CHAPTER 6

THE CONSERVATION EQUATIONS

6.1 LEIBNIZ' RULE FOR DIFFERENTIATION OF INTEGRALS

6.1.1 DIFFERENTIATION UNDER THE INTEGRAL SIGN

According to the fundamental theorem of calculus if f is a smooth function and the integral of f is

$$I(x) = \int_{constant}^{x} f(x')dx' \tag{6.1}$$

then the derivative of $I(x)$ is

$$\frac{dI}{dx} = f(x). \tag{6.2}$$

Similarly if

$$I(x) = \int_{x}^{constant} f(x')dx' \tag{6.3}$$

then

$$\frac{dI}{dx} = -f(x) \tag{6.4}$$

Suppose the function f depends on two variables and the integral is a definite integral.

$$I(t) = \int_{a}^{b} f(x', t)dx' \tag{6.5}$$

where a and b are constant. The derivative with respect to t is

$$\frac{dI(t)}{dt} = \int_a^b \frac{\partial}{\partial t} f(x', t)dx' \tag{6.6}$$

The order of the operations of integration and differentiation can be exchanged and so it is permissible to bring the derivative under the integral sign.

We are interested in applications to compressible flow and so from here on we will interpret the variable t as time. Now suppose that both the kernel of the integral and the limits of integration depend on time.

$$I(t, a(t), b(t)) = \int_{a(t)}^{b(t)} f(x', t)dx' \tag{6.7}$$

This situation is shown schematically below with movement of the boundaries indicated.

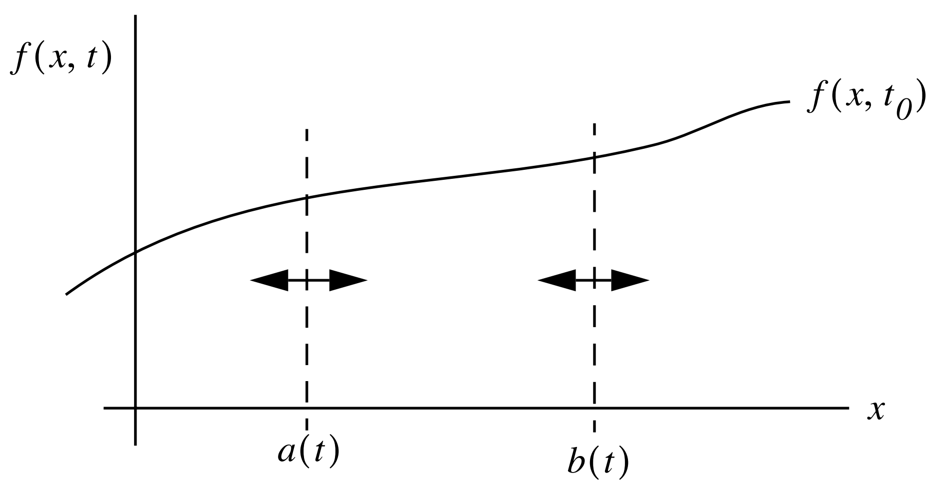


Figure 6.1 Integration with a moving boundary. The function $f(x, t)$ is shown at one instant in time.

Using the chain rule the substantial derivative of (6.7) is

$$\frac{DI}{Dt} = \frac{\partial I}{\partial t} + \frac{\partial I}{\partial a}\frac{da}{dt} + \frac{\partial I}{\partial b}\frac{db}{dt} \tag{6.8}$$

Now make use of the results in (6.2), (6.4) and (6.6). Equation (6.8) becomes,

$$\frac{DI}{Dt} = \int_{a(t)}^{b(t)} \frac{\partial}{\partial t} f(x', t)dx' + f(b(t), t)\frac{db}{dt} - f(a(t), t)\frac{da}{dt}. \tag{6.9}$$

The various terms in (6.9) can be interpreted as follows. The first term is the time rate of change of I due to the integrated time rate of change of $f(x, t)$ within the domain $[a, b]$. The second and third terms are the contributions to the time rate of change of I due to the movement of the boundaries enclosing more or less f at a given instant in time. The relation (6.9) is called *Leibniz' rule for the differentiation of integrals* after Gottfried Wilhelm Leibniz (1646-1716) who, along with Isaac Newton, is credited with independently inventing differential and integral calculus.

6.1.2 EXTENSION TO THREE DIMENSIONS

Let $F(x_1, x_2, x_3, t)$ be some field variable defined as a function of space and time and $V(t)$ be a time-dependent control volume that encloses some finite region in space at each instant of time. The time dependent surface of the control volume is $A(t)$.

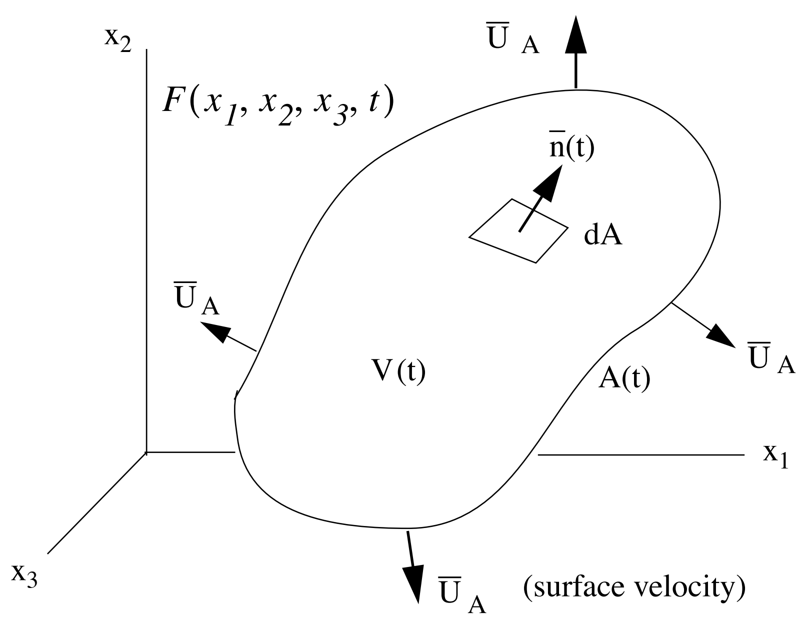


Figure 6.2 Control volume definition.

Leibniz' rule extended to three dimensions describes the time rate of change of the amount of F contained inside V.

$$\frac{D}{Dt}\int_{V(t)} F\,dV = \int_{V(t)} \frac{\partial F}{\partial t}dV + \int_{A(t)} F\,\bar{U}_A \bullet \bar{n}\,dA \qquad (6.10)$$

Note that when (6.9) is generalized to three dimensions the boundary term in (6.9) becomes a surface integral. Equation (6.10) can be expressed in words as follows.

$$\left\{\begin{matrix} \text{Rate of} \\ \text{change of the} \\ \text{total amount of } F \\ \text{inside } V \end{matrix}\right\} = \left\{\begin{matrix} \text{Rate of change} \\ \text{due to changes} \\ \text{of } F \text{ within } V \end{matrix}\right\} + \left\{\begin{matrix} \text{Rate of change due to} \\ \text{movement of the} \\ \text{surface } A \text{ enclosing} \\ \text{more or less } F \text{ within } V \end{matrix}\right\} \qquad (6.11)$$

The Leibniz relationship (6.10) is fundamental to the development of the transport theory of continuous media. The velocity vector $\bar{U}_A$ is that of the control volume surface itself. If the medium is a moving fluid the surface velocity $\bar{U}_A$ is specified independently of the fluid velocity $\bar{U}$. Consider a fluid with velocity vector $\bar{U}$ which is a function of space and time.

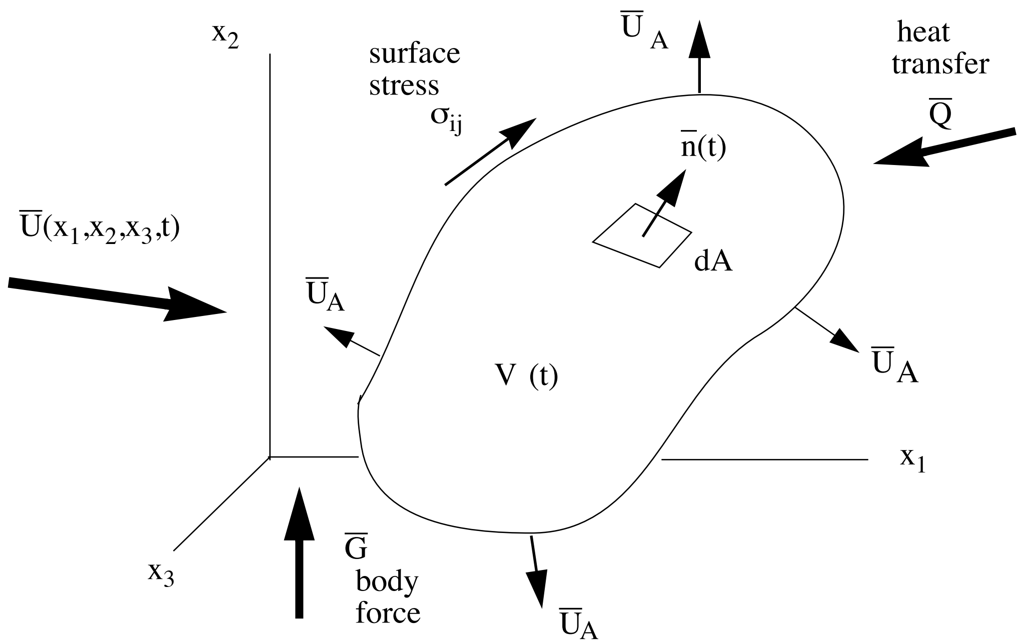


Figure 6.3 Control volume defined in a flow field subject to surface stresses, body forces and heat conduction.

