CHAPTER 13

COMPRESSIBLE THIN AIRFOIL THEORY

13.1 COMPRESSIBLE POTENTIAL FLOW

13.1.1 THE FULL POTENTIAL EQUATION

In compressible flow, both the lift and drag of a thin airfoil can be determined to a reasonable level of accuracy from an inviscid, irrotational model of the flow. Recall the equations developed in Chapter 6 governing steady, irrotational, homentropic ($\nabla s = 0$) flow in the absence of body forces.

$$\nabla(\rho\overline{U}) = 0$$
$$\nabla\left(\frac{\overline{U}\bullet\overline{U}}{2}\right) + \frac{\nabla P}{\rho} = 0 \qquad (13.1)$$
$$\frac{P}{P_0} = \left(\frac{\rho}{\rho_0}\right)^{\gamma}$$

The gradient of the isentropic relation is

$$\nabla P = a^2 \nabla\rho . \qquad (13.2)$$

Recall from the development in Chapter 6 that

$$\nabla\left(\frac{P}{\rho}\right) = \left(\frac{\gamma-1}{\gamma}\right)\frac{\nabla P}{\rho} \qquad (13.3)$$

Using (13.3) the momentum equation becomes

$$\nabla\left(\left(\frac{\gamma}{\gamma-1}\right)\frac{P}{\rho} + \frac{\overline{U}\bullet\overline{U}}{2}\right) = 0 \qquad (13.4)$$

Substitute (13.2) into the continuity equation and use (13.3). The continuity equation becomes

$$\overline{U}\bullet\nabla a^2 + (\gamma-1)a^2\nabla\bullet\overline{U} = 0 . \qquad (13.5)$$

Equate the Bernoulli integral to free stream conditions.

$$\frac{a^2}{\gamma - 1} + \frac{\bar{U} \bullet \bar{U}}{2} = \frac{a_\infty^2}{\gamma - 1} + \frac{U_\infty^2}{2} = \frac{a_\infty^2}{\gamma - 1}\left(1 + \frac{\gamma - 1}{2} M_\infty^2\right) = C_p T_t \tag{13.6}$$

Note that the momentum equation is essentially equivalent to the statement that the stagnation temperature T_t is constant throughout the flow. Using (13.6) we can write

$$\left(\frac{a^2}{\gamma - 1}\right) = h_t - \frac{\bar{U} \bullet \bar{U}}{2} \tag{13.7}$$

The continuity equation finally becomes

$$(\gamma - 1)\left(h_t - \frac{\bar{U} \bullet \bar{U}}{2}\right) \nabla \bullet \bar{U} - \bar{U} \bullet \nabla\left(\frac{\bar{U} \bullet \bar{U}}{2}\right) = 0 \tag{13.8}$$

The equations governing compressible, steady, inviscid, irrotational motion reduce to a single equation for the velocity vector $\bar{U}$. The irrotationality condition $\nabla \times \bar{U} = 0$ permits the introduction of a velocity potential.

$$\bar{U} = \nabla\Phi \tag{13.9}$$

and (13.8) becomes

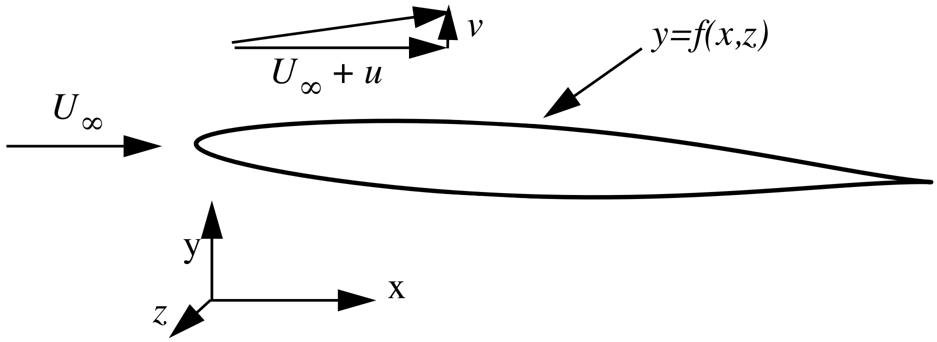
$$(\gamma - 1)\left(h_t - \frac{\nabla\Phi \bullet \nabla\Phi}{2}\right) \nabla^2 \Phi - \nabla\Phi \bullet \nabla\left(\frac{\nabla\Phi \bullet \nabla\Phi}{2}\right) = 0 . \tag{13.10}$$

For complex body shapes numerical methods are normally used to solve for Φ. However the equation is of relatively limited applicability. If the flow is over a thick airfoil or a bluff body for instance then the equation only applies to the subsonic Mach number regime at Mach numbers below the range where shocks begin to appear on the body. At high subsonic and supersonic Mach numbers where there are shocks then the homentropic assumption (13.2) breaks down. Equation (13.8) also applies to internal flows without shocks such as fully expanded nozzle flow.

13.1.2 The nonlinear small disturbance approximation

In the case of a thin airfoil that only slightly disturbs the flow, equation (13.8) can be simplified using small disturbance theory.

Consider the flow past a thin 3-D airfoil shown below.



The velocity field consists of a freestream flow plus a small disturbance

$$
\begin{aligned}
U &= U_\infty + u \\
V &= v \\
W &= w
\end{aligned}
\tag{13.11}
$$

where

$$
u/U_\infty \ll 1 \quad , \qquad v/U_\infty \ll 1 \quad , \qquad w/U_\infty \ll 1 \,. \tag{13.12}
$$

Similarly the state variables deviate only slightly from freestream values.

$$
\begin{aligned}
P &= P_\infty + P' \\
T &= T_\infty + T' \\
\rho &= \rho_\infty + \rho'
\end{aligned}
\tag{13.13}
$$

and

$$
a = a_\infty + a' \,. \tag{13.14}
$$

This decomposition of variables is substituted into equation (13.8). Various terms are

$$
\begin{aligned}
\frac{\bar{U} \bullet \bar{U}}{2} &= \frac{U_\infty^2}{2} + uU_\infty + \frac{u^2}{2} + \frac{v^2}{2} + \frac{w^2}{2} \\
\nabla \bullet \bar{U} &= u_x + v_y + w_z
\end{aligned}
\tag{13.15}
$$

and

$$\nabla\left(\frac{\overline{U} \bullet \overline{U}}{2}\right) = (u_x U_\infty + uu_x + vv_x + ww_x,\\ u_y U_\infty + uu_y + vv_y + ww_y,\\ u_z U_\infty + uu_z + vv_z + ww_z) \tag{13.16}$$

as well as

$$(\gamma - 1)\left(h_t - \frac{\overline{U} \bullet \overline{U}}{2}\right)\nabla \bullet \overline{U} =\\ (\gamma - 1)\left(h_t - \left(\frac{U_\infty^2}{2} + uU_\infty + \frac{u^2}{2} + \frac{v^2}{2} + \frac{w^2}{2}\right)\right)u_x +\\ (\gamma - 1)\left(h_t - \left(\frac{U_\infty^2}{2} + uU_\infty + \frac{u^2}{2} + \frac{v^2}{2} + \frac{w^2}{2}\right)\right)v_y +\\ (\gamma - 1)\left(h_t - \left(\frac{U_\infty^2}{2} + uU_\infty + \frac{u^2}{2} + \frac{v^2}{2} + \frac{w^2}{2}\right)\right)w_z \tag{13.17}$$

and, finally

$$\overline{U} \bullet \nabla\left(\frac{\overline{U} \bullet \overline{U}}{2}\right) = u_x U_\infty^2 + uu_x U_\infty + vv_x U_\infty + ww_x U_\infty +\\ uu_x U_\infty + u^2 u_x + uvv_x + uww_x +\\ vu_y U_\infty + vuu_y + v^2 v_y + vww_y +\\ wu_z U_\infty + wuu_z + wvv_z + w^2 w_z \quad . \tag{13.18}$$

Neglect terms in (13.17) and (13.18) that are of third order in the disturbance velocities. Now

$$(\gamma-1)\left(h_t-\frac{\overline{U}\bullet\overline{U}}{2}\right)\nabla\bullet\overline{U}-\overline{U}\bullet\nabla\left(\frac{\overline{U}\bullet\overline{U}}{2}\right)\cong$$
$$(\gamma-1)\left(h_t-\left(\frac{U_\infty^2}{2}+uU_\infty\right)\right)u_x+$$
$$(\gamma-1)\left(h_t-\left(\frac{U_\infty^2}{2}+uU_\infty\right)\right)v_y+ \tag{13.19}$$
$$(\gamma-1)\left(h_t-\left(\frac{U_\infty^2}{2}+uU_\infty\right)\right)w_z-(u_xU_\infty^2+uu_xU_\infty+vv_xU_\infty+ww_xU_\infty)-$$
$$uu_xU_\infty-vu_yU_\infty-wu_zU_\infty$$

or

$$(\gamma-1)\left(h_t-\frac{\overline{U}\bullet\overline{U}}{2}\right)\nabla\bullet\overline{U}-\overline{U}\bullet\nabla\left(\frac{\overline{U}\bullet\overline{U}}{2}\right)\cong$$
$$(\gamma-1)(h_\infty-uU_\infty)(u_x+v_y+w_z)-(u_xU_\infty^2+uu_xU_\infty+vv_xU_\infty+ww_xU_\infty) \tag{13.20}$$
$$-uu_xU_\infty-vu_yU_\infty-wu_zU_\infty$$

Recall that $(\gamma-1)h_\infty = a_\infty^2$. Equation (13.20) can be rearranged to read

$$(\gamma-1)\left(h_t-\frac{\overline{U}\bullet\overline{U}}{2}\right)\nabla\bullet\overline{U}-\overline{U}\bullet\nabla\left(\frac{\overline{U}\bullet\overline{U}}{2}\right)\cong$$
$$a_\infty^2(u_x+v_y+w_z)-U_\infty^2u_x-(\gamma+1)uu_xU_\infty-\ (vv_xU_\infty+ww_xU_\infty) \tag{13.21}$$
$$(\gamma-1)(uv_yU_\infty+uw_zU_\infty)-vu_yU_\infty-wu_zU_\infty$$

