\varphi^{DA}$ is group strategy-proof,
- $(\succeq, q)$ is set-Ergin acyclic and set-lexicographic.

Therefore, when contracts are added to an acceptant substitutable priority structure it is again necessary and sufficient for the priorities to be lexicographic and Ergin acyclic if a market designer wishes to guarantee efficiency and group strategy-proofness of deferred acceptance.

7 Conclusion

This paper extends “maximal domain” results of a general class of many-to-one matching mechanisms to include contracts. The analysis above gives a concrete methodology to study matching with contracts, and proves that student-lexicographic priorities are important in guaranteeing the existence of a stable and efficient matching. The main result of this paper shows that an acyclic and student-lexicographic priority structure, restrictions on the heterogeneity of school priority rankings across and within schools, respectively, is necessary and sufficient to guarantee stability and efficiency for any IIC mechanism.

To illustrate the practical application of the main result, I show how the main result applies to two famous mechanisms, student proposing deferred acceptance and top trading cycles. Therefore, I am able to generalize the main results of Ergin (2002) and Kesten (2006). Section 5 illustrates that the main result also extends to a many-to-many setting in which agents have multi-unit demand, thus extending the result of Kojima (2013). The final section extends the main result to allow for acceptant substitutable priorities, which allow schools to favor diversity of socio-economic status or selected major, and I generalize the results of Kumano (2009) regarding deferred acceptance when contracts are added.

One concern is that student-lexicographic priorities are too restrictive to have any real-world application. It seems unlikely to expect, as an example,
firms to give higher priority to a particular worker at all wage levels to another worker at any wage level (in the labor matching context of Kelso and Crawford (1982)). And while it may be the case that student-lexicographic priorities are too much to hope for in certain settings, and inefficiency or instability must be accepted, there are at least a handful of market that self-restrict with student-lexicographic priorities: college admissions in the United States, Canada and Scotland. These colleges allow a student to enroll in major $t$ and switch to major $u$ whenever she wishes. In this sense, a college is unable to deny any contract to a student if it admits the student under any other contract, exactly fulfilling the criterion of student-lexicographic priorities. One interesting observation is that this policy does not seem to extend to most other European colleges; students in Europe enroll in a particular major and are not generally allowed to switch. Why North American colleges have student-lexicographic priorities in practice and European colleges do not is a topic for further study.

A Appendix

Below I present proofs for several of the results found in the paper, as well as supplemental results and additional definitions.

Proof of Proposition 2: A priority structure is student-lexicographic if and only if every school $s$ ranks every contract from the top $q_s$ students above the other students and each student not in the top $q_s$ has her contracts ranked consecutively.

$(\Rightarrow)$ Assume the priority structure is student-lexicographic. This proof will be based on Example 3. Pick an arbitrary school $s$. Draw a dashed line as far down in $\succeq_s$ as possible such that there are contracts from no more than $q_s$ students above the line. Take all of the contracts $x^1, ..., x^T \in \mathcal{X}_s$ naming some student $i$. W.L.O.G. let $x^1 \succ_s x^2 \succ_s ... \succ_s x^T$. By assumption, there are three cases to consider:

1. All $x^t, t \in \{1, ..., T\}$ are above the dashed line, satisfying the requirement for $i$ as one of the top $q_s$ students.
2. All $x^t, t \in \{1, ..., T\}$ are below the dashed line. Since the priority structure is student-lexicographic, there is no $z \in \mathcal{X} \setminus \mathcal{X}_i$ such that $x^t \succ_s z \succ_s x^{t+1}$. Since $x^1 \succ_s x^2 \succ_s ... \succ_s x^T$ by assumption, all of student $i$’s contracts are ranked consecutively in $\succeq_s$.
3. For some $t \in \{1, ..., T\}$ $x^t$ is above the dashed line and $x^{t+1}$ is below the dashed line. By assumption, the dashed line is drawn as far down $\succeq_s$ as possible such that no more than $q_s$ students have contracts above the line, therefore, there exists $z \in \mathcal{X} \setminus \mathcal{X}_i$ such that $x^t \succ_s z \succ_s x^{t+1}$. Therefore, $x^t$ and $x^{t+1}$ are not
Proof of Proposition 4:

Case 1: generality: not cause a previously relevant contract to become irrelevant, that is from the preferences of students that the deletion of an irrelevant contract does not cause a previously relevant contract to become irrelevant, that is \( Q^X \setminus \{x\} \subseteq Q^X \setminus \{x\} \). To see \( Q^X \setminus \{x\} \subseteq Q^X \setminus \{x\} \), consider two cases, without loss of generality:

Case 2: The deletion of student \( i \)'s irrelevant contract does not make a previously irrelevant contract of \( i \)'s relevant. W.L.O.G. consider \( \{x, y, z\} = X_i \cap X_s \) for some \( i \) and \( s \), where \( x, y \) are irrelevant and \( z \) is relevant. Then \( zP_i X_i x \) and \( zP_i X_i y \). By assumption, all three contracts are indistinguishable to school \( s \). W.L.O.G. let \( y \succ_s X_i z \). Now suppose \( x \) is deleted. We know that \( zP_i X_i x \). Therefore, to show that \( y \) is still irrelevant, it suffices to show that \( z \) and \( y \) are indistinguishable under \( \succeq_{X_i \setminus \{x\}} \). If \( i \) was one of the “top \( q_s \)” students before, then he remains so after the deletion, since \( \{j \in I \setminus \{i\} : \exists w \in X_j \cap X_s \text{ such that } w \succ_{X_j \setminus \{x\}} z\} = \{j \in I \setminus \{i\} : \exists w \in X_j \cap X_s \text{ such that } w \succ_{X_j \setminus \{x\}} z\} \). If \( i \) was not one of the “top \( q_s \)” students before, then he remains so after the deletion and his contracts remain consecutively ranked, since \( \exists w \in X \setminus X_i \) such that \( z \succ_{X_i X_s} w \succ_{X_i X_s} y \Rightarrow \exists w \in X \setminus \{X_i \cup x\} \text{ such that } z \succ_{X_i \setminus \{x\}} w \succ_{X_i \setminus \{x\}} y \). Therefore, \( y \) is still irrelevant after the deletion of \( x \).

Case 2: The deletion of student \( i \)'s irrelevant contract does not make a previously irrelevant contract of \( j \)'s relevant. Let \( x \in X_i \cap X_s \) for some \( i \) and \( s \), and let \( \{y, v\} = X_j \cap X_s \) for some \( j \neq i \). Suppose \( x \) and \( v \) are irrelevant. W.L.O.G. let \( y \) be relevant and let \( v \succ_{X_s} y \). Then \( yP_j X_j v \). Now suppose \( x \) is deleted. It is still the case that \( yP_j X_j \{x\} \). It remains to show that \( y \) and \( v \) are indistinguishable under \( \succeq_{X_i \setminus \{x\}} \). If \( j \) was one of the “top \( q_s \)” students before, then he remains so after the deletion, since \( \{\ell \in I \setminus \{j\} : \exists w \in X_\ell \cap X_s \text{ such that } w \succ_{X_\ell \setminus \{x\}} y\} = \{\ell \in I \setminus \{j\} : \exists w \in X_\ell \cap X_s \text{ such that } w \succ_{X_\ell \setminus \{x\}} y\} \). If \( j \) was not one of the “top \( q_s \)” students before, then he remains so after the deletion and his contracts remain consecutively ranked, since \( \exists w \in X \setminus X_j \text{ such that } v \succ_s w \succ_s y \Rightarrow \exists w \in X \setminus \{X_i \cup x\} \text{ such that } v \succ_{X_i \setminus \{x\}} w \succ_{X_i \setminus \{x\}} y \). Therefore, \( v \) is still irrelevant after the deletion of \( x \).