Now let the velocity of each surface element of the control volume be the same as the velocity of the flow, $\bar{U} = \bar{U}_A$. In effect, we assume that the surface is attached to the fluid and therefore the control volume always contains the same set of fluid elements. This is called a *Lagrangian* control volume. In this case Leibnitz' rule becomes

$$\frac{D}{Dt}\int_{V(t)} F\,dV = \int_{V(t)} \frac{\partial F}{\partial t}dV + \int_{A(t)} F\,\bar{U} \bullet \bar{n}\,dA . \tag{6.12}$$

Use the Gauss theorem to convert the surface integral to a volume integral. The result is the Reynolds transport theorem.

$$\frac{D}{Dt}\int_{V(t)} F\,dV = \int_{V(t)} \left(\frac{\partial F}{\partial t} + \nabla \bullet (F\bar{U})\right) dV . \tag{6.13}$$

6.2 CONSERVATION OF MASS

Let $F = \rho$ where ρ is the density of the fluid. The Reynolds transport theorem gives

$$\frac{D}{Dt}\int_{V(t)} \rho dV = \int_{V(t)} \left(\frac{\partial \rho}{\partial t} + \nabla \bullet (\rho\bar{U})\right) dV . \tag{6.14}$$

The left-hand-side is the rate of change of the total mass inside the control volume. If there are no sources of mass within the control volume, the left-hand-side must be zero. Since the choice of control volume is arbitrary, the kernel of the right-hand-side must therefore be zero at every point in the flow.

Thus the continuity equation in the absence of mass sources is

$$\frac{\partial \rho}{\partial t} + \nabla \bullet (\rho\bar{U}) = 0 . \tag{6.15}$$

This equation, expressed in coordinate independent vector notation, is the same one that we derived in Chapter 1 using an infinitesimal, cubic, Eulerian control volume.

Expand (6.15)

$$\frac{\partial\rho}{\partial t} + \overline{U} \bullet \nabla\rho + \rho\nabla \bullet \overline{U} = 0 . \tag{6.16}$$

In terms of the substantial derivative the continuity equation is

$$\frac{D\rho}{Dt} + \rho\nabla \bullet \overline{U} = 0 \tag{6.17}$$

If the medium is *incompressible* then $\rho = constant$ and $\nabla \bullet \overline{U} = 0$.

6.3 Conservation of momentum

In this case the generic variable in Leibniz' rule is the vector momentum per unit volume, $F = \rho\overline{U}$. Momentum is convected about by the motion of the fluid itself and spatial variations of pressure and viscous stresses act as sources of momentum. Restricting ourselves to the motion of a continuous, viscous fluid (liquid or gas), the stress in a fluid is composed of two parts; a locally isotropic part proportional to the scalar pressure field and a non-isotropic part due to viscous friction. The stress tensor is

$$\sigma_{ij} = -P\delta_{ij} + \tau_{ij} \tag{6.18}$$

where P is the thermodynamic pressure, δ_{ij} is the Kronecker unit tensor defined in Chapter 3,

$$\bar{\bar{I}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{pmatrix} \delta_{ij} = 1 \;; & i = j \\ \delta_{ij} = 0 \;; & i \neq j \end{pmatrix} \tag{6.19}$$

and τ_{ij} is the viscous stress tensor. The net force acting on the control volume is the integral of the stress tensor, σ_{ij}, over the surface plus the integral of any body force vectors per unit mass, $\bar{G}$ (gravitational acceleration, electromagnetic acceleration, etc.), over the volume.

The isotropy of the pressure implies that it acts normal to any surface element in the fluid regardless of how it is oriented. The viscous part of the stress can take on many different forms. In Aeronautics and Astronautics we deal almost exclusively with Newtonian fluids discussed in Chapter 1 such as air or water where the viscous stress is linearly proportional to the rate-of-strain tensor of the flow.

The general form of the stress-rate-of-strain constitutive relation in Cartesian coordinates for a compressible Newtonian fluid is

$$\tau_{ij} = 2\mu S_{ij} - \left(\frac{2}{3}\mu - \mu_v\right)\delta_{ij}S_{kk} \tag{6.20}$$

where,

$$S_{ij} = \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right). \tag{6.21}$$

Recall that $S_{kk} = \nabla \bullet \bar{U}$. The stress components in cylindrical and spherical polar coordinates are given in Appendix 2.

Interestingly, there are actually two viscosity coefficients that are required to account for all possible stress fields that depend linearly on the rate-of-strain tensor. The so-called shear viscosity μ arises from momentum exchange due to molecular motion. A simple model of μ is described in Appendix I. The bulk viscosity μ_v is a little more mysterious. It contributes only to the viscous normal force and seems to arise from the exchange of momentum that can occur between colliding molecules and the internal degrees of freedom of the molecular system. Some typical values of the bulk viscosity are shown in Figure 6.4.

Fluid	$\mu \times 10^5$, kg/(m)(s)	μ_v/μ	$\kappa \times 10^2$, J/(m)(s)(K)	$\frac{\mu}{\rho} \times 10^5$, m^2/s	Pr
He	1.98	0	15.0	12.2	0.67
Ar	2.27	0	1.77	1.40	0.67
H_2	0.887	32	17.3	10.8	0.71
N_2	1.66	0.8	2.52	1.46	0.71
O_2	2.07	0.4	2.58	1.59	0.72
CO_2	1.50	1,000	1.66	0.837	0.75
Air	1.85	0.6	2.58	1.57	0.71
H_2O (*liquid*)	85.7	3.1	61	0.0857	6.0
Ethyl alcohol	110	4.5	18.3	0.14	15
Glycerine	134,000	0.4	29	109	11,000

Figure 6.4 Physical properties of some common fluids at one atmosphere and 298.15°K.

For monatomic gases that lack such internal degrees of freedom, $\mu_v = 0$. For some polyatomic gases such as CO_2 the bulk viscosity is much larger than the shear viscosity.

Recall the discussion of elementary flow patterns from Chapter 4. Any fluid flow can be decomposed into a rotational part and a straining part. According to the Newtonian model (6.20) only the straining part contributes to the viscous stress.

Although μ is called the shear viscosity it is clear from the diagonal terms in (6.20) that there are viscous normal force components proportional to μ. However they make no net contribution to the mean normal stress defined as

$$\sigma_{mean} = (1/3)\sigma_{ii} = -P + \mu_v S_{kk}. \tag{6.22}$$

This is not to say that viscous normal stresses are unimportant. They play a key role in many compressible flow phenomena we will study later especially shock waves.

A common assumption called Stokes' hypothesis is to assume that the "bulk viscosity" μ_v is equal to zero. Then only the so-called shear viscosity μ appears in the constitutive relation for the stress. While this assumption is strictly valid only for monatomic gases, it is applied very widely and works quite well, mainly because $\nabla \bullet \overline{U}$ tends to be relatively small in most situations outside of shock waves and high Mach number flow.

The rate of change of the total momentum inside the control volume is,

$$\frac{D}{Dt}\int_{V(t)} \rho\overline{U}dV = \int_{A(t)} (-P\bar{\bar{I}} + \bar{\bar{\tau}}) \bullet \bar{n}dA + \int_{V(t)} \rho\overline{G}dV . \qquad (6.23)$$

Use the Reynolds transport theorem to replace the left-hand-side of (6.23) and the Gauss theorem to convert the surface integral to a volume integral.

$$\int_{V(t)} \left(\frac{\partial\rho\overline{U}}{\partial t} + \nabla \bullet (\rho\overline{U}\,\overline{U} + P\bar{\bar{I}} - \bar{\bar{\tau}}) - \rho\overline{G}\right)dV = 0 . \qquad (6.24)$$

Since the equality must hold over an arbitrary control volume, the kernel must be zero at every point in the flow and we have the differential equation for conservation of momentum.

$$\frac{\partial\rho\overline{U}}{\partial t} + \nabla \bullet (\rho\overline{U}\,\overline{U} + P\bar{\bar{I}} - \bar{\bar{\tau}}) - \rho\overline{G} = 0 . \qquad (6.25)$$

This is the same momentum equation we derived in Chapter 1 except for the inclusion of the body force term.

6.4 CONSERVATION OF ENERGY

The energy per unit mass of a moving fluid element is $e + k$ where e is the internal energy per unit mass of the medium and

$$k = \frac{1}{2}U_iU_i = \frac{1}{2}(U_1^2 + U_2^2 + U_3^2) \qquad (6.26)$$

is the kinetic energy per unit mass. In this case we use $F = \rho(e + k)$ in the Leibniz rule.

The stress tensor acting over the surface does work on the control volume as do the body force vectors. In addition, there may be conductive heat flux, $\overline{Q}$, through the surface. There could also be sources of heat within the flow due to chemical reactions, radiative heating, etc. For a general fluid the internal energy per unit mass is a function of temperature and pressure $e = f(T, P)$.