Divide through by a_∞^2.

$$(\gamma - 1)\left(h_t - \frac{\overline{U} \bullet \overline{U}}{2}\right) \nabla \bullet \overline{U} - \overline{U} \bullet \nabla\left(\frac{\overline{U} \bullet \overline{U}}{2}\right) \cong$$

$$(1 - M_\infty^2)u_x + v_y + w_z - \frac{(\gamma + 1)M_\infty}{a_\infty} uu_x - \tag{13.22}$$

$$\frac{M_\infty}{a_\infty}((\gamma - 1)(uv_y + uw_z) + vu_y + wu_z + vv_x + ww_x)$$

This equation contains both linear and quadratic terms in the velocity disturbances and one might expect to be able to neglect the quadratic terms. But note that the first term becomes very small near $M_\infty = 1$. Thus in order to maintain the small disturbance approximation at transonic Mach numbers the uu_x term must be retained. The remaining quadratic terms are small at all Mach numbers and can be dropped. Finally the small disturbance equation is

$$(1 - M_\infty^2)u_x + v_y + w_z - \frac{(\gamma + 1)M_\infty}{a_\infty} uu_x = 0. \tag{13.23}$$

The velocity potential is written in terms of a freestream potential and a disturbance potential

$$\Phi = U_\infty x + \phi(x, y, z). \tag{13.24}$$

The small disturbance equation in terms of the disturbance potential becomes

$$(1 - M_\infty^2)\phi_{xx} + \phi_{yy} + \phi_{zz} = (\gamma + 1)\frac{M_\infty}{a_\infty}\phi_x\phi_{xx} \tag{13.25}$$

Equation (13.25) is valid over the whole range of subsonic, transonic and supersonic Mach numbers.

13.1.3 Linearized potential flow

If we restrict our attention to subsonic and supersonic flow, staying away from Mach numbers close to one, the nonlinear term on the right side of (13.25) can be dropped and the small disturbance potential equation reduces to the linear wave equation.

$$.\beta^2\phi_{xx} - (\phi_{yy} + \phi_{zz}) = 0 \tag{13.26}$$

where $\beta = \sqrt{M_\infty^2 - 1}$. In two dimensions

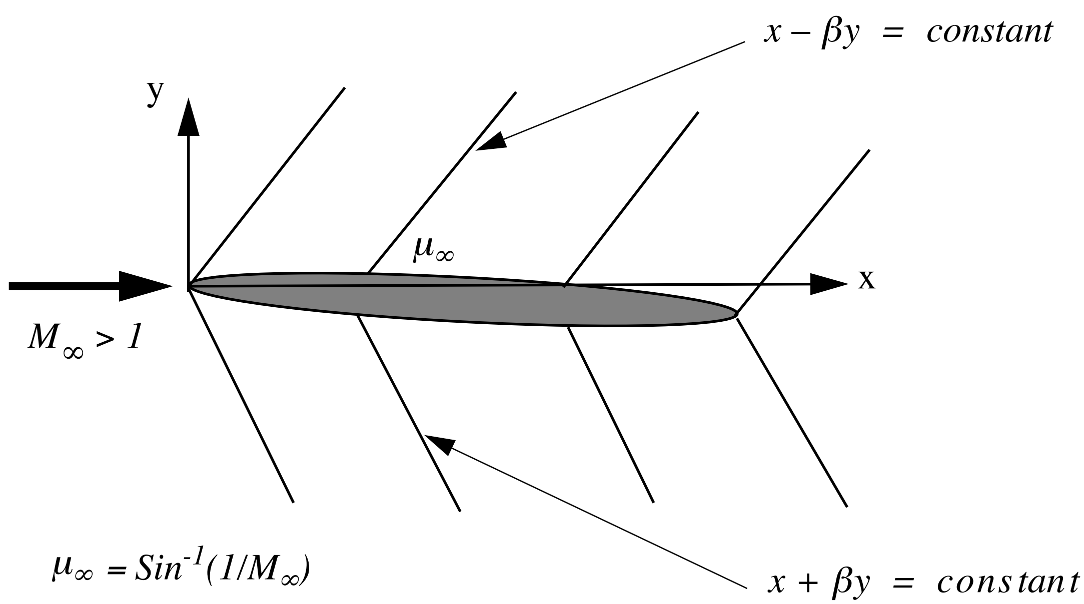
$$\beta^2\phi_{xx} - \phi_{yy} = 0. \tag{13.27}$$

The general solution of (13.27) can be expressed as a sum of two arbitrary functions

$$\phi(x, y) = F(x - \beta y) + G(x + \beta y). \tag{13.28}$$

Note that if $M_\infty^2 < 1$ the 2-D linearized potential equation (13.27) is an elliptic equation that can be rescaled to form Laplace's equation and (13.28) expresses the solution in terms of conjugate complex variables. In this case the subsonic flow can be analyzed using the methods of complex analysis. Presently we will restrict our attention to the supersonic case. The subsonic case is treated later in the chapter.

If $M_\infty^2 > 1$ then (13.27) is the 2-D wave equation and has solutions of hyperbolic type. Supersonic flow is analyzed using the fact that the properties of the flow are constant along the characteristic lines $x \pm \beta y = constant$. The figure below illustrates supersonic flow past a thin airfoil with several characteristics shown. Notice that in the linear approximation the characteristics are all parallel to one another and lie at the Mach angle μ_∞ of the free stream. Information about the flow is carried in the value of the potential assigned to a given characteristic and in the spacing between characteristics for a given flow change. Right-leaning characteristics carry the information about the flow on the upper surface of the wing and left-leaning characteristics carry information about the flow on the lower surface.



All properties of the flow, velocity, pressure, temperature, etc. are constant along the characteristics. Since disturbances only propagate along downstream running characteristics we can write the velocity potential for the upper and lower surfaces as

$$\begin{aligned}\phi(x, y) &= F(x - \beta y) \qquad y>0 \\ \phi(x, y) &= G(x + \beta y) \qquad y<0\end{aligned} \tag{13.29}$$

Let $y = f(x)$ define the coordinates of the upper surface of the wing and $y = g(x)$ define the lower surface. The full nonlinear boundary condition on the upper surface is

$$\left.\frac{v}{U}\right|_{y = f} = \frac{df}{dx}. \tag{13.30}$$

In the spirit of the thin airfoil approximation this boundary condition can be approximated by the linearized form

$$\left.\frac{v}{U_\infty}\right|_{y = 0} = \frac{df}{dx} \tag{13.31}$$

which we can write as

$$\left.\frac{\partial\phi(x, y)}{\partial y}\right|_{y = 0} = U_\infty\left(\frac{df}{dx}\right) \tag{13.32}$$

or

$$F'(x) = -\frac{U_\infty}{\beta}\left(\frac{df}{dx}\right). \tag{13.33}$$

On the lower surface the boundary condition is

$$G'(x) = \frac{U_\infty}{\beta}\left(\frac{dg}{dx}\right). \tag{13.34}$$

In the thin airfoil approximation the airfoil itself is, in effect, collapsed to a line along the x-axis, the velocity potential is extended to the line $y = 0$ and the surface boundary condition is applied at $y = 0$ instead of at the physical airfoil surface. The entire effect of the airfoil on the flow is accounted for by the vertical velocity perturbation generated by the local slope of the wing. The linearized boundary condition is valid on 2-D thin wings and on 3-D wings of that are of "thin planar form".

Recall that (13.26) is only valid for subsonic and supersonic flow and not for transonic flow where $(1 - M_\infty^2) \ll 1$.

13.1.4 The pressure coefficient

Let's work out the linearized pressure coefficient. The pressure coefficient is

$$C_P = \frac{P - P_\infty}{\frac{1}{2}\rho_\infty U_\infty^2} = \frac{2}{\gamma M_\infty^2}\left(\frac{P}{P_\infty} - 1\right). \tag{13.35}$$

The stagnation enthalpy is constant throughout the flow thus

$$\frac{T}{T_\infty} = 1 + \frac{1}{2C_p T_\infty}\left(U_\infty^2 - (U^2 + v^2 + w^2)\right). \tag{13.36}$$

Similarly the entropy is constant and thus the pressure and temperature are related by

$$\frac{P}{P_\infty} = \left(1 + \frac{1}{2C_P T_\infty}(U_\infty^2 - (U^2 + v^2 + w^2))\right)^{\frac{\gamma}{\gamma - 1}} \qquad (13.37)$$

and the pressure coefficient is

$$C_P = \frac{2}{\gamma M_\infty^2}\left\{\left(1 + \frac{1}{2C_p T_\infty}(U_\infty^2 - (U^2 + v^2 + w^2))\right)^{\frac{\gamma}{\gamma - 1}} - 1\right\}. \qquad (13.38)$$

The velocity term in (13.37) is small

$$U_\infty^2 - (U^2 + v^2 + w^2) = -(2uU_\infty + u^2 + v^2 + w^2). \qquad (13.39)$$

Now use the binomial expansion $(1 - \varepsilon)^n \cong 1 - n\varepsilon + n(n - 1)\varepsilon^2/2$ to expand the term in parentheses in (13.39). Note that the expansion has to be carried out to second order. The pressure coefficient is approximately

$$C_P \cong -\left(\frac{2u}{U_\infty} + (1 - M_\infty^2)\frac{u^2}{U_\infty^2} + \frac{v^2 + w^2}{U_\infty^2}\right). \qquad (13.40)$$

Equation (13.40) is a valid approximation for small perturbations in subsonic or supersonic flow.