\( \square \)

Proof of Proposition 3: Let \( \succeq_s \) be student-lexicographic. It follows easily from the preferences of students that the deletion of an irrelevant contract does not cause a previously relevant contract to become irrelevant, that is \( Q^X \setminus \{x\} \supseteq Q^X \setminus \{x\} \). To see \( Q^X \setminus \{x\} \supseteq Q^X \setminus \{x\} \), consider two cases, without loss of generality:

\[ \text{Proof of Proposition 3:} \]

\[ \{X, x, zP\} \]

Then was one of the “top \( q \)" students before, then he remains so after the deletion of \( x \). W.L.O.G. let \( y \succ_s x \). Therefore, to show that \( x \) is still irrelevant, it suffices to show that \( x \) and \( y \) are indistinguishable under \( \succeq_{X_i \setminus \{x\}} \). If \( i \) was one of the “top \( q_s \)” students before, then he remains so after the deletion, since \( \exists w \in X_i \cap X_s \text{ such that } w \succ_{X_i \setminus \{x\}} z \). If \( i \) was not one of the “top \( q_s \)” students before, then he remains so after the deletion and his contracts remain consecutively ranked, since \( \exists w \in X \setminus X_i \text{ such that } z \succ_{X_i X_s} w \succ_{X_i X_s} y \Rightarrow \exists w \in X \setminus \{X_i \cup x\} \text{ such that } z \succ_{X_i \setminus \{x\}} w \succ_{X_i \setminus \{x\}} y \). Therefore, \( x \) is still irrelevant after the deletion of \( x \).

\( \square \)

Proof of Proposition 4:

\[ \square \]
(⇒) Let \( \preceq, q \) be student-lexicographic. Then by definition, every distinct \( x, y \in X_i \cap X_s \) are indistinguishable to school \( s \) for all \( i \in I, s \in S \). By transitivity and antisymmetry of \( R_i^X \) there is a unique \( x \in X_i \cap X_s \) such that \( x \preceq_i y \) for all \( y \in X_i \cap X_s \). Therefore, \( x \) is the unique relevant contract between student \( i \) and school \( s \).

(⇐) I prove the contrapositive. Suppose that for some \( R_i^X \) and some \( i \in I, s \in S \) there are distinct relevant contracts \( x, y \in X_i \cap X_s \). Therefore, \( (\preceq, q) \) is not student-lexicographic.

\[ \square \]

**Proof of Remark 1**: Suppose \( (\preceq, q) \) is student-lexicographic. Then by Proposition 4, there is at most one relevant contract between each school \( s \) and each student \( i \). Assuming \( X_i \cap X_s \) is non-empty, it must be the case that the \( R_i^X \)-maximal element of \( X_i \cap X_s \) is the unique relevant contract between \( i \) and \( s \). Therefore, \( (R_i^X \setminus x^i, X \setminus x^i) = (R_i^X \setminus y^i, X \setminus y^i) \). Since by construction \( \bar{\varphi}(R_i^X \setminus y^i, X \setminus y^i) = \varphi(R_i^X \setminus x^i, X \setminus x^i), \bar{\varphi} \) is an extension of \( \varphi \). Noting that \( \bar{\varphi} \) considers only the most preferred contracts of students, it is clear that it ignores all irrelevant contracts (and possibly some relevant ones, too).

\[ \square \]

**Proof of Proposition 5**: Let \( \mathcal{P}^\varphi \subset \mathcal{P}^X \) be the set of all priority structures which do not violate \( \varphi \)-acyclicity. I show that if \( \mathcal{P}^\varphi \neq \emptyset \) then \( \mathcal{P}^\varphi \cap \mathcal{L} \neq \emptyset \). This proof is by construction of a homogeneous priority structure that is student-lexicographic. Since homogeneity implies \( \varphi \)-acyclicity, this is sufficient to show the desired result. Suppose there are \( f_{(s,i)} = |X_i \cap X_s| \) possible contracts between each student \( i \) and each school \( s \). Consider any homogeneous priority structure \( (\succeq^X, q) \) where \( |X^i \cap X^s| = \min(f_{(s,i)}, 1) \), that is, there is at most one contract between each school and each student. Note that this is equivalent to a non-contract setting. Let \( T = \max f_{(s,i)} \) and consider a sequence \( \succeq^X, ..., \succeq^{X_T} \) such that \( |X^i \cap X^s| = \min(f_{(s,i), t}) \) for all \( s \in S, i \in I \) and \( t \in \{1, ..., T\} \). For each step \( t \), add any new contracts between student \( i \) and school \( s \) directly after the most recently added contract between \( i \) and \( s \) in school \( s \)'s priority ranking \( \succeq^X \). Clearly \( X^T = X \), and \( (\succeq^{X_T}, q) \) is student-lexicographic since all contracts ranking the same student are ranked consecutively in each school’s priorities. To show that \( (\succeq^{X_T}, q) \) is homogeneous, I make use of the following observation which comes from comparing the definition of homogeneity to the definition of student-lexicographic priorities:

**Observation 2**: Consider two contracts \( x, y \in X_s \) which are indistinguishable to school \( s \). Then \( x \) is involved in a \( \varphi \)-cycle if and only if \( y \) is involved in a \( \varphi \)-cycle (where a \( \varphi \)-cycle is any cycle that violates \( \varphi \)-acyclicity).
Since \((\succeq^{X_T}, q)\) is student-lexicographic (all contracts involving the same student are indistinguishable), by invoking the previous observation we see that \((\succeq^{X_T}, q)\) is homogeneous if and only if \((\succeq^{X_I}, q)\) is homogeneous, which is true by assumption. Therefore, \((\succeq^{X_T}, q)\) \(\in P_\varphi \cap L\).

\(\square\)

### A.1 Proof of Main Result

Before proving Theorem 2, I provide several lemmas to simplify the proof of the theorem.

**Lemma 1**: For any \((R, \succeq, X)\) involving a student-lexicographic priority structure let \(Q^X\) be the set of all irrelevant contracts. If mechanism \(\bar{\varphi}\) satisfies IIC, then for any subset \(W \subseteq Q^X\), \(\bar{\varphi}(R, \succeq, X) = \bar{\varphi}(R \setminus W, \succeq \setminus W, X \setminus W)\).

**Proof**: The proof follows by induction on the size of \(W\) and the definition of IIC, after noting that the order of deletion of irrelevant contracts does not matter (Proposition 3).

**Lemma 2**: Suppose \(\bar{\varphi}\) satisfies IIC and is contract neutral. For any \((R, \succeq, X)\) involving a student-lexicographic priority structure and for all \(j \in J \subseteq I\), suppose \(j\) switches the rankings of \(x\) and \(y\) where \(x, y \in X_j \cap X_s\) for some \(s \in S\) with \(xP_j y\) and \(x, y\) indistinguishable to school \(s\). Call this misreport \(R^*_j\). Then \(\bar{\varphi}_i(R, \succeq, X) = \bar{\varphi}_i((R_{I \setminus J}, R^*_j), \succeq, X)\) for all \(i \in I \setminus J\), and \(\bar{\varphi}_j(R, \succeq, X) \neq \bar{\varphi}_j((R_{I \setminus J}, R^*_j), \succeq, X)\) for \(j \in J\) only if \(\bar{\varphi}_j(R, \succeq, X) = x\), in which case \(\bar{\varphi}_j((R_{I \setminus J}, R^*_j), \succeq, X) = y\).

**Proof**: The proof follows by induction on the size of \(J\) and the definition of contract neutrality, after noting that the order of deletion of irrelevant contracts does not matter (Proposition 3).

**First Claim of Main Theorem**: Let \(P_\varphi \subseteq P^X\) be the set of all priority structures which do not violate \(\varphi\)-acyclicity and let \(\bar{\varphi}\) satisfy independence of irrelevant contracts. Then \(\bar{\varphi}(R, \succeq, X)\) is stable and efficient for any \(R^X \in R^X\) if and only if \((\succeq, q) \in P_\varphi \cap L\).