The rate of change of the energy inside the control volume in Figure 6.3 is

$$\frac{D}{Dt}\int_{V(t)} \rho(e+k)dV = \int_{A(t)} ((-P\bar{\bar{I}} + \bar{\bar{\tau}}) \bullet \overline{U} - \overline{Q}) \bullet \bar{n}dA + \int_{V(t)} (\rho\overline{G} \bullet \overline{U})dV . \qquad (6.27)$$

The Reynolds transport theorem and Gauss' theorem lead to

$$\int_{V(t)} \left(\frac{\partial\rho(e+k)}{\partial t} + \nabla \bullet \left(\rho\overline{U}\left(e + \frac{P}{\rho} + k\right) - \bar{\bar{\tau}} \bullet \overline{U} + \overline{Q}\right) - \rho\overline{G} \bullet \overline{U}\right) dV = 0 . \qquad (6.28)$$

Since the equality holds over an arbitrary volume, the kernel must be zero and we have the differential equation for conservation of energy.

$$\frac{\partial\rho(e+k)}{\partial t} + \nabla \bullet \left(\rho\overline{U}\left(e + \frac{P}{\rho} + k\right) - \bar{\bar{\tau}} \bullet \overline{U} + \overline{Q}\right) - \rho\overline{G} \bullet \overline{U} = 0 . \qquad (6.29)$$

The sum of enthalpy and kinetic energy

$$\boxed{h_t = e + \frac{P}{\rho} + k = h + \frac{1}{2}U_iU_i} \qquad (6.30)$$

is called the *stagnation or total enthalpy* and plays a key role in the transport of energy in compressible flow systems. Take care to keep in mind that the flow *energy* is purely the sum of internal and kinetic energy, $e + k$. Some typical gas transport properties at 300K and one atmosphere are shown in Figure 6.4.

According to Fick's law, the heat flux vector in a linear heat conducting medium is:

$$Q_i = -\kappa(\partial T / \partial x_i) \qquad (6.31)$$

where κ is the thermal conductivity.

The rightmost column in Figure 6.4 is the Prandtl number

$$Pr = \frac{\mu C_p}{\kappa} \tag{6.32}$$

The Prandtl number can be thought of as comparing the rate at which momentum is transported by viscous diffusion to the rate at which temperature diffuses through conductivity. For most gases the Prandtl number is around 0.7. This number is close to one due to the fact that heat and momentum transport are accomplished by the same basic mechanism of molecular collision with lots of space between molecules. Liquids are often characterized by large values of the Prandtl number and the underlying mechanisms of heat and momentum transport in a condensed fluid are much more complex.

6.5 SUMMARY - DIFFERENTIAL FORM OF THE EQUATIONS OF MOTION

The coordinate-independent form of the equations of motion is

$$\boxed{\begin{gathered}\frac{\partial\rho}{\partial t} + \nabla \bullet (\rho\overline{U}) = 0 \\ \\ \frac{\partial\rho\overline{U}}{\partial t} + \nabla \bullet (\rho\overline{U}\,\overline{U} + P\bar{\bar{I}} - \bar{\bar{\tau}}) - \rho\overline{G} = 0 \\ \\ \frac{\partial\rho(e+k)}{\partial t} + \nabla \bullet \left(\rho\overline{U}\left(e + \frac{P}{\rho} + k\right) - \bar{\bar{\tau}} \bullet \overline{U} + \overline{Q}\right) - \rho\overline{G} \bullet \overline{U} = 0\end{gathered}} \; . \tag{6.33}$$

Using index notation the same equations in Cartesian coordinates are

$$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho U_i) = 0$$

$$\frac{\partial\rho U_i}{\partial t} + \frac{\partial}{\partial x_j}(\rho U_i U_j + P\delta_{ij} - \tau_{ij}) - \rho G_i = 0 \qquad . \qquad (6.34)$$

$$\frac{\partial\rho(e + k)}{\partial t} + \frac{\partial}{\partial x_i}\left(\rho U_i\left(e + \frac{P}{\rho} + k\right) - \tau_{ij}U_j + Q_i\right) - \rho G_i U_i = 0$$

The equations of motion in cylindrical and spherical polar coordinates are given in Appendix 2.

6.6 Integral form of the equations of motion

In deriving the differential form of the conservation equations (6.33) we used a general Lagrangian control volume where the surface velocity is equal to the local fluid velocity and the surface always encloses the same set of fluid elements. In Chapter 1 we derived the same equations on a rectangular Eulerian control volume. It is important to recognize that these partial differential equations are valid at every point and at every instant in the flow of a compressible continuum and are completely independent of the particular control volume approach (Eulerian or Lagrangian) used to derive them.

With the equations of motion in hand we will now reverse the process and work out the integral form of these equations on control volumes that are adapted to solving useful problems. In this endeavor, it is useful to consider other kinds of control volumes where the control surface may be stationary or where part of the control surface is stationary and part is moving but not necessarily attached to the fluid. Recall the general form of the Leibniz rule.

$$\frac{D}{Dt}\int_{V(t)} F\,dV = \int_{V(t)} \frac{\partial F}{\partial t}dV + \int_{A(t)} F\,\bar{U}_A \bullet \bar{n}\,dA . \qquad (6.35)$$

6.6.1 Integral equations on an Eulerian control volume

The simplest case to consider is the Eulerian control volume used in Chapter 1 where $\bar{U}_A = 0$. This is a stationary volume fixed in space through which the fluid moves. In this case the Leibniz rule reduces to

$$\frac{d}{dt}\int_V F\,dV = \int_V \frac{\partial F}{\partial t}dV\,. \tag{6.36}$$

Note that, since the Eulerian volume is fixed in space and not time dependent, the lower case form of the derivative d/dt is used in (6.36). See for comparison (6.6) and (6.9).

Let $F = \rho$ in Equation (6.36)

$$\frac{d}{dt}\int_V \rho dV = \int_V \frac{\partial \rho}{\partial t}dV \tag{6.37}$$

Use (6.15) to replace $\partial\rho/\partial t$ in (6.37).

$$\frac{d}{dt}\int_V \rho dV = -\int_V \nabla \bullet (\rho\bar{U})dV \tag{6.38}$$

Now use the Gauss theorem to convert the volume integral on the right-hand-side of (6.38) to a surface integral. The integral form of the mass conservation equation valid on a finite Eulerian control volume of arbitrary shape is

$$\frac{d}{dt}\int_V \rho dV + \int_A \rho\bar{U} \bullet \bar{n}dA = 0 \tag{6.39}$$

The integral equations for conservation of momentum and energy are derived in a similar way using (6.25), (6.29), and (6.36). The result is

$$\frac{d}{dt}\int_V \rho dV + \int_A \rho \overline{U} \bullet \bar{n} dA = 0$$

$$\frac{d}{dt}\int_V \rho \overline{U} dV + \int_A (\rho \overline{U}\,\overline{U} + P\bar{\bar{I}} - \bar{\bar{\tau}}) \bullet \bar{n} dA - \int_V \rho \overline{G} dV = 0 \qquad (6.40)$$

$$\frac{d}{dt}\int_V \rho(e+k) dV + \int_A \left(\rho \overline{U}\left(e + \frac{P}{\rho} + k\right) - \bar{\bar{\tau}} \bullet \overline{U} + \overline{Q}\right) \bullet \bar{n} dA - \int_V (\rho \overline{G} \bullet \overline{U}) dV = 0$$

6.6.2 Mixed Eulerian-Lagrangian control volumes

More general control volumes where part of the surface may be at rest and other parts may be attached to the fluid are of great interest especially in the analysis of propulsion systems. Now use the general form of the Liebniz rule with $F = \rho$.

$$\frac{D}{Dt}\int_{V(t)} \rho dV = \int_{V(t)} \frac{\partial \rho}{\partial t} dV + \int_{A(t)} \rho \, \overline{U}_A \bullet \bar{n} dA \qquad (6.41)$$

Use (6.15) to replace $\partial\rho / \partial t$ in (6.41) and use the Gauss theorem to convert the volume integral to a surface integral. The result is the integral form of mass conservation on an arbitrary moving control volume.

$$\frac{D}{Dt}\int_{V(t)} \rho dV = -\int_{A(t)} \rho \overline{U} \bullet \bar{n} dA + \int_{A(t)} \rho \, \overline{U}_A \bullet \bar{n} dA \qquad (6.42)$$

The integral equations for conservation of momentum and energy on a general, moving, finite control volume are derived in a similar way using (6.25), (6.29), and (6.35). Finally, the most general integrated form of the conservation equations is

$$\frac{D}{Dt}\int_{V(t)} \rho dV + \int_{A(t)} \rho(\bar{U} - \bar{U}_A) \bullet \bar{n} dA = 0$$