For 2-D flows over planar bodies it is sufficient to retain only the first term in (13.40) and we use the expression

$$C_P \cong -2\frac{u}{U_\infty}. \qquad (13.41)$$

For 3-D flows over slender approximately axisymmetric bodies we must retain the last term and so

$$C_P \cong -\left(\frac{2u}{U_\infty} + \frac{v^2 + w^2}{U_\infty^2}\right). \qquad (13.42)$$

As was discussed above, if the airfoil is a 2-D shape defined by the function $y = f(x)$ the boundary condition at the surface is

$$\frac{df}{dx} = \frac{v}{U_\infty + u} = \tan\theta \tag{13.43}$$

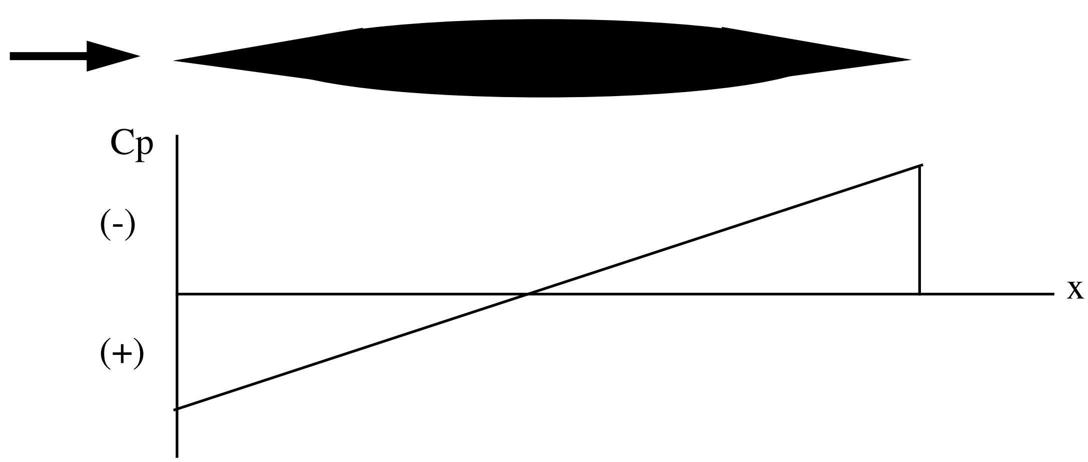
where θ is the angle between the airfoil surface and the horizontal. For a thin airfoil this is accurately approximated by

$$\frac{df}{dx} \cong \frac{\phi_y}{U_\infty} \cong \theta . \tag{13.44}$$

For a thin airfoil *in supersonic flow* the wall pressure coefficient is

$$C_{Pwall} = \frac{2}{(M_\infty^2 - 1)^{1/2}}\left(\frac{df}{dx}\right) . \tag{13.45}$$

where $dU^2/U^2 = -\left(2/(M_\infty{}^2 - 1)^{1/2}\right)d\theta$, which is valid for $M_\infty \geq 1$, has been used. In the thin airfoil approximation in supersonic flow the local pressure coefficient is determined by the local slope of the wing. The figure below shows the wall pressure coefficient on a thin, symmetric biconvex wing.



This airfoil will have shock waves at the leading and trailing edges and at first sight this would seem to violate the isentropic assumption. But for small disturbances the shocks are weak and the entropy changes are negligible.

13.1.5 Drag coefficient of a thin symmetric airfoil

A thin, 2-D, symmetric airfoil is situated in a supersonic stream at Mach number M_∞ and zero angle of attack. The y-coordinate of the upper surface of the airfoil is given by the function

$$y(x) = A\,Sin\left(\frac{\pi x}{C}\right) \tag{13.46}$$

where C is the airfoil chord. The airfoil thickness to chord ratio is small, $2A/C \ll 1$. Determine the drag coefficient of the airfoil.

Solution

The drag integral is

$$D = 2\int_0^C (P - P_\infty) Sin(\alpha) dx \tag{13.47}$$

where the factor of 2 accounts for the drag of both the upper and lower surfaces and α is the local angle formed by the upper surface tangent to the airfoil and the x-axis. Since the airfoil is thin the angle is small and we can write the drag coefficient as

$$C_D = \frac{D}{\frac{1}{2}\rho_\infty U_\infty^2 C} = 2\int_0^1 \left(\frac{P - P_\infty}{\frac{1}{2}\rho_\infty U_\infty^2}\right)(\alpha) d\left(\frac{x}{C}\right) \tag{13.48}$$

The local tangent is determined by the local slope of the airfoil therefore $Tan(\alpha) = dy/dx$ and for small angles $\alpha = dy/dx$. Now the drag coefficient is

$$C_D = \frac{D}{\frac{1}{2}\rho_\infty U_\infty^2 C} = 2\int_0^1 \left(\frac{P - P_\infty}{\frac{1}{2}\rho_\infty U_\infty^2}\right)\left(\frac{dy}{dx}\right) d\left(\frac{x}{C}\right) \tag{13.49}$$

The pressure coefficient on the airfoil is given by thin airfoil theory (13.45) as

$$C_P = \frac{P - P_\infty}{\frac{1}{2}\rho_\infty U_\infty^2} = \frac{2}{\sqrt{M_\infty^2 - 1}}\left(\frac{dy}{dx}\right) \tag{13.50}$$

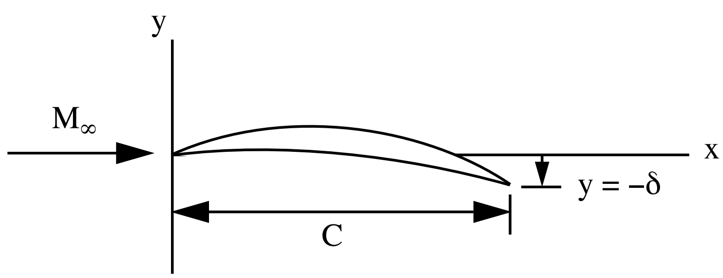
and so the drag coefficient becomes

$$C_D = \frac{4}{\sqrt{M_\infty^2 - 1}}\int_0^1 \left(\frac{dy}{dx}\right)^2 d\left(\frac{x}{C}\right) = \frac{4A^2\pi}{C^2\sqrt{M_\infty^2 - 1}}\int_0^\pi Cos^2\left(\frac{\pi x}{C}\right) d\left(\frac{\pi x}{C}\right) = \frac{2A^2\pi^2}{C^2\sqrt{M_\infty^2 - 1}} \tag{13.51}$$

The drag coefficient is proportional to the square of the wing thickness-to-chord ratio.

13.1.6 Thin airfoil with lift and camber at a small angle of attack

A thin, cambered, 2-D, airfoil is situated in a supersonic stream at Mach number M_∞ and a small angle of attack as shown below.



The y-coordinate of the upper surface of the airfoil is given by the function

$$f\left(\frac{x}{C}\right) = A\tau\left(\frac{x}{C}\right) + B\sigma\left(\frac{x}{C}\right) - \frac{\delta x}{C} \tag{13.52}$$

and the y-coordinate of the lower surface is

$$g\left(\frac{x}{C}\right) = -A\tau\left(\frac{x}{C}\right) + B\sigma\left(\frac{x}{C}\right) - \frac{\delta x}{C} \tag{13.53}$$

where $2A/C \ll 1$ and $\delta/C \ll 1$. The airfoil surface is defined by a dimensionless thickness function

$$\tau\left(\frac{x}{C}\right) \quad ; \qquad \tau(0) = \tau(1) = 0, \tag{13.54}$$

a dimensionless camber function

$$\sigma\left(\frac{x}{C}\right) \quad ; \quad \sigma(0) = \sigma(1) = 0 \tag{13.55}$$

and the angle of attack

$$Tan(\alpha) = -\frac{\delta}{C} \tag{13.56}$$

Determine the lift and drag coefficients of the airfoil. Let

$$\xi = \frac{x}{C} \tag{13.57}$$

Solution

The lift integral is

$$L = \int_0^C (P_{lower} - P_\infty) Cos(\alpha_{lower}) dx - \int_0^C (P_{upper} - P_\infty) Cos(\alpha_{upper}) dx \tag{13.58}$$

where α is the local angle formed by the tangent to the surface of the airfoil and the x-axis. Since the angle is everywhere small we can use $Cos(\alpha) \cong 1$. The lift coefficient is

$$C_L = \frac{D}{\frac{1}{2}\rho_\infty U_\infty^2 C} = \int_0^1 C_{P_{lower}} d\xi - \int_0^1 C_{P_{upper}} d\xi \tag{13.59}$$

The pressure coefficient on the upper surface is given by thin airfoil theory as

$$C_{P_{upper}} = \frac{2}{C\sqrt{M_\infty^2 - 1}}\left(\frac{df}{d\xi}\right) = \frac{2}{C\sqrt{M_\infty^2 - 1}}\left(A\frac{d\tau}{d\xi} + B\frac{d\sigma}{d\xi} - \delta\right) \tag{13.60}$$

and on the lower surface the pressure coefficient is

$$C_{P_{lower}} = -\frac{2}{C\sqrt{M_\infty^2 - 1}}\left(\frac{dg}{d\xi}\right) = -\frac{2}{C\sqrt{M_\infty^2 - 1}}\left(-A\frac{d\tau}{d\xi} + B\frac{d\sigma}{d\xi} - \delta\right). \tag{13.61}$$

Substitute into the lift integral

$$C_L = \frac{-2}{C\sqrt{M_\infty^2 - 1}}\left(\int_0^1 \frac{df}{d\xi}d\xi + \int_0^1 \frac{dg}{d\xi}d\xi\right) = \frac{-2}{C\sqrt{M_\infty^2 - 1}}\left(\int_0^{-\delta} df + \int_0^{-\delta} dg\right) \tag{13.62}$$

In the thin airfoil approximation the lift is independent of the airfoil thickness.

$$C_L = \frac{4}{\sqrt{M_\infty^2 - 1}}\left(\frac{\delta}{C}\right) \tag{13.63}$$

The drag integral is

$$D = \int_0^C (P_{upper} - P_\infty)Sin(\alpha_{upper})dx + \int_0^C (P_{lower} - P_\infty)Sin(-\alpha_{lower})dx \tag{13.64}$$

Since the airfoil is thin the angle is small and we can write the drag coefficient as

$$C_D = \frac{D}{\frac{1}{2}\rho_\infty U_\infty^2 C} = \int_0^1 \left(\frac{P_{upper} - P_\infty}{\frac{1}{2}\rho_\infty U_\infty^2}\right)(\alpha_{upper})d\xi + \int_0^1 \left(\frac{P_{lower} - P_\infty}{\frac{1}{2}\rho_\infty U_\infty^2}\right)(-\alpha_{lower})d\xi \tag{13.65}$$

The local tangent is determined by the local slope of the airfoil therefore

$Tan(\alpha) = dy/dx$ and for small angles $\alpha = dy/dx$.