**Proof**:

\((\Rightarrow)\) By Proposition 4 there is at most one relevant contract (with respect to the true preferences \(R^X\)) between each school and each student. Therefore, by Lemma 1 and the fact that \(\bar{\varphi}\) is an extension of \(\varphi\) we know that \(\bar{\varphi}(R, \succeq, X) = \bar{\varphi}(R \setminus Q, \succeq \setminus Q, X \setminus \{Q\}) = \varphi(R \setminus Q, \succeq \setminus Q, X \setminus \{Q\})\). Since \((\succeq, q)\) is \(\varphi\)-acyclic we know that
Proof: \( R \) group strategy-proof for any \( \phi \) of irrelevant contracts and contract neutrality. Then \( \bar R \) structures which do not violate \( \phi \). Let \( \phi \)

Second Claim of Main Theorem: Let \( P^\phi \subset P^X \) be the set of all priority structures which do not violate \( \phi \)-acyclicity, and let \( \bar \phi \) satisfy independence of irrelevant contracts and contract neutrality. Then \( \bar \phi(R, \succeq, X) \) is stable and group strategy-proof for any \( R^X \in R^X \) if and only if \( (\succeq, q) \in P^\phi \cap L \).

Proof:

(⇒) By Proposition 4 there is at most one relevant contract (with respect to the true preferences \( R^X \)) between each school and each student. Therefore, by Lemma 1 and the fact that \( \bar \phi \) is an extension of \( \phi \) we know that \( \bar \phi(R_i, \succeq, X) = \bar \phi(R^X \setminus Q, \succeq^X \setminus Q, X \setminus \{Q\}) = \phi(R^X \setminus Q, \succeq^X \setminus Q, X \setminus \{Q\}) \). Since \( (\succeq^X \setminus Q, q) \) is \( \phi \)-acyclic we know that \( (\succeq^X \setminus Q, q) \) is also \( \phi \)-acyclic. Therefore, \( \bar \phi(R_i, \succeq, X) \) yields a stable and efficient matching for any \( R^X \in R^X \). To see that \( \bar \phi \) is group strategy-proof, let \( \bar \phi \) be any untrue preference profile. Suppose \( xP_i^X, y \) for some \( i \in I \) and \( x, y \in \mathcal{X}_i \setminus \mathcal{X}_s \) for some \( s \in S \). By contract neutrality and Lemma 2 it is a weakly dominant strategy for each subset \( J \subset I \) of students to correctly report the order of contracts naming the same school. This implies that for all \( i \) if \( x, y \in \mathcal{X}_i \setminus \mathcal{X}_s \) and \( x \) is relevant, then \( xP_i^X, y \). By IIC, we can therefore restrict attention only to relevant contracts. Now suppose by way of contradiction that there exists a non-empty subset \( I' \subset I \) such that \( \bar \phi_i \left((R_i \setminus J', R_i), \succeq, X\right) \) \( R_i^X, \bar \phi_i \left(R, \succeq, X\right) \) for all
\( i \in I' \) and \( \bar{\varphi}_i \left( (R_{I \setminus I'}, R_{I'}'), \succeq, \mathcal{X} \right) P^X_i \bar{\varphi}_i (R_i, \succeq, \mathcal{X}) \) for some \( i \in I' \). But \( \bar{\varphi} \left( (R_{I \setminus I'}, R_{I'}'), \succeq, \mathcal{X} \right) = \varphi \left( \left( R_{X \setminus Q}, R_{X'} \setminus Q \right), \succeq^{X \setminus Q}, \mathcal{X} \setminus Q \right) = \varphi \left( \left( R_{X \setminus Q}, R_{X'} \setminus Q \right), \succeq^{X \setminus Q}, \mathcal{X} \setminus Q \right) \). This contradicts \( \varphi \) being group-strategy proof for since the \( \varphi \)–acyclicity of \( (\succeq^X, q) \) implies the acyclic-
ity of \( (\succeq^X, q) \).

\( \Leftarrow \) This proof is nearly identical to the proof of the First Claim of Main Theorem. Consider Example 2, and suppose that student \( j \) misreports by stating that she has no acceptable contracts. The final matching for any IIC mechanism is \( (i, s, t) \). Student \( j \) is indifferent, but student \( i \) is strictly better off.

\[ \square \]

### A.2 Proof of Theorem 1

Theorem 1 is, in large part, a special case of Theorem 2. I will first prove the following are all equivalent:

- \( \bar{\varphi}^{DA} (\cdot, \succeq, \mathcal{X}) \) is efficient,
- \( \bar{\varphi}^{DA} (\cdot, \succeq, \mathcal{X}) \) is group strategy-proof, and
- the priority structure is Ergin acyclic and student-lexicographic.