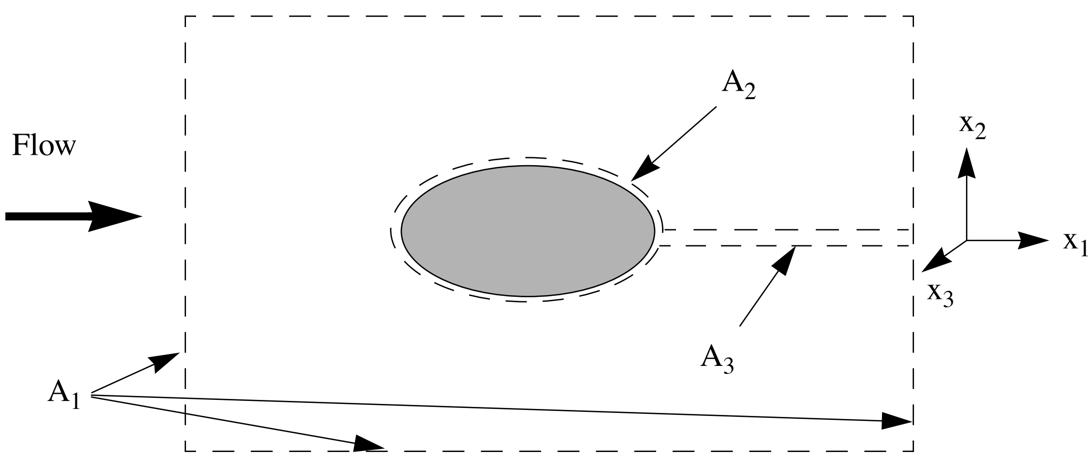
$$\frac{D}{Dt}\int_{V(t)} \rho \bar{U} dV + \int_{A(t)} (\rho \bar{U}(\bar{U} - \bar{U}_A) + P\bar{\bar{I}} - \bar{\bar{\tau}}) \bullet \bar{n} dA - \int_{V(t)} \rho \bar{G} dV = 0$$

$$\frac{D}{Dt}\int_{V(t)} \rho(e+k) dV + \int_{A(t)} (\rho(e+k)(\bar{U} - \bar{U}_A) + P\bar{\bar{I}} \bullet \bar{U} - \bar{\bar{\tau}} \bullet \bar{U} + \bar{Q}) \bullet \bar{n} dA - \int_{V(t)} (\rho \bar{G} \bullet \bar{U}) dV = 0 \qquad (6.43)$$

Remember, $\bar{U}_A$ is the velocity of the control volume surface and can be selected at the convenience of the user.

6.7 APPLICATIONS OF CONTROL VOLUME ANALYSIS

6.7.1 EXAMPLE 1 - SOLID BODY AT REST IN A STEADY FLOW.



This is a simply connected Eulerian control volume where the segment A_2 surrounding (and attached to) the body is connected by a cut to the surrounding boundary A_1. All fluxes on the cut A_3 cancel and therefore make no contribution to the integrated conservation laws. There is no mass injection through the surface of the body thus

$$\int_{A_1} (\rho \bar{U}) \bullet \bar{n} dA = 0 . \tag{6.44}$$

Momentum fluxes integrated on A_1 are directly related to the lift and drag forces exerted on the body. The integrated momentum equation gives

$$\int_{A_1} (\rho \bar{U}\bar{U} + P\bar{\bar{I}} - \bar{\bar{\tau}}) \bullet \bar{n} dA + \int_{A_2} (P\bar{\bar{I}} - \bar{\bar{\tau}}) \bullet \bar{n} dA = 0 \tag{6.45}$$

$$Drag = \left. \int_{A_2} (P\bar{\bar{I}} - \bar{\bar{\tau}}) \bullet \bar{n} dA \right|_{x_1} \quad ; \quad Lift = \left. \int_{A_2} (P\bar{\bar{I}} - \bar{\bar{\tau}}) \bullet \bar{n} dA \right|_{x_2} . \tag{6.46}$$

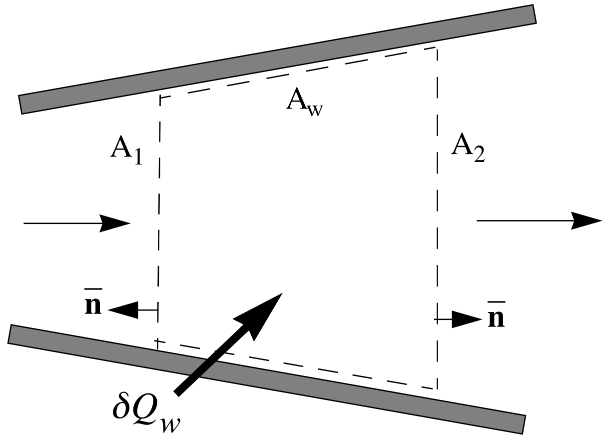
where the integrals are the drag and lift *by the flow on the body*. The integral momentum balance in the streamwise direction is

$$\left. \int_{A_1} (\rho \bar{U}\bar{U} + P\bar{\bar{I}} - \bar{\bar{\tau}}) \bullet \bar{n} dA \right|_{x_1} + Drag = 0 . \tag{6.47}$$

and in the vertical direction

$$\left. \int_{A_1} (\rho \bar{U}\bar{U} + P\bar{\bar{I}} - \bar{\bar{\tau}}) \bullet \bar{n} dA \right|_{x_2} + Lift = 0 . \tag{6.48}$$

6.7.2 EXAMPLE 2 - CHANNEL FLOW WITH HEAT ADDITION



Heat addition to the compressible flow shown above occurs through heat transfer through the channel wall, Q. There is no net mass addition to the control volume.

$$\int_{A_1}(\rho\overline{U})\bullet\bar{n}dA+\int_{A_2}(\rho\overline{U})\bullet\bar{n}dA = 0. \tag{6.49}$$

The flow is steady with no body forces. In this case, the energy balance is,

$$\int_{A}(\rho\overline{U}(e+k)+P\overline{U}-\bar{\bar{\tau}}\bullet\overline{U}+\overline{Q})\bullet\bar{n}dA = 0. \tag{6.50}$$

The contribution of the viscous stresses to the energy balance is zero along the wall because of the no-slip condition, $\overline{U}\big|_{A_w} = 0$ and, as long as the streamwise velocity gradients are not large, the term $-\bar{\bar{\tau}}\bullet\overline{U}$ is very small on the upstream and downstream faces, A_1, A_2. In this approximation, The energy balance becomes,

$$\int_{A}\rho\overline{U}\left(e+\frac{P}{\rho}+k\right)\bullet\bar{n}dA = -\int_{A}\overline{Q}\bullet ndA. \tag{6.51}$$

The effects of heat transfer through the wall and conduction through the upstream and downstream faces, A_1 and A_2 are accounted for by the change in the flux of stagnation enthalpy. Heat transfer through the upstream and downstream faces is usually small and so most of the conductive heat transfer into the flow is through the wall.

$$-\int_{A} \overline{Q} \bullet \bar{n} dA \cong -\int_{A_w} \overline{Q} \bullet \bar{n} dA = \delta Q . \tag{6.52}$$

The energy balance for this case reduces to,

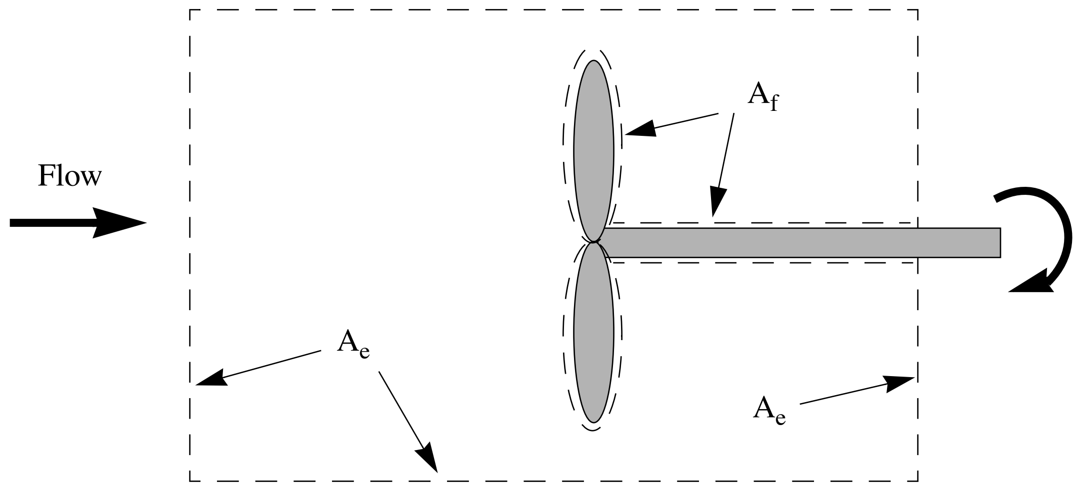
$$\int_{A_2} \rho h_t \overline{U} \bullet \bar{n} dA + \int_{A_1} \rho h_t \overline{U} \bullet \bar{n} dA = \delta Q . \tag{6.53}$$

When the vector multiplication is carried out (6.53) becomes

$$\int_{A_2} \rho_2 U_2 h_{t2} dA - \int_{A_1} \rho_1 U_1 h_{t1} dA = \delta Q . \tag{6.54}$$

The heat added to the flow is directly and simply related to the change of the integrated flow rate of stagnation enthalpy along the channel.

6.7.3 Example 3 - Stationary flow about a rotating fan.



This is the prototype example for propellers, compressors and turbines. In contrast to a steady flow, a stationary flow is one where time periodic motions such as the rotation of the fan illustrated above do occur, but the properties of the flow averaged over one fan rotation period or one blade passage period are constant. This will be the case if the fan rotation speed is held constant.