Now the drag coefficient is

$$C_D = \frac{D}{\frac{1}{2}\rho_\infty U_\infty^2 C} = \frac{1}{C}\int_0^1 C_{P_{upper}}\left(\frac{dy_{upper}}{d\xi}\right)d\xi + \frac{1}{C}\int_0^1 C_{P_{lower}}\left(-\frac{dy_{lower}}{d\xi}\right)d\xi \tag{13.66}$$

The drag coefficient becomes

$$C_D = \frac{2}{C^2\sqrt{M_\infty^2 - 1}}\left(\int_0^1 \left(A\frac{d\tau}{d\xi} + B\frac{d\sigma}{d\xi} - \delta\right)^2 d\xi + \int_0^1 \left(-A\frac{d\tau}{d\xi} + B\frac{d\sigma}{d\xi} - \delta\right)^2 d\xi\right) \tag{13.67}$$

Note that most of the cross terms cancel. The drag coefficient breaks into several terms.

$$C_D = \frac{4}{\sqrt{M_\infty^2 - 1}}\left(\left(\frac{A}{C}\right)^2\int_0^1 \left(\frac{d\tau}{d\xi}\right)^2 d\xi + \left(\frac{B}{C}\right)^2\int_0^1 \left(\frac{d\sigma}{d\xi}\right)^2 d\xi - \left(\frac{B}{C}\right)\frac{\delta}{C}\int_0^1 \left(\frac{d\sigma}{d\xi}\right)d\xi + \left(\frac{\delta}{C}\right)^2\int_0^1 d\xi\right) \tag{13.68}$$

The third term in (13.68) is

$$\left(\frac{B}{C}\right)\frac{\delta}{C}\int_0^1\left(\frac{d\sigma}{d\xi}\right)d\xi = \left(\frac{B}{C}\right)\frac{\delta}{C}\int_0^1 d\sigma = \left(\frac{B}{C}\right)\frac{\delta}{C}[\sigma(1)-\sigma(0)] = 0 \qquad (13.69)$$

Finally

$$C_D = \frac{4}{\sqrt{M_\infty^2-1}}\left(\left(\frac{A}{C}\right)^2\int_0^1\left(\frac{d\tau}{d\xi}\right)^2 d\xi + \left(\frac{B}{C}\right)^2\int_0^1\left(\frac{d\sigma}{d\xi}\right)^2 d\xi + \left(\frac{\delta}{C}\right)^2\right). \qquad (13.70)$$

In the thin airfoil approximation, the drag is a sum of the drag due to thickness, drag due to camber and drag due to lift.

13.2 SIMILARITY RULES FOR HIGH SPEED FLIGHT

The figure below shows the flow past a thin symmetric airfoil at zero angle of attack. The fluid is assumed to be inviscid and the flow Mach number $U_{\infty 1}/a_\infty$, where $a_\infty^2 = \gamma P_\infty/\rho_\infty$ is the speed of sound, is assumed to be much less than one. The airfoil chord is c and the maximum thickness is t_1. The subscript one is applied in anticipation of the fact that we will shortly scale the airfoil to a new shape with subscript two and the same chord.

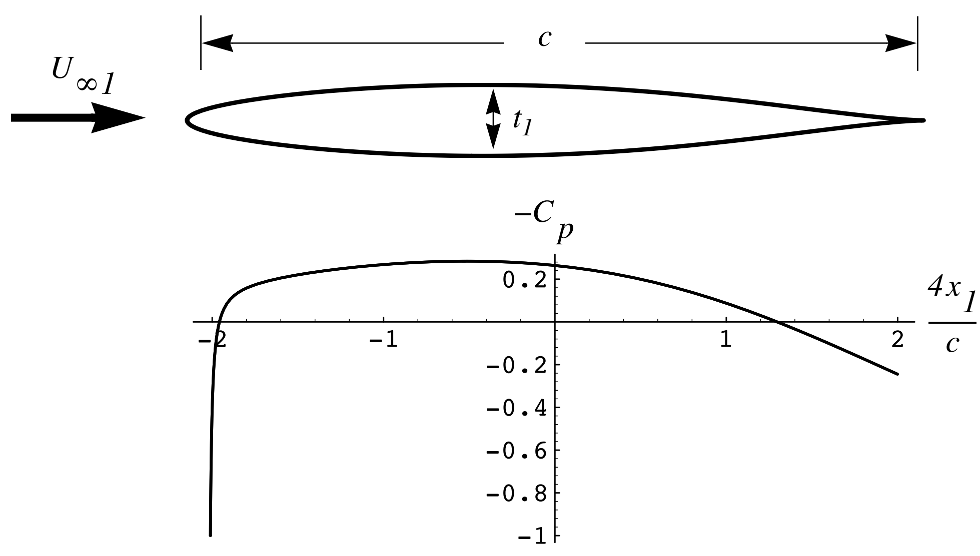


Figure 13.1 Pressure variation over a thin symmetric airfoil in low speed flow.

The surface pressure distribution is shown below the wing, expressed in terms of the pressure coefficient.

$$C_{P1} = \frac{P_s - P_\infty}{\frac{1}{2}\rho_\infty U_{\infty 1}^2}. \tag{13.71}$$

The pressure and flow speed throughout the flow satisfy the Bernoulli relation, in particular, near the airfoil surface,

$$P_\infty + \frac{1}{2}\rho_\infty U_{\infty 1}^2 = P_{s1} + \frac{1}{2}\rho_\infty U_{s1}^2. \tag{13.72}$$

The pressure is high at the leading edge where the flow stagnates, then as the flow accelerates about the body, the pressure falls rapidly at first, then more slowly reaching a minimum at the point of maximum thickness. From there the surface velocity decreases and the pressure increases continuously to the trailing edge. In the absence of viscosity, the flow is irrotational.

$$\nabla \times u_1 = 0 \tag{13.73}$$

this permits the velocity to be described by a potential function.

$$u_1 = \nabla\phi_1 \tag{13.74}$$

when this is combined with the condition of incompressibility, $\nabla \bullet u_1 = 0$, the result is Laplace's equation.

$$\frac{\partial^2 \phi_1}{\partial^2 x_1} + \frac{\partial^2 \phi_1}{\partial^2 y_1} = 0. \tag{13.75}$$

Let the shape of the airfoil surface in (x_1, y_1) be given by

$$\frac{y_1}{c} = \tau_1 g[x_1/c] \tag{13.76}$$

where $\tau_1 = t_1/c$ is the thickness to chord ratio of the airfoil. The boundary conditions that the velocity potential must satisfy are,

$$\left(\frac{\partial\phi_1}{\partial y_1}\right)_{y_1 = c\tau_1 g\left(\frac{x_1}{c}\right)} = U_{\infty 1}\left(\frac{dy_1}{dx_1}\right)_{body} = U_\infty\tau_1\frac{dg[x_1/c]}{d(x_1/c)}$$
$$\phi_{1_{x_1 \to \infty}} = U_{\infty 1} \qquad (13.77)$$

Any number of methods of solving for the velocity potential are available including the use of complex variables. In the following we are going to restrict the airfoil to be thin, $\tau_1 \ll 1$. In this context we will take the velocity potential to be a perturbation potential so that.

$$u_1 = U_{\infty 1} + u_1' \ ; \quad v_1 = v_1' \qquad (13.78)$$

or

$$u_1 = U_{\infty 1} + \frac{\partial\phi_1}{\partial x_1} \ ; \quad v_1 = \frac{\partial\phi_1}{\partial y_1}. \qquad (13.79)$$

The boundary conditions on the perturbation potential in the thin airfoil approximation are,

$$\left.\begin{aligned}\left(\frac{\partial\phi_1}{\partial y_1}\right)_{y_1 = 0} &= U_{\infty 1}\left(\frac{dy_1}{dx_1}\right)_{body} = U_\infty\tau_1\frac{dg[x_1/c]}{d(x_1/c)} \\ \phi_{1_{x_1 \to \infty}} &= 0\end{aligned}\right\}. \qquad (13.80)$$

The surface pressure coefficient in the thin airfoil approximation is,

$$C_{P1} = -\frac{2}{U_{\infty 1}}\left(\frac{\partial\phi_1}{\partial x_1}\right)_{y_1 = 0}. \qquad (13.81)$$

Note that the boundary condition on the vertical velocity is now applied on the line $y_1 = 0$. In effect the airfoil has been replaced with a line of volume sources whose strength is proportional to the local slope of the actual airfoil. This sort of approximation is really unnecessary in the low Mach number limit but it is essential when the Mach number is increased and compressibility effects come in to play. Equally, it is essential in this example where we will map a compressible flow to the incompressible case.

13.2.1 Subsonic flow $M_{\infty} < 1$

Now imagine a second flow at a free stream velocity, U_{∞_2} in a new space (x_2, y_2) over a new airfoil of the same shape (defined by the function $g[x/c]$) but with a new thickness ratio $\tau_2 = t_2/c \ll 1$. Part of what we need to do is to determine how τ_1 and τ_2 are related to one another. The boundary conditions that the new perturbation velocity potential must satisfy are,

$$\begin{aligned} \left(\frac{\partial\phi_2}{\partial y_2}\right)_{y_2 = 0} &= U_{\infty 2}\left(\frac{dy_2}{dx_2}\right)_{body} = U_{\infty 2}\tau_2\frac{dg[x_1/c]}{d(x_1/c)} \\ \phi_{2_{x_2 \to \infty}} &= 0 \end{aligned} \tag{13.82}$$

In this second flow the Mach number has been increased to the point where compressibility effects begin to occur: the density begins to vary significantly and the pressure distribution begins to deviate from the incompressible case. As long as the Mach number is not too large and shock waves do not form, the flow will be nearly isentropic. In this instance the 2-D steady compressible flow equations are,

$$\begin{aligned} u_2\frac{\partial u_2}{\partial x_2} + v_2\frac{\partial u_2}{\partial y_2} + \frac{1}{\rho}\frac{\partial P}{\partial x_2} &= 0 \\ u_2\frac{\partial v_2}{\partial x_2} + v_2\frac{\partial v_2}{\partial y_2} + \frac{1}{\rho}\frac{\partial P}{\partial y_2} &= 0 \\ u_2\frac{\partial \rho}{\partial x_2} + v_2\frac{\partial \rho}{\partial y_2} + \rho\frac{\partial u_2}{\partial x_2} + \rho\frac{\partial v_2}{\partial y_2} &= 0 \\ \frac{P}{P_\infty} &= \left(\frac{\rho}{\rho_\infty}\right)^{\gamma} \end{aligned} \tag{13.83}$$