#### Lemma 3: \( \bar{\varphi}^{DA} \) satisfies IIC and CN and is an extension of \( \varphi^{DA} \).

**Proof of Lemma 3:** Suppose W.L.O.G. that \( \{x, y\} = X_i \cap X_s \) for some \( i \in I \) and \( s \in S \) and suppose \( y \) is irrelevant. Then \( x \preceq y \).

**Independence of Irrelevant Contracts:**

Case 1: There is no step in which student \( i \) proposes either contract. Then the mechanism terminates, and is unaffected by the existence of either contract.

Case 2: At some step \( \ell \) student \( i \) proposes \( x \) and he never proposes \( y \). Since contract \( y \) is never proposed, the mechanism is unaffected by the existence of \( y \).

Case 3: At some step \( \ell \) student \( i \) proposes contract \( x \) and at some step \( m > \ell \) student \( i \) proposes \( y \). Then \( x \) is rejected by school \( s \) at some step \( k \), with \( \ell \leq k < m \). Since \( y \) is irrelevant, either \( x \) and \( y \) are indistinguishable to school \( s \) or \( x \succeq_s y \). If \( x \) is rejected, then at step \( m \), \( C_s(A^n_s \cup \{y\}) = C_s(A^n_s) \) by transitivity of \( \succeq_s \) and therefore \( y \) is also rejected as step \( m \). It is also known that the order in which students propose does not affect the final matching, and that the proposal of a rejected contract does not impact the final matching (see, for example, Kominers and Sönmez (2013)). Therefore, the mechanism is unaffected by the existence of \( y \).
Contract Neutrality:
Consider any \((R, \succeq, X)\) such that W.L.O.G. \(\{x, y\} = X_i \cap X_s\) for some \(i \in I\) and \(s \in S\). Suppose \(x\) and \(y\) are indistinguishable to school \(s\) and \(x \succeq y\). Since, \(\phi^{DA}\) satisfies IIC, W.L.O.G. I assume \(x\) is a relevant contract. Let \(R\) be a preference profile and let \(R^*_i\) switch the order of contracts \(x\) and \(y\).

Case 1: If student \(i\) never proposes either contract, then at some step \(\ell\), he proposes a contract \(z \succeq x\) and \(z\) is accepted. But \(z \succeq x\) is rejected. Therefore, \(\overline{\phi}_i((R \cap I, R^*_i), \succeq, X) = \overline{\phi}_i(R, \succeq, X)\) for all \(i\).

Case 2: At some step \(\ell\) student \(i\) proposes contract \(x\) and at some step \(m \geq \ell\), \(x\) is rejected. By transitivity of \(\succ_i\), \(y\) is also rejected if it is proposed, that is, if at some step \(j > m\) contract \(y\) is proposed, then \(C_s(A^*_i \cup \{y\}) = C_s(A^*_i)\). Now consider \(\overline{\phi}_i((R \cap I, R^*_i), \succeq, X)\). Since all other agents make the same proposal as before, when contract \(y\) is proposed, it is rejected, and \(x\) is also rejected if it is proposed. Therefore, \(\overline{\phi}_i((R \cap I, R^*_i), \succeq, X) = \overline{\phi}_i(R, \succeq, X)\) for all \(i\).

Case 3: At some step \(\ell\) student \(i\) proposes contract \(x\) and it is accepted. That is, at every \(t \geq \ell\), \(x \in C_s(A^*_i)\). Now consider \(\overline{\phi}_i((R \cap I, R^*_i), \succeq, X)\). Since \(x\) and \(y\) are indistinguishable to school \(s\), \(y\) is accepted when it is proposed at some step \(\ell\), and never rejected. All other players are unaffected. That is, \(y \in C_s(A^*_i)\) for all \(t \geq \ell\).

