APPROXIMATE CALCULATION OF THE LAMINAR BOUNDARY LAYER

by

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INTRODUCTION

The steady two-dimensional flow of viscous incompressible fluid in the boundary layer along a solid boundary, which is governed by Prandtl's approximation to the full equations of motion, presents a problem which in general is as intractable as any in applied mathematics. The problem, however, has such an immediate and necessary application that approximate methods of varying accuracy which go beyond the formal processes of expansions in series and so on, have been devised for the rapid calculation of the principal characteristics of the laminar boundary-layer, the variation of pressure along the surface being known. Such methods usually represent approximately the boundary-layer velocity distribution at any point by one of a known family of distributions whose spacing along the surface is determined by some means, often by the use of Kármán's momentum equation.

In the first main part of this paper, Sections 3-5, all known velocity distributions from exact and approximate solutions are collected and compared in a manner which shows clearly their potentialities or limitations when used as the bases of approximate methods. This critical comparison explains that it is possible for an approximate method to yield the exact values of the more important characteristics of a flow, and enables a method of calculation to be constructed in the second part of the paper, Sections 6 and 7.

The momentum equation, of which a full discussion is given in Sections 1 and 2, is the basis of the construction, but is not necessary for the final working, of the new method. The method, which is shown to give good results and to be simple and speedy in application, can be used with confidence in regions of increasing pressure and, in particular, predicts separation of flow with good accuracy.

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Notation

Note:

U', U'' are derivatives of U with respect to x
f' f'', f''' are derivatives of f with respect to η
F', F'', F''' are derivatives of F with respect to η.

x, y	distances parallel and perpendicular to the plane boundary $y=0$
u, v	velocity components parallel and perpendicular to the boundary
U	the stream velocity
v_0	the value of v at $y=0$
ψ	the stream function
δ^*	$\int_0^\infty \left(1-\frac{u}{U}\right) dy$, the displacement thickness
θ	$\int_0^\infty \frac{u}{U}\left(1-\frac{u}{U}\right) dy$, the momentum thickness
δ	a length of the order of δ^*
c	a chord length
η	a parameter, variously defined, denoting distance perpendicular to the boundary
Θ	θ/δ
H	$\frac{\delta^*}{\theta}$
ν	the kinematic coefficient of viscosity
U_0, β_0, β_1, k	various constants occurring in definitions for U
λ, A,	numerical parameters
f	a function of η defined in Section 4 (*ii*)
$f(\lambda)$	$\frac{\delta}{U}\left(\frac{\partial u}{\partial y}\right)_{y=0}$
$I(\lambda)$, $J(\lambda)$	certain functions of λ
l	$\frac{\theta}{U}\left(\frac{\partial u}{\partial y}\right)_{y=0}$
m	$\frac{\theta^2}{U}\left(\frac{\partial^2 u}{\partial y^2}\right)_{y=0}$
k	an index of x/c
β	$\frac{2k}{k+1}$
Y	$(\frac{1}{2}(k+1))^{\frac{1}{2}}\eta$
$F(Y)$	$(\frac{1}{2}(k+1))^{\frac{1}{2}} f(\eta)$
K	a suction parameter
ϕ	a function of $K\eta$

$L(m)$	$2[(H(m)+2)m+l(m)]$
M	a function of θ^2 and U
δ	an x-interval
a_1, a_2, a_3, a_4	various coefficients
F	a function of various parameters according to context
t	$\dfrac{u}{U}$
R_θ	$\dfrac{U\theta}{\nu}$
ξ	$\dfrac{\beta_1 x}{\beta_0}$
S_1, S_2	certain sums

1. PRELIMINARY DISCUSSION OF EXISTING METHODS

(*i*) Prandtl's equation of motion of steady two-dimensional boundary-layer flow is

$$\frac{u\partial u}{\partial x}+\frac{v\partial v}{\partial y}=UU'+\nu\frac{\partial^2 u}{\partial y^2} \qquad (1)$$

(u, v) being velocity components, (x, y) Cartesian co-ordinates, U the stream velocity just outside the boundary layer and $y=0$ the solid boundary. ν is the kinematic viscosity.