Let

$$u_2 = U_{\infty 2} + u'_2 \;; \quad v_2 = v'_2 \;; \quad \rho = \rho_\infty + \rho' \;; \quad P = P_\infty + P' \tag{13.84}$$

where the primed quantities are assumed to be small compared to the free stream conditions. When quadratic terms in the equations of motion are neglected the equations (13.83) reduce to,

$$\left(1 - \frac{\rho_\infty U_{\infty 2}^2}{\gamma P_\infty}\right)\frac{\partial u'_2}{\partial x_2} + \frac{\partial v'_2}{\partial y_2} = 0\,. \tag{13.85}$$

Introduce the perturbation velocity potential

$$u'_2 = \frac{\partial \phi_2}{\partial x_2} \quad ; \qquad v'_2 = \frac{\partial \phi_2}{\partial y_2}\,. \tag{13.86}$$

The equation governing the disturbance flow becomes,

$$(1 - M_{\infty 2}^2)\frac{\partial^2 \phi_2}{\partial x_2^2} + \frac{\partial^2 \phi_2}{\partial y_2^2} = 0\,. \tag{13.87}$$

Notice that (13.87) is valid for both sub and supersonic flow in the thin airfoil approximation.

Since the flow is isentropic, the pressure and velocity disturbances are related to lowest order by,

$$P' + \rho_\infty U_{\infty 2} u_2' = 0 \tag{13.88}$$

and the surface pressure coefficient retains the same basic form as in the incompressible case,

$$C_{P2} = -\frac{2}{U_{\infty 2}}\left(\frac{\partial \phi_2}{\partial x_2}\right)_{y_2 = 0}\,. \tag{13.89}$$

Since we are at a finite Mach number, this last relation is valid only within the thin airfoil, small disturbance approximation and therefore may be expected to be invalid near the leading edge of the airfoil where the velocity change is of the order of the free stream velocity. For example for the “thin” airfoil depicted in Figure

13.1 which is actually not all that thin, the pressure coefficient is within $-0.2 < C_P < 0.2$ except over a very narrow portion of the chord near the leading edge.

Equation (13.87) can be transformed to Laplace's equation, (13.75) using the following change of variables.

$$x_2 = x_1 \quad ; \qquad y_2 = \frac{1}{\sqrt{1 - M_{\infty 2}^2}} y_1 \quad ; \qquad \phi_2 = \frac{1}{A}\left(\frac{U_{\infty 2}}{U_{\infty 1}}\right)\phi_1 \tag{13.90}$$

where, at the moment, A is an arbitrary constant. The velocity potentials are related by,

$$\phi_2[x_2, y_2] = \frac{1}{A}\left(\frac{U_{\infty 2}}{U_{\infty 1}}\right)\phi_1[x_1, y_1] \tag{13.91}$$

or,

$$\phi_1[x_1, y_1] = A\left(\frac{U_{\infty 1}}{U_{\infty 2}}\right)\phi_2\left[x_1, \frac{1}{\sqrt{1 - M_{\infty 2}^2}} y_1\right] \tag{13.92}$$

and the boundary conditions transform as,

$$\left.\begin{aligned} \left(\frac{\partial \phi_1}{\partial y_1}\right)_{y_1 = 0} &= U_{\infty 1}\left(\frac{A\tau_2}{\sqrt{1 - M_{\infty 2}^2}}\right)\frac{dg[x_1/c]}{d(x_1/c)} \\ \phi_{1_{x_1 \to \infty}} &= \phi_{2_{x_2 \to \infty}} = 0 \end{aligned}\right\}. \tag{13.93}$$

The transformation between flows one and two is completed by the correspondence,

$$\tau_1 = \frac{A\tau_2}{\sqrt{1 - M_{\infty 2}^2}} \tag{13.94}$$

or

$$\left(\frac{t_1}{c}\right) = \frac{A}{\sqrt{1-M_{\infty 2}^2}}\left(\frac{t_2}{c}\right). \tag{13.95}$$

Finally the transformed pressure coefficient is

$$C_{P1} = AC_{P2}. \tag{13.96}$$

These results may be stated as follows. The solution for incompressible flow over a thin airfoil with shape $g[x_1/c]$ and thickness ratio, t_1/c at velocity U_1 is identical to the subsonic compressible flow at velocity, U_2 and Mach number M_2 over an airfoil with a similar shape but with the thickness ratio,

$$\frac{t_2}{c} = \frac{\sqrt{1-M_{\infty 2}^2}}{A}\left(\frac{t_1}{c}\right). \tag{13.97}$$

The pressure coefficient for the compressible case is derived by adjusting the incompressible coefficient using $C_{P2} = C_{P1}/A$. This result comprises several different similarity rules that can be found in the aeronautical literature depending on the choice of the free constant, A.

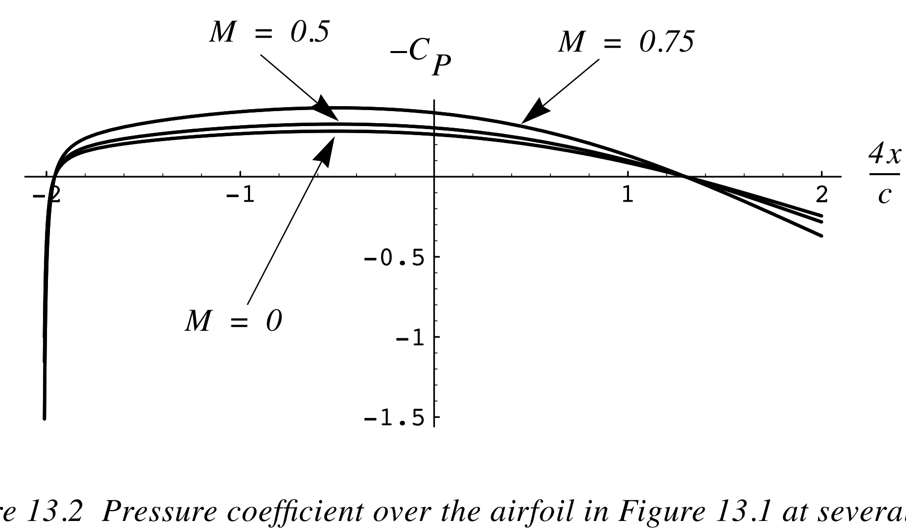


Figure 13.2 Pressure coefficient over the airfoil in Figure 13.1 at several Mach numbers as estimated using the Prandtl-Glauert rule (13.99)

Perhaps the one of greatest interest is the so-called Prandtl-Glauert rule that describes the variation of pressure coefficient with Mach number for a body of a given shape and thickness ratio. In this case we select,

$$A = \sqrt{1 - M_{\infty 2}^2} \tag{13.98}$$

so that the two bodies being compared in (13.97) have the same shape and thickness ratio. The pressure coefficient for the compressible flow is,

$$C_{P2} = \frac{C_{P1}}{\sqrt{1 - M_{\infty 2}^2}}. \tag{13.99}$$

Several scaled profiles are shown in Figure 13.2. Keep in mind the lack of validity of (13.99) near the leading edge where the pressure coefficient is scaled to inaccurate values.

13.2.2 Supersonic similarity $M_\infty > 1$

All the theory developed in the previous section can be extended to the supersonic case by simply replacing $1 - M_\infty^2$ with $M_\infty^2 - 1$. In this instance the mapping is between the equation,

$$(M_{\infty 2}^2 - 1)\frac{\partial^2 \phi_2}{\partial x_2^2} - \frac{\partial^2 \phi_2}{\partial y_2^2} = 0 \tag{13.100}$$

and the simple wave equation

$$\frac{\partial^2 \phi_1}{\partial x_1^2} - \frac{\partial^2 \phi_1}{\partial y_1^2} = 0 \tag{13.101}$$

A generalized form of the pressure coefficient valid for subsonic and supersonic flow is,

$$\frac{C_p}{A} = F\left[\frac{\tau}{A\sqrt{\left|1 - M_\infty^2\right|}}\right] \tag{13.102}$$

where A is taken to be a function of $\left|1 - M_\infty^2\right|$.

13.2.3 Transonic similarity, $M_\infty \cong 1$

When the Mach number is close to one, the simple linearization used to obtain (13.87) from (13.83) loses accuracy. In this case the equations (13.83) reduce to the nonlinear equation

$$(1-M_{\infty 1}^2)\frac{\partial u'_1}{\partial x_1}+\frac{\partial v'_1}{\partial y_1}-\frac{(\gamma_1+1)M_{\infty 1}^2}{U_{\infty 1}}u'_1\frac{\partial u'_1}{\partial x_1} = 0. \qquad (13.103)$$

In terms of the perturbation potential,

$$(1-M_{\infty 1}^2)\frac{\partial^2 \phi_1}{\partial x_1^2}+\frac{\partial^2 \phi_1}{\partial y_1^2}-\frac{(\gamma_1+1)M_{\infty 1}^2}{U_{\infty 1}}\frac{\partial \phi_1}{\partial x_1}\frac{\partial^2 \phi_1}{\partial x_1^2} = 0. \qquad (13.104)$$

This equation is invariant under the change of variables,

$$x_2 = x_1 \;; \qquad y_2 = \frac{\sqrt{1-M_{\infty 1}^2}}{\sqrt{1-M_{\infty 2}^2}}y_1 \;; \qquad \phi_2 = \frac{1}{A}\left(\frac{U_{\infty 2}}{U_{\infty 1}}\right)\phi_1 \qquad (13.105)$$

where

$$A = \left(\frac{1+\gamma_2}{1+\gamma_1}\right)\left(\frac{1-M_{\infty 1}^2}{1-M_{\infty 2}^2}\right)\left(\frac{M_{\infty 2}^2}{M_{\infty 1}^2}\right). \qquad (13.106)$$