\(\overline{\phi}^{DA}\) is an extension of \(\phi^{DA}\):
Invoking Proposition 4, it suffices to suppose the priority structure is student-lexicographic. Then every \(x, y \in X_i \cap X_s\) are indistinguishable to school \(s\), and let \(x \succeq y\). By IIC and the fact that neither order of proposals, nor rejection of proposed contracts affects \(\overline{\phi}^{DA}\), the desired result is achieved.

Combining Theorem 2, Proposition 1 (the main result of Ergin (2002)) and Lemma 3, I am able to formally state the domain of priority structures over which \(\overline{\phi}^{DA}\) is stable, efficient and group strategy-proof. Noting that \(\overline{\phi}^{DA}\) is always stable (the choice set satisfies the substitutes condition of Hatfield and Milgrom (2005)) yields the desired result.

\[\square\]

To show that there exists a (unique) stable and efficient matching for any \(R\) if and only if the priority structure is both Ergin acyclic and student-lexicographic, it is only necessary to note that \(\overline{\phi}^{DA}\) yields the (unique) student optimal stable matching (Hatfield and Milgrom (2005)). This concludes the proof of Theorem 1.

\[\square\]

A.3 Sections 5 and 6 Definitions and Proofs
Proof of Corollary 2:
The “necessity” argument is shown in Theorem 2 (since single-unit demand is a special case of multi-unit demand). The “sufficiency” argument is almost identical to that which is presented in the proof of Theorem 2 and noting that \( \bar{\varphi}^{MDA} \) meets the necessary regularity conditions.

\[\square\]

**Definition 19**: A set of feasible contracts \( Z \subset X_s \) is irrelevant with respect to \( X \) if

1. \( \exists \) feasible \( Y \subset X_s \) with \( Y \) indistinguishable from \( Z \) by school \( s \) and \( \forall i \in I_Z \) and \( z_i \in X_i \cap Z, y_i \in X_i \cap Y, y_i R_i^{X_i} z_i \),
2. \( \exists \) feasible \( Y \subset X_s \) with \( Y \succ^X Z \) and \( \forall i \in I_Z \) and \( z_i \in X_i \cap Z, y_i \in X_i \cap Y, y_i R_i^{X_i} z_i \), or
3. \( \emptyset \neq P_i z_i \) for some \( i \in I_Z \) and \( z_i \in X_i \cap Z \).

**Definition 20**: For any \( (R, \succeq, X) \) let \( Q^X \) be the set of all irrelevant sets of contracts and let \( Z \in Q^X \). Define \( \succeq^{X \setminus Z} \) as the priority structure of the schools once \( Z \) is deleted. A mechanism \( \bar{\varphi} \) satisfies independence of irrelevant sets of contracts (IIASC) if \( \bar{\varphi}(R, \succeq, X) = \bar{\varphi}(R^{X \setminus Z}, \succeq^{X \setminus Z}, X \setminus Z) \).

**Definition 21**: Let \( \bar{\varphi} \) satisfy IIASC. For any \( (R, \succeq, X) \) suppose there exist \( y, z \in X_i \cap X_s \) where \( x \) and \( y \) are part of two indistinguishable sets \( Y, Z \), respectively, and \( y P_i z \). Let \( R^*_i \) reverse the rankings of \( y \) and \( z \). Then \( \bar{\varphi} \) is set-contract neutral (SCN) if \( \bar{\varphi}_j ((R_{P_i}, R^*_i), \succeq, X) = \bar{\varphi}_j (R, \succeq, X) \) for all \( j \neq i \), and

\[
\bar{\varphi}_i ((R_{P_i}, R^*_i), \succeq, X) = \begin{cases} 
\bar{\varphi}_i (R, \succeq, X) & \text{if } \bar{\varphi}_i (R, \succeq, X) \neq y \\
z & \text{if } \bar{\varphi}_i (R, \succeq, X) = y 
\end{cases}
\]

**Proof of Theorem 3**: Follows the same logic as the Proof of Main Result.

\[\square\]

**References**