Here we make use of a mixed Eulerian-Lagrangian control volume. The Lagrangian part is attached to and moves with the fan blade surfaces and fan axle. Remember the fluid is viscous and subject to a no slip condition at the solid surface. The Eulerian surface elements are the upstream and downstream faces of the control volume as well as the cylindrical surrounding surface aligned with the axis of the fan. We will assume that the fan is adiabatic, $\overline{Q} = 0$ and there is no mass injection through the fan surface. The integrated mass fluxes are zero.

$$\int_{A_e} (\rho \overline{U}) \bullet \bar{n} dA = 0 \tag{6.55}$$

Momentum fluxes integrated on A_e are equal to the surface forces exerted by the flow on the fan.

$$\int_{A_e} (\rho \overline{U}\,\overline{U} + P\bar{\bar{I}} - \bar{\bar{\tau}}) \bullet \bar{n} dA + \int_{A_f} (\rho \overline{U}(\overline{U} - \overline{U}_A) + P\bar{\bar{I}} - \bar{\bar{\tau}}) \bullet \bar{n} dA = 0 \tag{6.56}$$

or

$$\int_{A_e} (\rho \bar{U}\bar{U} + P\bar{\bar{I}} - \bar{\bar{\tau}}) \bullet \bar{n} dA + \bar{F} = 0 \tag{6.57}$$

where the vector force *by the flow on the fan* is

$$\bar{F} = \int_{A_f} (P\bar{\bar{I}} - \bar{\bar{\tau}}) \bullet \bar{n} dA \tag{6.58}$$

Note that the flow and fan velocity on A_f are the same due to the no-slip condition

$$(\bar{U} - \bar{U}_A)\Big|_{A_f} = 0. \tag{6.59}$$

For stationary flow, the integrated energy fluxes on A_e are equal to the *work/sec done by the flow on the fan*.

$$\int_{A_e} (\rho \bar{U}(e+k) + P\bar{U} - \bar{\bar{\tau}} \bullet \bar{U}) \bullet \bar{n} dA + \int_{A_f} (P\bar{U} - \bar{\bar{\tau}} \bullet \bar{U}) \bullet \bar{n} dA = 0. \tag{6.60}$$

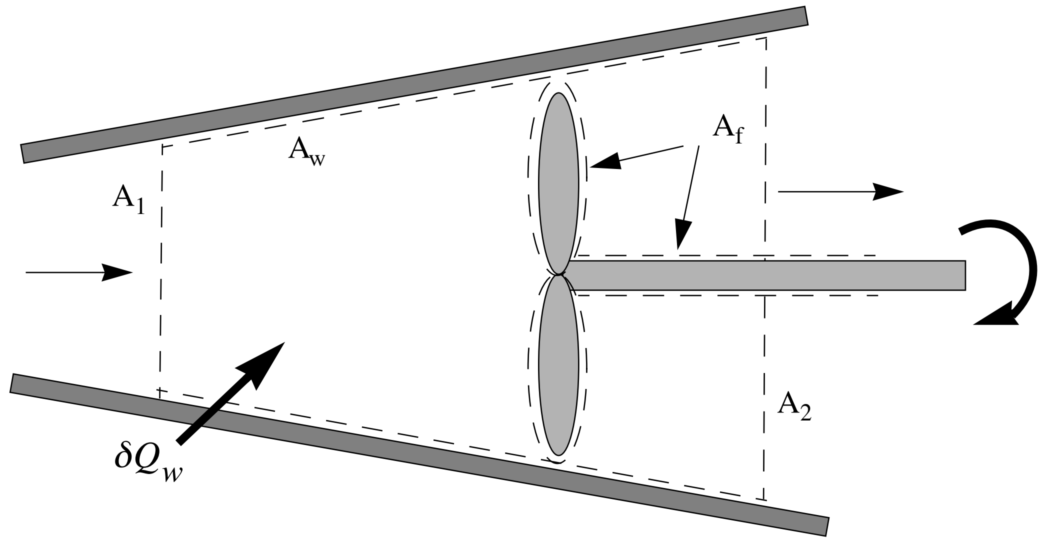
$$Work = \int_{A_f} (P\bar{U} - \bar{\bar{\tau}} \bullet \bar{U}) \bullet \bar{n} dA = \delta W \tag{6.61}$$

If the flow is adiabatic, and viscous normal stresses are neglected on A_e and A_f the energy equation becomes,

$$\int_{A_e} \rho\left(e + \frac{P}{\rho} + k\right)\bar{U} \bullet \bar{n} dA + \delta W = 0 \tag{6.62}$$

6.7.4 Example 4 - Combined heat transfer and work

In a general situation where there is heat transfer and work done



the energy equation has the concise form,

$$\int_{A_2} \rho_2 U_2 h_{t2} dA - \int_{A_1} \rho_1 U_1 h_{t1} dA = \delta Q - \delta W . \qquad (6.63)$$

The effects of heat addition and work done are both accounted for by changes in the stagnation enthalpy of the flow. In general, changes due to thermal exchange are irreversible whereas changes due to the work done can be very nearly reversible except for entropy changes due to viscous friction.

6.8 Stagnation enthalpy, temperature and pressure

6.8.1 Stagnation enthalpy of a fluid element

It is instructive to develop an equation for the stagnation enthalpy change of a fluid element in a general unsteady flow. Using the energy equation

$$\frac{\partial \rho(e+k)}{\partial t} + \nabla \bullet \left(\rho \overline{U} \left(e + \frac{P}{\rho} + k \right) - \bar{\bar{\tau}} \bullet \overline{U} + \overline{Q} \right) - \rho \overline{G} \bullet \overline{U} = 0 \qquad (6.64)$$

along with the continuity equation

$$\frac{D\rho}{Dt} = -\rho\nabla \bullet \overline{U} \tag{6.65}$$

and the identity

$$\frac{D}{Dt}\left(\frac{P}{\rho}\right) = \frac{1}{\rho}\frac{DP}{Dt} - \frac{P}{\rho^2}\frac{D\rho}{Dt} \tag{6.66}$$

one can show that

$$\rho\frac{Dh_t}{Dt} = \nabla \bullet (\bar{\bar{\tau}} \bullet \overline{U} - \overline{Q}) + \rho\overline{G} \bullet \overline{U} + \frac{\partial P}{\partial t}. \tag{6.67}$$

Equation (6.67) shows that changes in the stagnation enthalpy of a fluid element may be due to heat conduction as well as the work done by body forces and work by viscous forces. In addition, nonsteady changes in the pressure can also change h_t.

In most aerodynamic problems body forces are negligible. Exceptions can occur at low speeds where density changes due to thermal gradients lead to gravity driven flows.

The work done by viscous forces is usually small (the work is zero at a wall where $\overline{U} = 0$) and energy transport by heat conduction, Q, is often small. In this case the stagnation enthalpy of a fluid element is preserved if the flow is steady, $Dh_t/Dt = 0$.

The last term in (6.67) can be quite large in many nonsteady processes. For example, large temperature changes can occur in unsteady vortex formation in the wake of a bluff body in high speed flow due to this term.

If the flow is steady, inviscid and non-heat-conducting then (6.67) reduces to

$$\overline{U} \bullet \nabla(h_t + \Psi) = 0. \tag{6.68}$$

where the gravitational potential is related to the gravitational force by $\overline{G} = \nabla\Psi$.

6.8.2 BLOWDOWN FROM A PRESSURE VESSEL REVISITED

In Chapter 2 we looked at the blowdown of gas from an adiabatic pressure vessel through a small nozzle. The situation is shown below.

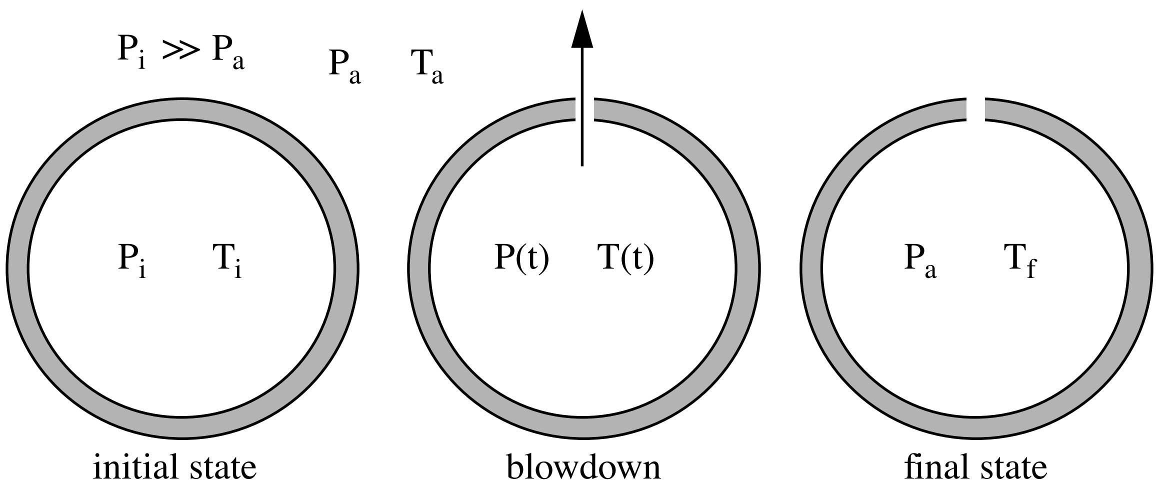


Figure 6.5 A adiabatic pressure vessel exhausting to the surroundings.

The stagnation enthalpy of the flow inside the pressure vessel is governed by (6.67). If we drop the body force term and assume the divergence term involving heat transfer and viscous work are small then (6.67) simplifies to

$$\rho\frac{Dh_t}{Dt} = \frac{\partial P}{\partial t}. \tag{6.69}$$

If we further assume that the vessel is very large and the hole is very small then the contribution of the flow velocity to the stagnation enthalpy can be neglected since most of the gas is almost at rest. Under these conditions the temperature and pressure can be regarded as approximately uniform over the interior of the vessel. With these assumptions (6.69) simplifies to

$$\rho\frac{dh}{dt} = \frac{dP}{dt} \tag{6.70}$$

Equation (6.70) is the Gibbs equation for an isentropic process. If the gas is calorically perfect the final temperature is given by

$$\frac{T_f}{T_i} = \left(\frac{P_a}{P_i}\right)^{\frac{\gamma - 1}{\gamma}}. \tag{6.71}$$

This is the result we were led to in Chapter 2 when we assumed the process in the vessel was isentropic. The difference is that by beginning with (6.67) the assumptions needed to reach (6.71) are clarified. This example nicely illustrates the role of the unsteady pressure term in (6.67).