Notice that, due to the nonlinearity of the transonic equation (13.104), the constant A is no longer arbitrary. The pressure coefficient becomes,

$$C_{P1} = \left(\frac{1+\gamma_2}{1+\gamma_1}\right)\left(\frac{1-M_{\infty 1}^2}{1-M_{\infty 2}^2}\right)\left(\frac{M_{\infty 2}^2}{M_{\infty 1}^2}\right)C_{p2} \qquad (13.107)$$

and the thickness ratios are related by,

$$\frac{t_2}{c} = \left(\frac{1+\gamma_1}{1+\gamma_2}\right)\left(\frac{1-M_{\infty 2}^2}{1-M_{\infty 1}^2}\right)^{\frac{3}{2}}\left(\frac{M_{\infty 1}^2}{M_{\infty 2}^2}\right)\frac{t_1}{c}. \tag{13.108}$$

In the transonic case, it is not possible to compare the same body at different Mach numbers or bodies with different thickness ratios at the same Mach number except by selecting gases with different γ. For a given gas it is only possible to map the pressure distribution for one airfoil to an airfoil with a different thickness ratio at a different Mach number. A generalized form of (13.102) valid from subsonic to sonic to supersonic Mach numbers is,

$$\frac{C_P((\gamma+1)M_\infty^2)^{1/3}}{\tau^{2/3}} = F\left[\frac{1-M_\infty^2}{(\tau(\gamma+1)M_\infty^2)^{2/3}}\right]. \tag{13.109}$$

Prior to the advent of supercomputers capable of solving the equations of high speed flow, similarity methods and wind tunnel correlations were the only tools available to the aircraft designer and these methods played a key role in the early development of transonic and supersonic flight.

13.3 The effect of sweep

Return to the continuity equation (13.8) written as

$$\nabla\bullet\overline{U} - \frac{1}{a^2}\overline{U}\bullet\nabla\left(\frac{\overline{U}\bullet\overline{U}}{2}\right) = 0 \tag{13.110}$$

Using the irrotationality condition $\nabla\times\overline{U} = 0$, (13.110) can expressed as

(13.111)

$$\left(1-\frac{U^2}{a^2}\right)\frac{\partial U}{\partial x} + \left(1-\frac{V^2}{a^2}\right)\frac{\partial V}{\partial y} + \left(1-\frac{W^2}{a^2}\right)\frac{\partial W}{\partial z} - 2\left(\frac{UV}{a^2}\frac{\partial U}{\partial y} + \frac{WV}{a^2}\frac{\partial V}{\partial z} + \frac{UW}{a^2}\frac{\partial U}{\partial z}\right) = 0$$

The figure below shows a plan view of of a long slender wing with the free stream flow approaching at a sweep angle β. The component of the free stream flow parallel to the long axis of the wing is assumed to play little role in determining the pressure variation over the wing except at the wing tips.

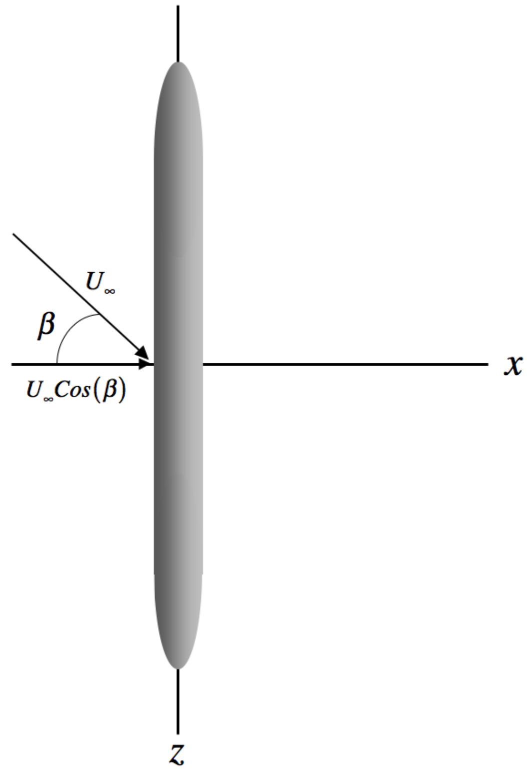


Figure 13.3 Slender wing with free stream approaching at sweep angle β

Therefore the flow field in the (x, y) plane normal to the long axis of the wing will be nearly two-dimensional so that $\partial W/\partial z$ and $\partial U/\partial z$ can be assumed to be very small. Now

$$\left(1-\frac{U^2}{a^2}\right)\frac{\partial U}{\partial x}+\left(1-\frac{V^2}{a^2}\right)\frac{\partial V}{\partial y}-2\frac{V}{a}\left(\frac{U}{a}\frac{\partial U}{\partial y}+\frac{W}{a}\frac{\partial V}{\partial z}\right) = 0 \qquad (13.112)$$

Assume small disturbances.

$$U = U_\infty Cos(\beta) + u$$
$$V = v$$
$$W = U_\infty Sin(\beta) + w \qquad (13.113)$$
$$a = a_\infty + a'$$

Neglect terms that are cubic in the small disturbances. Equation (13.112) becomes

(13.114)

$$\left(1 - \frac{U_\infty^2 Cos^2(\beta)}{a_\infty^2}\right)\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} - 2\frac{V}{a_\infty}\left(\frac{U_\infty Cos(\beta)}{a_\infty}\frac{\partial U}{\partial y} + \frac{U_\infty Sin(\beta)}{a_\infty}\frac{\partial V}{\partial z}\right) = 0$$

The quantity V/a_∞ is very small and we can neglect terms in (13.114) that are quadratic in the disturbances. Introduce the disturbance potential. Equation (13.111) reduces to

$$(1 - M_\infty^2 Cos^2(\beta))\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \qquad (13.115)$$

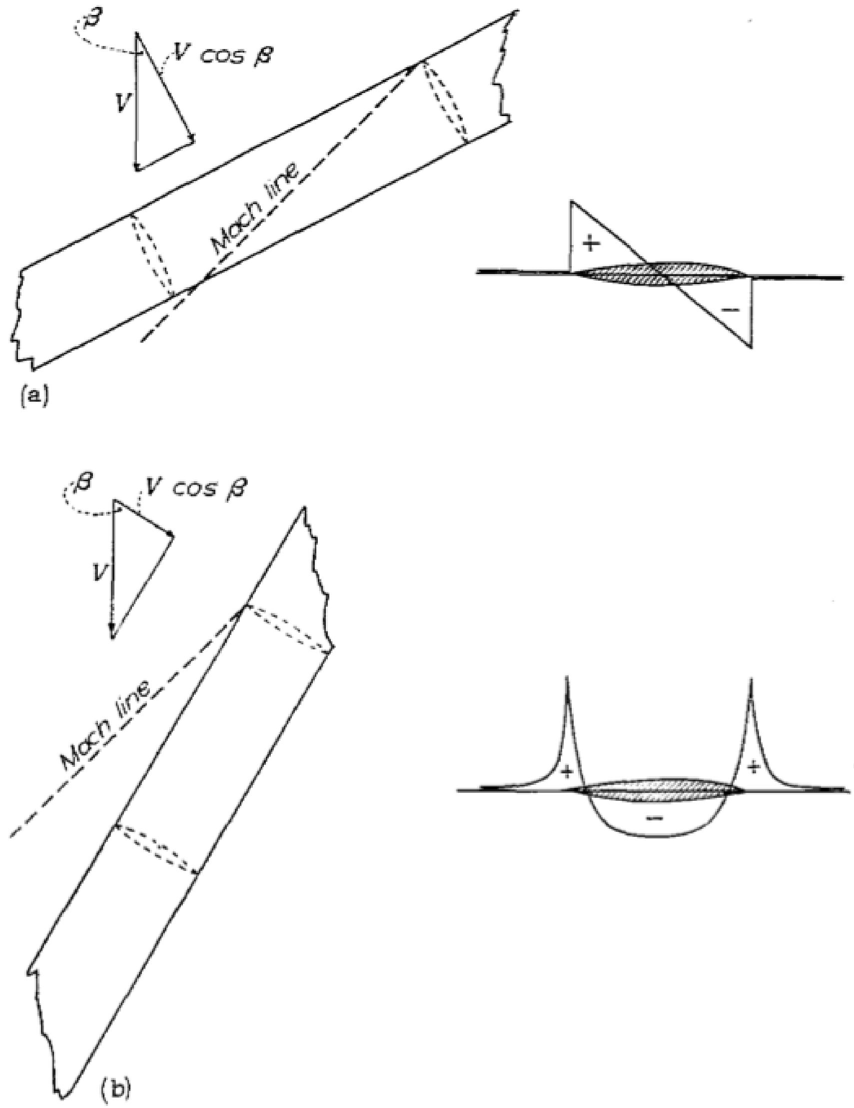
For the swept wing the disturbance potential is governed by the component of the free stream velocity normal to the leading edge of the wing $U_\infty Cos(\beta)$. If $M_\infty^2 Cos^2(\beta) > 1$ then the normal flow is supersonic and we would use supersonic theory developed above to relate the pressure coefficient to the slope of the airfoil surface in the (x, y) plane.

If $M_\infty^2 Cos^2(\beta) < 1$ then the normal flow is subsonic and we can use the mapping (13.90) to transform (13.115) to Laplace's equation. The methods of subsonic thin airfoil theory can be used to determine the $M = 0$ disturbance potential ϕ_1. The pressure coefficient of the $M = 0$ solution is used with the Prandtl-Glauert rule (13.99) to determine the pressure coefficient on the swept wing.

$$C_{P_{M_\infty Cos(\beta)}} = \frac{C_{P1}}{\sqrt{1 - M_\infty^2 Cos^2(\beta)}} \qquad (13.116)$$

The best discussion of this topic that I know of appears in NACA Technical Report 863 by R. T. Jones published in 1945. The rest of the discussion of this topic comprises images taken directly from this report. Where Jones refers to equations (4) and (6) he is referring to Laplaces equation and the wave equation.