If equation (1) is integrated from $y=0$ to $y=\infty$, the first integral or momentum equation is obtained as†

$$\frac{d\theta}{dx}=-(H+2)\frac{U'\theta}{U}+\frac{\nu}{U^2}\left(\frac{\partial u}{\partial y}\right)_{y=0} \qquad (2)$$

in which the momentum thickness $\theta=\int_0^\infty \frac{u}{U}\left(1-\frac{u}{U}\right)dy$, the displacement thickness $\delta^*=\int_0^\infty \left(1-\frac{u}{U}\right)dy$, $H=\delta^*/\theta$ and the equation of continuity $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$ is used.

The equation of motion (1) becomes, when $y=0$

$$0=UU'+\nu\left(\frac{\partial^2 u}{\partial y^2}\right)_{y=0} \qquad (3)$$

Of the methods which use equation (2) directly, Pohlhausen's[1] is the simplest, which assumes a family of velocity distributions analytically defined. It is well

† It should be noted that, in the case of a porous surface through which there is a normal velocity v_0, an extra term $\frac{v_0}{U}$ should be added to the right hand side of the equation

known to give poor results in regions of rising pressure. Various other authors have tried to improve and extend his method by assuming different families of distributions.

It is possible to devise other methods for giving a correspondence between a known family of velocity distributions and a certain flow. For example, Howarth[2] uses the distributions which occur in the solution of $U = \beta_0 - \beta_1 x$ as the basis of an approximate method for regions of rising pressure. Falkner[3] uses the distributions corresponding to the flows $U = U_0 (x/c)^k$ as a basis for an elaborate method in which the momentum equation is incidentally used. In Ref. 4, he simplifies this work and also suggests a form for θ in terms of U directly. Young and Winterbottom[5] also gave θ as a function of the stream velocity U in a form which is close to that derived in this paper.

The Kármán momentum equation, being an exact integral of the equation of motion, should be able to yield good results but, without knowledge of any exact solutions, such an equation would be practically useless. It is one of the intentions of the present paper to show how known exact solutions can themselves be given accurately by correct use of the momentum equation. A method of solution is finally derived which is virtually the best possible under the general conditions of the use of the momentum equation.

(*ii*) The assumption is usually made that to a sufficient degree of accuracy the velocity distribution $\frac{u}{U} = \frac{u}{U}(x, y)$ can be represented by the single infinity of distributions

$$\frac{u}{U} = F\left(\frac{y}{\delta}, \lambda\right) \quad . \quad . \quad . \quad . \quad . \quad (4)$$

δ is some length of the order of the boundary layer thickness, λ is a function of x and F is a function of two variables which satisfies the usual boundary conditions:

$$F(0, \lambda) = 0; \quad F\left(\frac{y}{\delta}, \lambda\right) = 1, \quad y \geqslant A\delta.$$

If A is finite, the approximation assumes that the stream velocity U is actually reached at a finite distance from the boundary.

Equation (3) imposes another condition upon F which is

$$0 = UU' + \nu \frac{U}{\delta^2} F''(0, \lambda) \quad \text{or} \quad -\frac{\delta^2 U'}{\nu} = F''(0, \lambda), \quad . \quad . \quad (5)$$

in which dashes on F denote differentiation with respect to y/δ. (5) is an identity which must be satisfied at all points of the boundary and which suggests that λ can be conveniently defined by $\lambda = -\delta^2 U'/\nu$. The significance of this relation is that it relates the thickness of the boundary layer and the stream-velocity gradient to λ or to the "shape" of the velocity distribution.

From (4), expressions are obtainable for θ, H and $\left(\frac{\partial u}{\partial y}\right)_{y=0}$ in the form: $\theta = \delta\,\Theta(\lambda)$, $\left(\frac{\partial u}{\partial y}\right)_{y=0} = \frac{U}{\delta} f(\lambda)$, $H = H(\lambda)$, in which Θ, H, f are functions of λ depending on the form of F originally chosen. By substituting these expressions in (2) and with $\lambda = -\frac{\delta^2 U'}{\nu}$ we obtain, after some re-arrangement:

$$\tfrac{1}{2}\frac{d}{dx}\left(\frac{\lambda\Theta^2}{U'}\right) + (H+2)\frac{\lambda\Theta^2}{U} + \frac{\Theta f}{U} = 0$$

which can be reduced to

$$\frac{d\lambda}{dx} = I(\lambda)\frac{U''}{U'} + J(\lambda)\frac{U'}{U} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (6)$$

$I(\lambda)$, $J(\lambda)$ being determinable functions.