6.8.3 STAGNATION ENTHALPY AND TEMPERATURE IN STEADY FLOW

The figure below shows the path of a fluid element in *steady flow* stagnating at the leading edge of an airfoil.

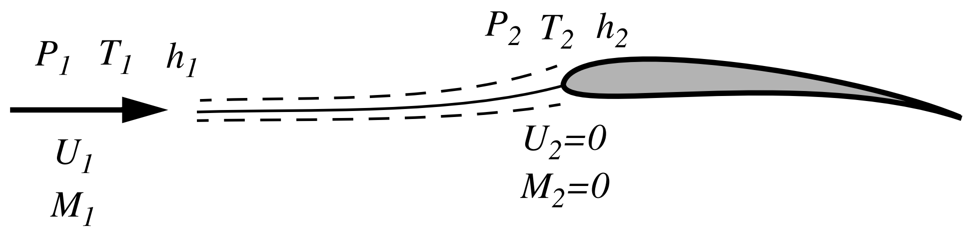


Figure 6.6 Schematic of a stagnation process in steady flow

The dashed lines outline a small streamtube surrounding the stagnation streamline. Let's interpret this problem in two ways, first as an extension of section 5.7 example 2 and then in the context of equation (6.67).

The result in example 2 applies to the flow in the streamtube.

$$\int_{A_2} \rho_2 U_2 h_{t2} dA - \int_{A_1} \rho_1 U_1 h_{t1} dA = \delta Q \qquad (6.72)$$

If we assume the airfoil is adiabatic and that there is no net heat loss or gain through the stream tube then

$$\int_{A_2} \rho_2 U_2 h_{t2} dA = \int_{A_1} \rho_1 U_1 h_{t1} dA \qquad (6.73)$$

Since the mass flow at any point in the stream tube is the same then one can conclude that the enthalpy per unit mass is also the same at any point along the stream tube (at least in an average sense across the tube which we can make arbitrarily narrow).

According to (6.67) the stagnation enthalpy of the fluid element can also change due to viscous work which we neglected in example 3. However as the element decelerates on its approach to the airfoil leading edge where viscous forces might be expected to play a role, the velocity becomes small and goes to zero at the stagnation point so one can argue that the viscous work terms are also small. An exception where this assumption needs to be re-examined is at very low Reynolds number where the viscous region can extend a considerable distance from the airfoil.

By either argument we can conclude that to a good approximation in a steady flow

$$h_{t1} = e_1 + \frac{P_1}{\rho_1} + k_1 = h_1 + \frac{1}{2}U_1^2 = h_2 + \frac{1}{2}U_2^2 = h_2 \tag{6.74}$$

With viscous work neglected, the stagnation enthalpy is conserved along an adiabatic path. When a fluid element is brought to rest adiabatically the enthalpy at the rest state is the stagnation enthalpy at the initial state.

The stagnation temperature is defined by the enthalpy relation.

$$h_t - h = \int_T^{T_t} C_p dT = \frac{1}{2}U_i U_i \tag{6.75}$$

The stagnation temperature is the temperature reached by an element of gas brought to rest adiabatically. For a calorically perfect ideal gas with constant specific heat in the range of temperatures between T and T_t (6.75) can be written

$$C_p T_t = C_p T + \frac{1}{2}U_i U_i . \tag{6.76}$$

In Figure 6.6 we would expect $T_2 = T_{t1}$ to the extent that the heat capacity is constant. Divide through by $C_p T$ and introduce the speed of sound $a^2 = \gamma RT$. Equation (6.76) becomes,

$$\boxed{\frac{T_t}{T} = 1 + \left(\frac{\gamma - 1}{2}\right) M^2} \tag{6.77}$$

where M is the Mach number of the fluid element.

6.8.4 FRAMES OF REFERENCE

In Chapter 1 we discussed how the momentum and kinetic energy of a fluid element change when the frame of reference is transformed from fixed to moving coordinates. Clearly since the stagnation temperature (6.76) depends on the kinetic energy then it too depends on the frame of reference. The stagnation temperatures in fixed and moving coordinates are

$$\begin{aligned} C_p T_t &= C_p T + \frac{1}{2} U_i U_i \\ C_p T_t' &= C_p T + \frac{1}{2} U_i' U_i' \end{aligned} \tag{6.78}$$

The transformation of the kinetic energy is

$$\frac{1}{2} U_i' U_i' = \frac{1}{2} U_i U_i + \frac{1}{2} \dot{X}(\dot{X} - 2U) + \frac{1}{2} \dot{Y}(\dot{Y} - 2V) + \frac{1}{2} \dot{Z}(\dot{Z} - 2W) \tag{6.79}$$

where $(\dot{X}, \dot{Y}, \dot{Z})$ are the velocity components of the moving origin of coordinates. The thermodynamic variables density, temperature and pressure do not change and so the transformation of the stagnation temperature is

$$C_p T_t' = C_p T_t + \frac{1}{2} \dot{X}(\dot{X} - 2U) + \frac{1}{2} \dot{Y}(\dot{Y} - 2V) + \frac{1}{2} \dot{Z}(\dot{Z} - 2W) \tag{6.80}$$

6.8.5 STAGNATION PRESSURE

The stagnation pressure is defined using the Gibbs equation for an ideal gas.

$$ds = C_p \frac{dT}{T} - R \frac{dP}{P} \tag{6.81}$$

Note that while C_p and C_v depend on temperature, the gas constant $C_p - C_v$ is independent of temperature as long as there are no chemical reactions that might change the molecular weight of the gas. Integrate (6.81) along an isentropic path $ds = 0$ from state 1 to state 2.

$$R\int_{P_1}^{P_2}\frac{dP}{P} = \int_{T_1}^{T_2} C_p(T)\frac{dT}{T} \tag{6.82}$$

which becomes

$$\frac{P_2}{P_1} = Exp\left(\frac{1}{R}\int_{T_1}^{T_2} C_p(T)\frac{dT}{T}\right) \tag{6.83}$$

If an element of flowing gas at pressure P and temperature T is brought to rest *adiabatically and isentropically* to a temperature T_t the stagnation pressure P_t is defined by

$$\frac{P_t}{P} = Exp\left(\frac{1}{R}\int_{T}^{T_t} C_p(T)\frac{dT}{T}\right) \tag{6.84}$$

If the heat capacities are constant (6.84) reduces to the isentropic relation

$$\boxed{\frac{P_t}{P} = \left(\frac{T_t}{T}\right)^{\frac{\gamma}{\gamma-1}} = \left(1 + \left(\frac{\gamma-1}{2}\right)M^2\right)^{\frac{\gamma}{\gamma-1}}}. \tag{6.85}$$

If the stagnation path in Figure 6.6 is isentropic as well as adiabatic and the gas is calorically perfect then one would expect

$$P_2 = P_{t1} = P_1\left(1 + \left(\frac{\gamma-1}{2}\right)M_1^{\ 2}\right)^{\frac{\gamma}{\gamma-1}}. \tag{6.86}$$

The stagnation state is just that, a thermodynamic state, and changes in the stagnation state of a material are described by the Gibbs equation.

$$T_t\frac{Ds}{Dt} = \frac{Dh_t}{Dt} - \frac{1}{\rho_t}\frac{DP_t}{Dt} \tag{6.87}$$

where for an ideal gas $P_t = \rho_t R T_t$. Note that there is no distinction between the entropy and stagnation entropy. They are one and the same and the identity

$$\frac{1}{T_t}\frac{Dh_t}{Dt} - \frac{1}{\rho_t T_t}\frac{DP_t}{Dt} = \frac{1}{T}\frac{Dh}{Dt} - \frac{1}{\rho T}\frac{DP}{Dt} \tag{6.88}$$

is occasionally useful.

Let's return to Figure 6.6 for a moment and ask what pressure and temperature would one measure at the stagnation point of the airfoil in a real experiment. In practice it is impossible to make the airfoil truly adiabatic and so some heat loss would be expected through conduction. In addition, as the gas approaches the airfoil, gradients in temperature arise that tend to conduct heat away from the stagnation point. In addition, gradients in temperature and velocity near the stagnation point tend to produce an increase in entropy. As a result, the measured temperature and pressure both tend to be lower than predicted. The difference may vary by anywhere from a fraction of a percent to several percent depending on the Reynolds number and Mach number of the flow.