The derivation of equations (4) and (6) is actually a special case of a more general statement, namely, that the component of translation of a cylindrical body in the direction of its long axis has no effect on the motion of a frictionless fluid. In the case of a wing of constant section moving through still fluid, the flow is determined by the normal components of velocity of its solid boundaries and these components in turn are completely specified by the component of motion in planes perpendicular to the axis $V \cos \beta$. When the normal component of velocity $V \cos \beta$ is less than sonic, then the wing-section flows are determined by solutions of Laplace's equation. As is well known, these flows show no pressure drag due to thickness of the airfoil. On the other hand, if the normal component exceeds the velocity of sound, the flow patterns are of a different type and are characterized by plane sound waves. In this case a pressure drag arises and the suction force at the leading edge disappears (fig. 2 (a)).



(a) $V \cos \beta > c$.
(b) $V \cos \beta < c$.

FIGURE 2.—Effect of leading-edge angle on pressure distribution.

A physical explanation of the occurrence of smooth flow patterns and pressure distributions at supersonic velocities is as follows: If V is greater than c but $V \cos \beta$ is less, then the angle of sideslip or sweepback is greater than the Mach angle (see fig. 2 (b)) and the airfoil will lie behind the characteristic lines along which pressure influences are transmitted (Mach lines). Thus, although the fluid directly upstream from a given section can receive no pressure signal from this section, the flow behaves as though it did receive such signals because of the successive influence of similar sections farther upstream along the airfoil. The streamlines will thus be caused to curve and follow paths appropriate to a subsonic flow, although the speed is everywhere supersonic.

Figure 3 illustrates the effect of sweepback on the change in cross section of a stream tube passing near the upper surface of a cambered airfoil. As is well known, the equations of fluid motion show a reduction in the area of a stream tube in the region of increased velocity above the airfoil when the velocity of flight is subsonic but show an increase in the cross section when the velocity of flight is supersonic. In figure 3 the component normal to the leading edge $V \cos \beta$ is subsonic; and hence in section view the streamlines, following the pattern for subsonic velocities, appear to contract as they flow over the upper surface. In plan view, however, the resolution of velocities shows that the flow lines bend as they pass over the wing in such a way as to increase the stream-tube area. In case the velocity of flight is supersonic, the latter effect must predominate, as is required by the equations of motion.

The order of magnitude of the pressure-drag coefficient and its variation with angle of sweepback are indicated by figure 4. The calculations were made by applying the Ackeret theory and formulas (4) and (5) to a wing of infinite aspect ratio. A simple biconvex wing section was assumed and the angle of attack was varied so as to maintain a constant lift coefficient of 0.5. The calculations were made for a Mach number of 1.4, with the result that at 45° the angle of sweepback becomes equal to the Mach angle and the factor

$$\frac{1}{\sqrt{\left(\frac{V \cos \beta}{c}\right)^2 - 1}}$$

becomes infinite. At this point the pressure drag due to thickness becomes infinite and the drag due to angle of attack (shown by the curve marked $\frac{t}{c}=0$) vanishes.

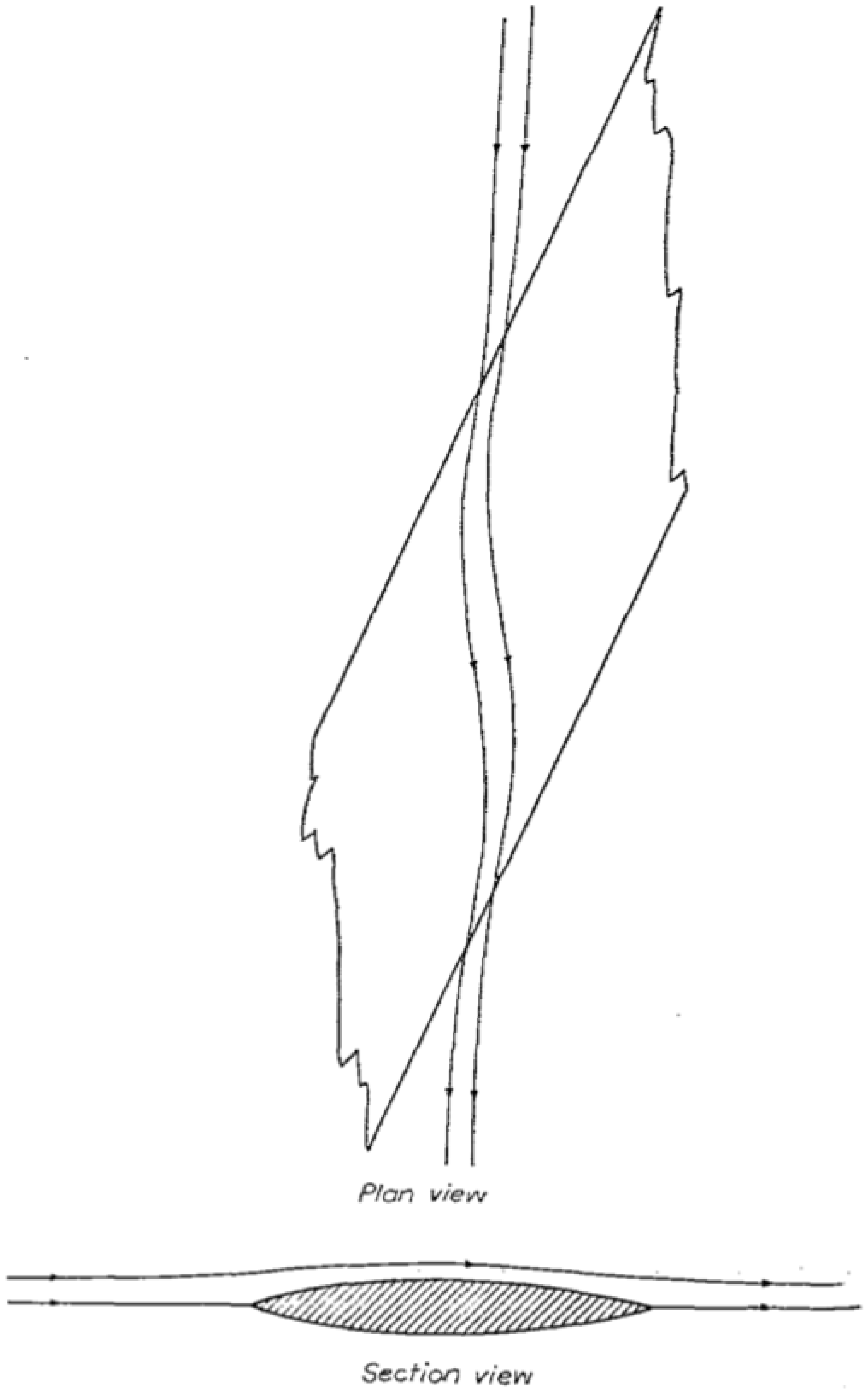


FIGURE 3.—Change in area of stream tube over upper surface of sweptback wing.

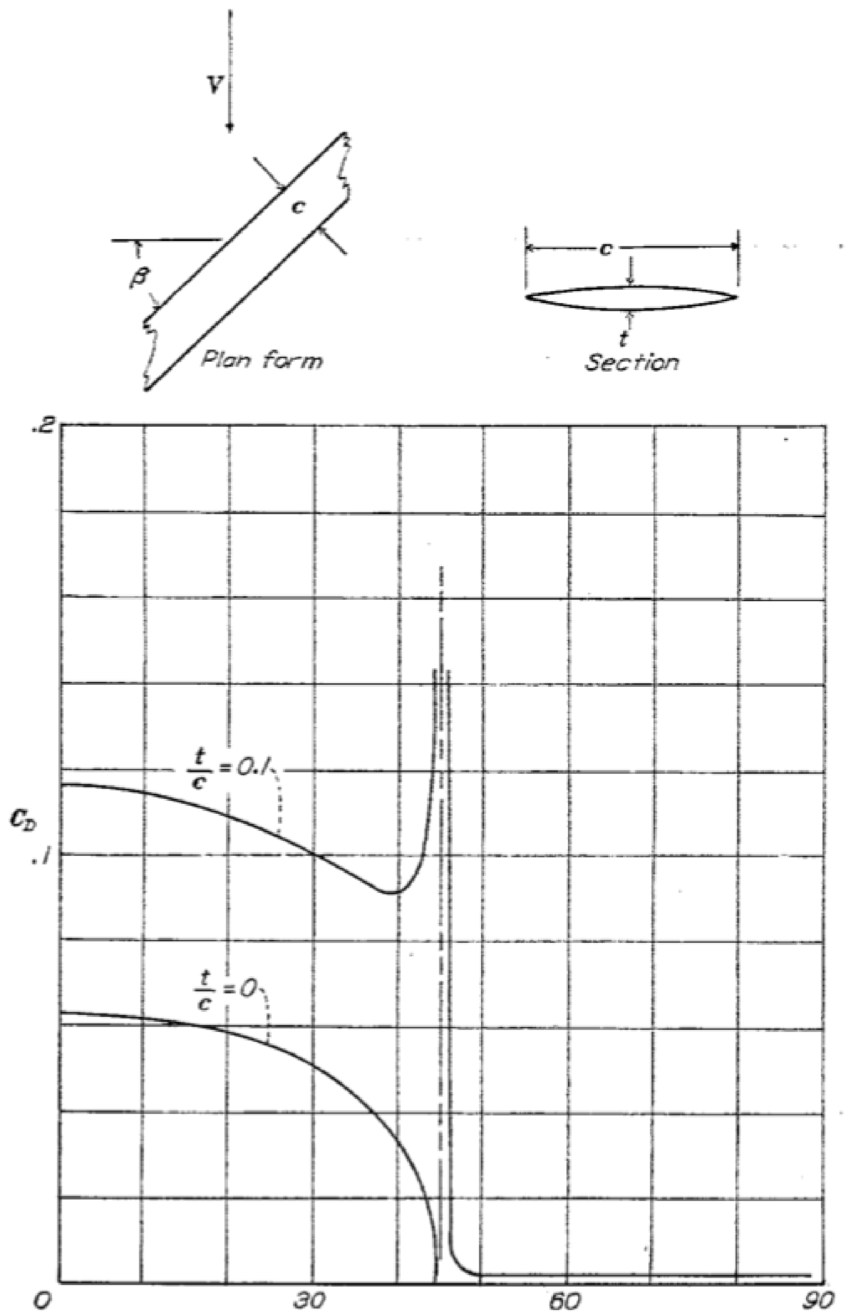


FIGURE 4.—Variation of pressure drag with angle of sweepback for infinite aspect ratio. $M=1.4$; $C_L=0.5$.