6.8.6 TRANSFORMING THE STAGNATION PRESSURE BETWEEN FIXED AND MOVING FRAMES

The stagnation pressure and stagnation temperature in the fixed and moving frames are

$$\frac{P_t}{P} = \left(\frac{T_t}{T}\right)^{\frac{\gamma}{\gamma-1}}$$
$$\frac{P_t'}{P} = \left(\frac{T_t'}{T}\right)^{\frac{\gamma}{\gamma-1}} \tag{6.89}$$

Divide out the static pressure and temperature in (6.89).

$$\frac{P_t'}{P_t} = \left(\frac{T_t'}{T_t}\right)^{\frac{\gamma}{\gamma-1}} \tag{6.90}$$

Substitute (6.80) into (6.90). The transformation of stagnation pressure is

$$P_t' = P_t\left(1 + \frac{\dot{X}(\dot{X} - 2U) + \dot{Y}(\dot{Y} - 2V) + \dot{Z}(\dot{Z} - 2W)}{2C_pT_t}\right)^{\frac{\gamma}{\gamma - 1}}. \tag{6.91}$$

6.9 Symmetries of the incompressible Navier-Stokes equations

The Navier-Stokes equations governing incompressible flow are,

$$\left.\begin{aligned} \frac{\partial u_j}{\partial x_j} &= 0 \quad ; \quad \textit{sum over } j = 1, 2, 3 \\ \frac{\partial u_i}{\partial t} + u_j\frac{\partial u_i}{\partial x_j} + \frac{\partial p}{\partial x_i} - \nu\frac{\partial^2 u_i}{\partial x_j^2} &= 0 \quad ; \quad i = 1,2,3 \end{aligned}\right\} \tag{6.92}$$

where p is the kinematic pressure *pressure/density* and ν is the kinematic viscosity *viscosity/density*. We transform (6.92) according to the following infinitesimal group.

$$\left.\begin{aligned} \tilde{x}_i &= x_i + s\xi_i[\boldsymbol{x}, t] \\ \tilde{t} &= t + s\tau[\boldsymbol{x}, t] \\ \tilde{u}_i &= u_i + s\eta_i[\boldsymbol{x}, t] \\ \tilde{p} &= p + s\zeta[\boldsymbol{x}, t] \end{aligned}\right\} \tag{6.93}$$

Applying this infinitesimal transformation to the incompressible Navier Stokes equations leads to the following set of operators.

1) Invariance under translation in time

$$X^1 = \frac{\partial}{\partial t} \tag{6.94}$$

2) An arbitrary function of time added to the pressure, $g[t]$.

$$X^2 = g[t]\frac{\partial}{\partial p} \tag{6.95}$$

3) Rotation about the z-axis

$$X^3 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} + v\frac{\partial}{\partial u} - u\frac{\partial}{\partial v} \tag{6.96}$$

4) Rotation about the x-axis

$$X^4 = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z} + w\frac{\partial}{\partial v} - v\frac{\partial}{\partial w} \tag{6.97}$$

5) Rotation about the y-axis

$$X^5 = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z} + w\frac{\partial}{\partial u} - u\frac{\partial}{\partial w} \tag{6.98}$$

6) Nonuniform translation in the x-direction, $a[t]$ is an arbitrary, twice differentiable function of time. Simple translation in x corresponds to $a[t] = const$.

$$X^6 = a[t]\frac{\partial}{\partial x} + \left(\frac{da}{dt}\right)\frac{\partial}{\partial u} - x\left(\frac{d^2 a}{dt^2}\right)\frac{\partial}{\partial p} \tag{6.99}$$

7) Nonuniform translation in the y-direction, $b[t]$ is an arbitrary, twice differentiable function.

$$X^7 = b[t]\frac{\partial}{\partial y} + \left(\frac{db}{dt}\right)\frac{\partial}{\partial v} - y\left(\frac{d^2 b}{dt^2}\right)\frac{\partial}{\partial p} \tag{6.100}$$

8) Nonuniform translation in the z-direction, $c[t]$ is an arbitrary, twice differentiable function.

$$X^8 = c[t]\frac{\partial}{\partial z} + \left(\frac{dc}{dt}\right)\frac{\partial}{\partial w} - z\left(\frac{d^2 c}{dt^2}\right)\frac{\partial}{\partial p} \tag{6.101}$$

9) The one-parameter dilation group of the equation

$$X^9 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} - u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v} - w\frac{\partial}{\partial w} - 2p\frac{\partial}{\partial p} \tag{6.102}$$

The finite form of the dilation group corresponding to the infinitesimal operator X_9 is,

$$\left.\begin{aligned}\tilde{x}_i &= e^{s}x_i\\ \tilde{t} &= e^{2s}t\\ \tilde{u}_i &= e^{-s}u_i\\ \tilde{p} &= e^{-2s}p\end{aligned}\right\}\qquad(6.103)$$

Note that the stretching in all three coordinate directions is the same. If the kinematic viscosity in (6.92) is set to zero, the full equations reduce to the incompressible Euler equations which are invariant under X^1 to X^8 and a two-parameter dilation group in space and time. This is,

$$\left.\begin{aligned}\tilde{x}_i &= e^{s}x_i\\ \tilde{t} &= e^{\frac{s}{k}}t\\ \tilde{u}_i &= e^{s\left(1-\frac{1}{k}\right)}u_i\\ \tilde{p} &= e^{s\left(2-\frac{2}{k}\right)}p\end{aligned}\right\}\qquad(6.104)$$

with group parameters s and k. The group (6.104) is the main symmetry group governing elementary turbulent shear flows.

Occasionally exact solutions of the full Navier-Stokes equations are discovered and when they are, it is virtually always the case that the problem is invariant under one or more of the above groups. Some of the most interesting solutions are those invariant under the dilation group, (6.103) and in later chapters we will describe two famous examples. First, we consider the implications of the invariance under the non-uniform translation groups, (6.99), (6.100) and (6.101).

6.9.1 Accelerating Frames of reference in incompressible, uniform density flow

The finite form of the infinite dimensional groups corresponding to nonuniform translation in three space directions X^6, X^7 and X^8 can be written concisely as,

$$\left.\begin{aligned}\tilde{x}^i &= x^i + a^i[t] \\ \tilde{t} &= t \\ \tilde{u}^i &= u^i + \frac{da^i}{dt} \\ \tilde{p} &= p - x^j \frac{d^2 a^j}{dt^2} + g[t]\end{aligned}\right\} \tag{6.105}$$

The arbitrary functions translating the coordinates imply that the Navier-Stokes equations are invariant for all moving observers as long as the observer moves irrotationally. An observer translating and accelerating arbitrarily in three dimensions will sense the same equations of motion as an observer at rest. This invariance implies a great degree of flexibility in the choice of the observer used to view a flow. For example, one may wish to move with a particular fluid element. Or, if some convecting vortical feature happens to be of interest, then one is free to select a frame of reference attached to that feature. This has been used in Figure 6.7 to view the wake of a circular cylinder in a frame where the eddying motions in the wake become apparent. Flow fields are commonly studied this way however there is a danger in attaching too much dynamical significance to the flow patterns seen by any specific observer since the choice of the frame of reference is itself arbitrary and the flow patterns seen by different observers may differ dramatically as they do in Figure 6.7.

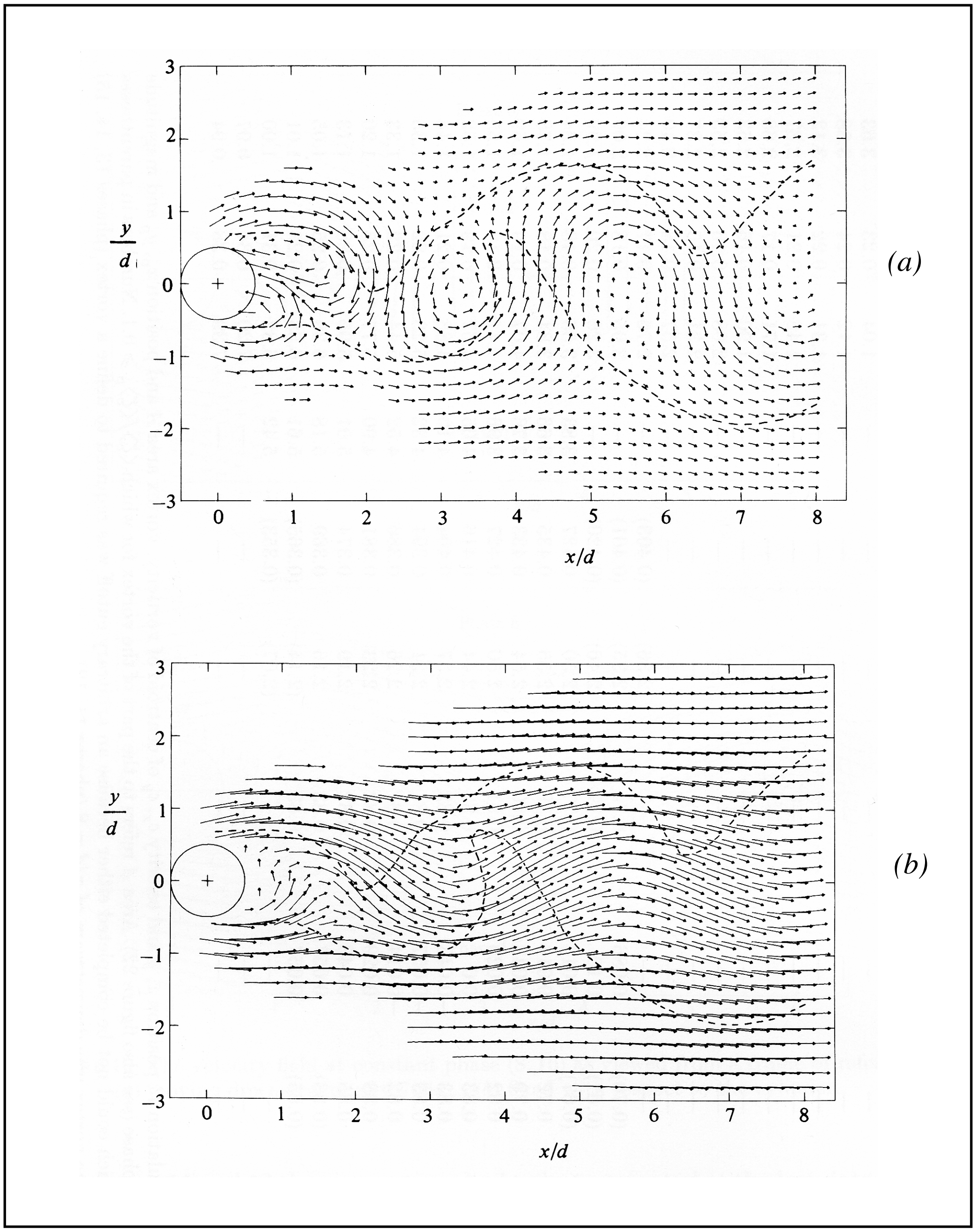


Figure 6.7 Velocity vector field in the wake of a circular cylinder as viewed by two observers; (a) frame of reference moving downstream at $0.755U_\infty$, (b) frame of reference fixed with respect to the cylinder. The dashed contour roughly corresponds to the instantaneous boundary of turbulence.

The term added to the pressure in (6.105) represents a spatially uniform effective body force induced by the acceleration of the observer. This force is purely hydrostatic in nature in that it is exactly balanced everywhere by the rate of change of the velocity field (the derivative of the translation term in the transformation of the velocity) and has no dynamical significance; it produces no differential changes in the flow field. It does suggest that when we examine the compressible equations of motion, they will probably not admit this symmetry. Similarly, the transformation (6.105) would not be admitted by incompressible problems involving variable density and/or a free boundary.

6.10 PROBLEMS

Problem 1 - Work out the time derivative of the following integral.

$$I(t) = \int_{t^2}^{Sin(t)} e^{xt} dx \tag{6.106}$$

Obtain dI/dt in two ways: (1) by directly integrating, then differentiating the result and (2) by applying Leibniz' rule (6.9) then carrying out the integration.

Problem 2 - In Chapter 2 Problem 2 we worked out a hypothetical incompressible steady flow with the velocity components

$$(U, V) = (\cos x \cos y, \sin x \sin y). \tag{6.107}$$

This 2-D flow clearly satisfies the continuity equation (conservation of mass), could it possibly satisfy conservation of momentum for an inviscid fluid? To find out work out the substantial derivatives of the velocity components and equate the results to the partial derivatives of the pressure that appear in the momentum equation. The differential of the pressure is

$$dP = \frac{\partial P}{\partial x}dx + \frac{\partial P}{\partial y}dy. \tag{6.108}$$

Show by the cross derivative test whether a pressure field exists that could enable (6.107) to satisfy momentum conservation. If such a pressure field exists work it out.

Problem 3 - Consider steady flow in one dimension where $\overline{U} = (U(x), 0, 0)$ and all velocity gradients are zero except

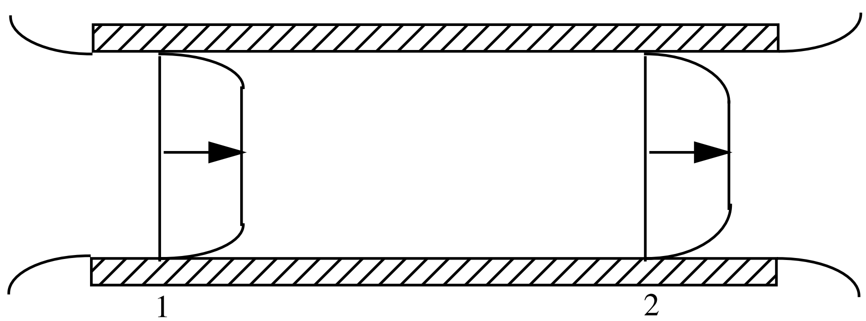
$$A_{11} = \frac{\partial U}{\partial x} \tag{6.109}$$

Work out the components of the Newtonian viscous stress tensor τ_{ij}. Note the role of the bulk viscosity.

Problem 4 - A cold gas thruster on a spacecraft uses Helium (atomic weight 4) at a chamber temperature of 300K and a chamber pressure of one atmosphere. The gas exhausts adiabatically through a large area ratio nozzle to the vacuum of space. Estimate the maximum speed of the exhaust gas.

Problem 5 - Work out equation (6.67).

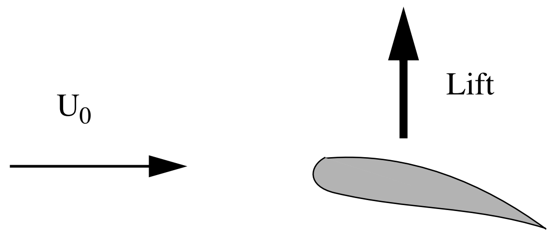
Problem 6 - Steady flow through the empty test section of a wind tunnel with parallel walls and a rectangular cross-section is shown below. Use a control volume balance to relate the integrated velocity and pressure profiles at stations 1 and 2 to an integral of the wall shear stress.



State any assumptions used.

Problem 7 - Use a control volume balance to show that the drag of a circular cylinder at low Mach number can be related to an integral of the velocity and stress profile in the wake downstream of the cylinder. Be sure to use the continuity equation to help account for the x-momentum convected out of the control volume through the upper and lower surfaces. State any assumptions used.

Problem 8 - Use a control volume balance to evaluate the lift of a three dimensional wing in an infinite steady stream. Assume the Mach number is low enough so that there are no shock waves formed.



1) Select an appropriate control volume.

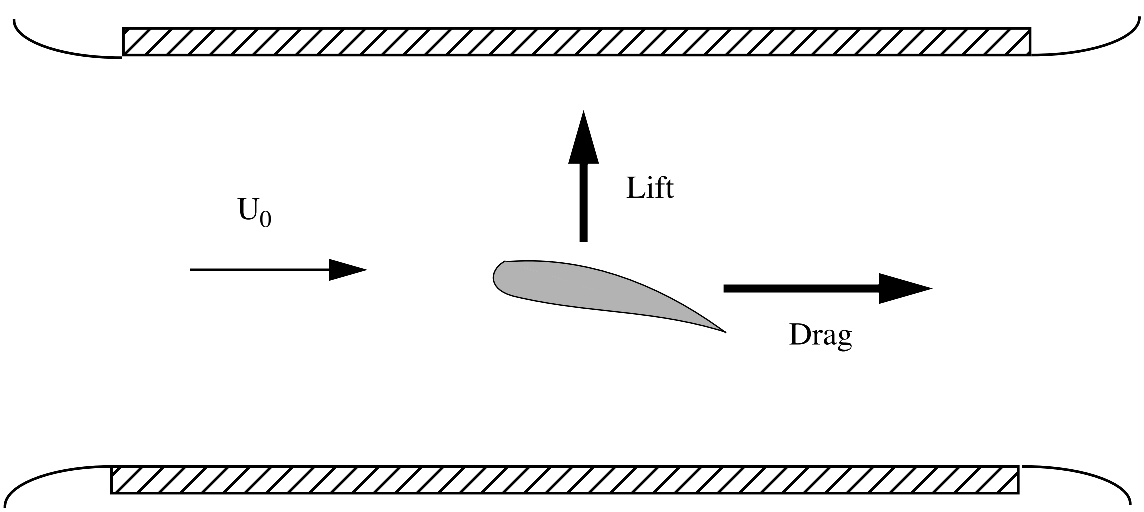
2) Write down the integral form of the mass conservation equation.

3) Write down the integral form of the momentum conservation equation.

4) Evaluate the various terms on the control volume boundary so as to express the lift of the wing in terms of an integral over the downstream wake.

5) Why did I stipulate that there are no shock waves? Briefly state any other assumptions that went in to your solution.

Problem 9 - Suppose a model 3-D wing is contained in a finite sized wind tunnel test section with horizontal and vertical walls as shown below.



What would a test engineer have to measure to determine lift and drag in the absence of sensors on the model or a mechanical balance for directly measuring forces? Consider a control volume that coincides with the wind tunnel walls.

Problem 10 - Transform each of the following equations using the following four parameter dilation group.

$$\tilde{x}_j = e^a x_j \quad ; \quad \tilde{t} = e^b t \quad ; \quad \tilde{u}_i = e^c u_i$$
$$\tilde{p} = e^d p \quad ; \quad \tilde{\rho} = \rho \tag{6.110}$$

i) The incompressible Navier-Stokes equations

$$\frac{\partial u_i}{\partial t} + \frac{\partial}{\partial x_k}\left(u_i u_k + \frac{p}{\rho}\delta_{ik}\right) - \nu \frac{\partial^2 u_i}{\partial x_k \partial x_k} = 0 \quad ; \quad \frac{\partial u_k}{\partial x_k} = 0 \tag{6.111}$$

ii) The Stokes equations for slow flow

$$\frac{\partial u_i}{\partial t} + \frac{\partial}{\partial x_i}\left(\frac{p}{\rho}\right) - \nu \frac{\partial u_i}{\partial x_k \partial x_k} = 0 \quad ; \quad \frac{\partial u_k}{\partial x_k} = 0 \tag{6.112}$$

iii) The Euler equations for inviscid flow

$$\frac{\partial u_i}{\partial t} + \frac{\partial}{\partial x_k}\left(u_i u_k + \frac{p}{\rho}\delta_{ik}\right) = 0 \quad ; \quad \frac{\partial u_k}{\partial x_k} = 0 \tag{6.113}$$

How do the group parameters a, b, c, d have to be related to one another in order for the given equations to be invariant?