In the case of a wing of finite aspect ratio, it seems probable that in the regions of the center section and the tips pressure drags of the same order as those indicated for these sections by the Ackeret theory will appear. If the wing is of sufficiently high aspect ratio, however, the fraction of the wing area affected will be negligible and the pressure drag will be nearly that given in figure 4. The other drags involved are: (1) skin-friction drag, which may be of the order of 0.01, and (2) induced drag, which for an aspect ratio of 8 is also about 0.01.

WINGS OF FINITE SPAN AND THICKNESS

Schlichting (reference 10) proposes a trapezoidal plan form with tips cut away at the Mach angle as the ideal supersonic wing, since in this case the wake has no influence on the lifting surface and the drag is no greater than that of a wing of infinite span. In the plan forms proposed by Schlichting, however, the resultant force remains at right angles to the chord; hence the pressure drag is equal to the lift times the angle of attack. With this type of flow there is no favorable effect of aspect ratio.

It is interesting to note that a favorable interference may be obtained by separating the wing into lifting elements and staggering the elements in a rearward direction behind the Mach lines as in figure 5. In the staggered arrangement the upflow outside the vortices trailing from element A will be effective at the position of B and, although the lift of each element is at right angles to its chord, the upflow permits the angle of attack of element B to be reduced for the same lift and hence the lift-drag ratio will be improved.

According to Munk's stagger theorem (reference 11) the over-all drag of a lifting system in an incompressible flow would not be altered by changing the relative positions of the lifting elements along the direction of flight. In the type of flow considered by Munk, therefore, a reduction in the drag

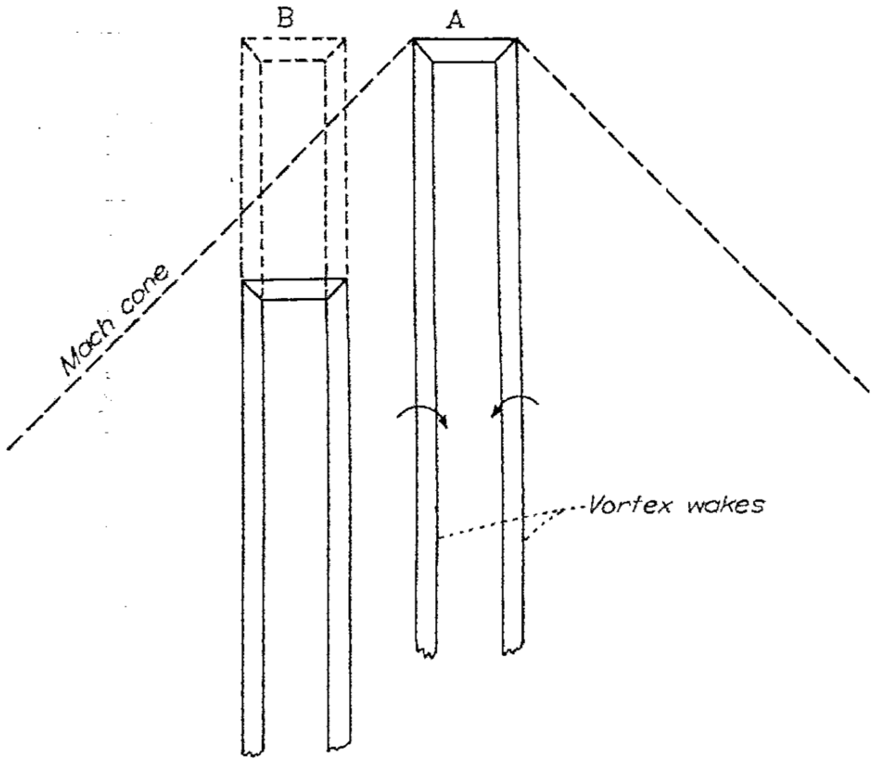


FIGURE 5.—Staggered lifting elements in supersonic flow.

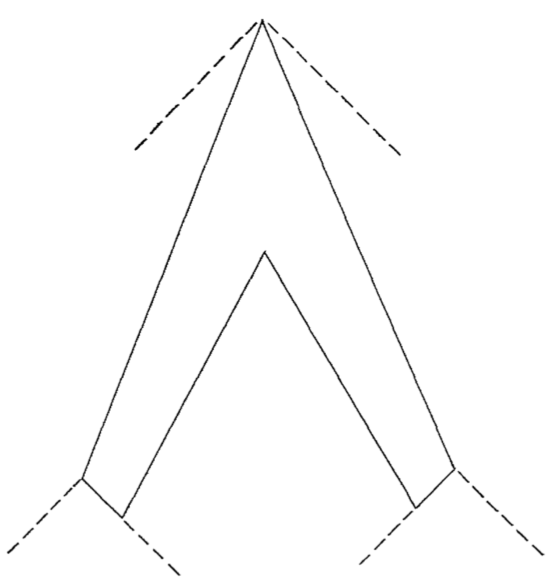


FIGURE 6.—Wing with tips cut away along the Mach lines.

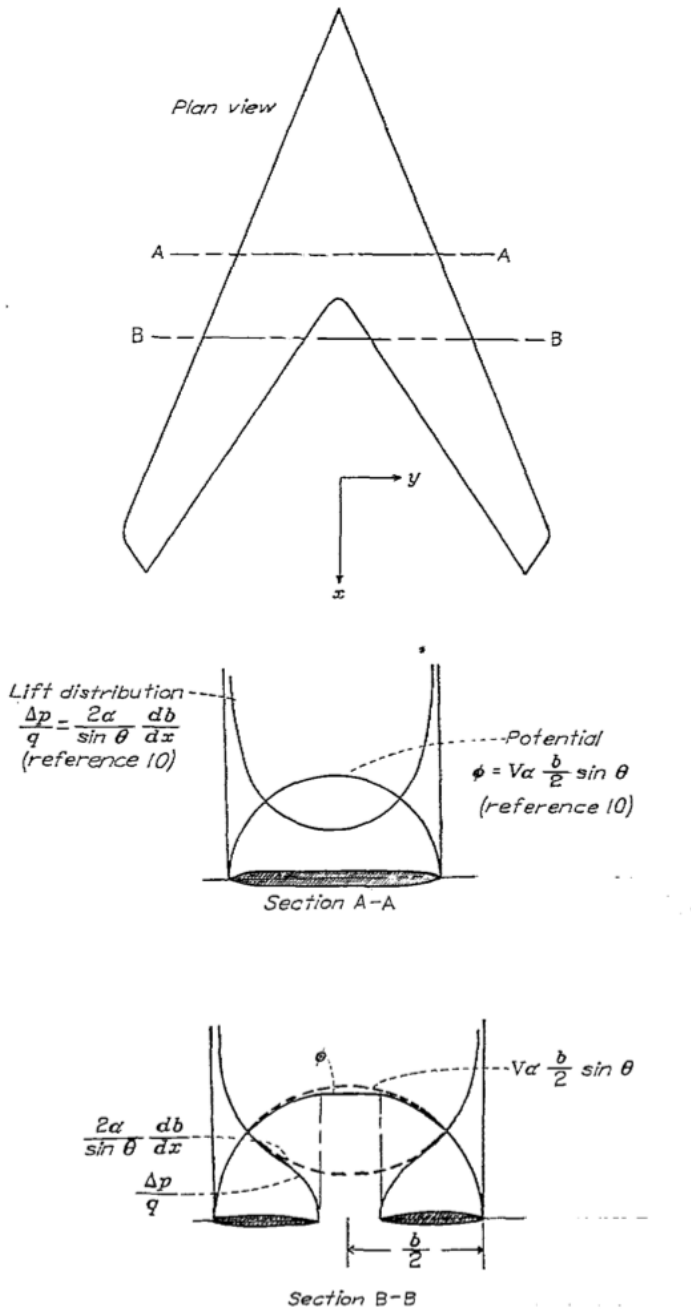


FIGURE 7.—Approximate distribution of lift near vertex of wing with large angle of sweepback.

Finite thickness is expected to result in a pressure drag on those sections near the center of the wing and further study is also required to establish the flow due to thickness in this region. Some insight into the problem of flow near the center section may be furnished by the known solutions for supersonic flow in three dimensions (reference 13). Finite thickness may also cause pressure drag in regions where the flow is two-dimensional if the induced velocities are great enough to cause shock waves. This effect may be avoided by increasing the angle of sweepback so that the normal component of velocity not only is subsonic but is less than the critical speed of the airfoil sections. This principle may also be applied to wings designed for subsonic speeds near the speed of sound.

LANGLEY MEMORIAL AERONAUTICAL LABORATORY,
NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS,
LANGLEY FIELD, VA., *June 23, 1945.*

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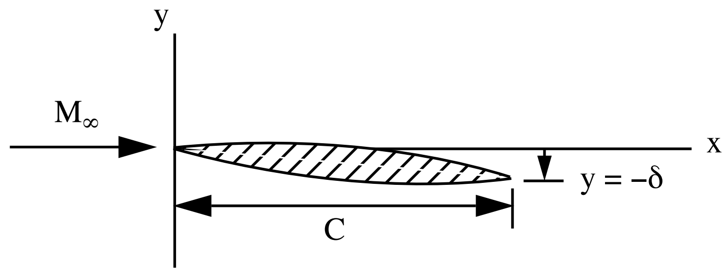
13.4 PROBLEMS

Problem 1 - A thin, 2-D, airfoil is situated in a supersonic stream at Mach number M_∞ and a small angle of attack as shown below.



The y-coordinate of the upper surface of the airfoil is given by the function

$$f(x) = A\frac{x}{C}\left(1 - \frac{x}{C}\right) - \frac{\delta}{C}x$$

and the y-coordinate of the lower surface is

$$g(x) = -A\frac{x}{C}\left(1 - \frac{x}{C}\right) - \frac{\delta}{C}x$$

where $2A/C \ll 1$ and $\delta/C \ll 1$. Determine the lift and drag coefficients of the airfoil.