(6) can be integrated to obtain the distribution of λ with respect to x, whence all the other characteristics follow at once.

(*iii*) The method given above is the standard method of using the momentum equation. In (6), λ is the dependent variable—some writers have made δ^2 the dependent variable, which is a slight alteration but has no effect on the comments made below.

First, to carry out the whole process from the beginning is tedious. A large amount of algebra is involved in obtaining $\Theta(\lambda)$, $H(\lambda)$, $f(\lambda)$, $I(\lambda)$ and $J(\lambda)$. When (6) has been integrated numerically, further work is needed to find the distributions of θ, H and $\left(\frac{\partial u}{\partial y}\right)_{y=0}$ with x.

In equation (6), the quantity U'' cannot be evaluated with much accuracy from an experimental set of readings of the pressure distribution along a surface. Further there is a singularity in the equation at $U=0$ and it is necessary to find the correct value of λ to be taken at a stagnation point.

The introduction of δ as a convenient but unrequired parameter involves not only general complication in the handling of the equations, but additional computation to obtain θ and H.

Lastly, there is always the possibility that the solution requires values of λ for which the method is unworkable (for example, when $\lambda > 12$ in Pohlhausen's method), and also that for some values of λ equation (4) no longer gives a reasonable approximation since F may have a maximum greater than unity.

Next to be considered is the general approach to the equations (2) and (3). These equations are regarded as simultaneous equations in δ and λ, which are the parameters in terms of which the required variables are expressed. It happens that the second equation allows the very simple relation, $\lambda = -\delta^2 U'/\nu$ between δ and λ and it is natural to make analytic use of this to turn the momentum equation into an equation for either δ or λ.

The present author considers this to be rather the wrong approach to the equations in that it obscures the initial assumptions of a certain family of distributions and confuses the true relationship between the two equations (2) and (3). In what follows, a different approach will be explained and a simplified treatment of the momentum equation given.

2. THE NEW APPROACH

(*i*) In general, the intention is to avoid the introduction of unnecessary parameters such as δ, to avoid all algebraic work (except, of course, in the actual integration of the equation), to simplify to a great extent this integration, to ensure that the method can always be worked, and to avoid the necessity for knowledge of U''. These aims will be attained largely by the use of θ as the principal dependent variable.

The main purpose of any approximate method is to give as accurately as possible the distribution of skin friction (and therewith the point of separation, if it exists) and of the momentum or displacement thickness. A good representation of the velocity distributions is not necessary. In the method to be presented, a synthesis of all known accurate solutions enables the first purpose to be achieved with the greatest degree of accuracy possible, and a good representation of the velocity distribution can also be obtained simply.

(*ii*) The essential features of equations (2) and (3) are the two terms

$$\left(\frac{\partial u}{\partial y}\right)_{y=0} \text{ and } \left(\frac{\partial^2 u}{\partial y^2}\right)_{y=0} \quad . \quad . \quad . \quad . \quad . \quad (7)$$

From the second equation $\nu\left(\frac{\partial^2 u}{\partial y^2}\right)_{y=0}$ is equal to $-UU'$. To integrate the momentum equation the value of $\left(\frac{\partial u}{\partial y}\right)_{y=0}$ is required. Thus the fundamental requirement is a relation between the two terms in (7). For if some such relation is assumed (*e.g.* by the use of a family of velocity distributions such as (4)) then $\left(\frac{\partial u}{\partial y}\right)_{y=0}$ is known as a function of x since the value, $-\frac{UU'}{\nu}$, of $\left(\frac{\partial^2 u}{\partial y^2}\right)_{y=0}$ is known.

The two terms (7) indicate the behaviour of the velocity distribution at the boundary. Its general "shape" elsewhere is indicated by the value of H which also occurs in the momentum equation. This equation therefore is not only concerned with boundary values, but relates these with the velocity distribution as a whole. For this reason, the momentum equation is capable of giving good results. Thus a relation between H and $\left(\frac{\partial^2 u}{\partial y^2}\right)_{y=0}$ is also required.

The construction and use of these required relationships is now considered.

3. THE COMPARISON OF BOUNDARY LAYER VELOCITY DISTRIBUTIONS

All known velocity distributions are now compared by classifying in a particular way the values of their first and second derivatives at $y=0$ which we have seen, following (7), are of particular importance.

We put $$\left(\frac{\partial u}{\partial y}\right)_{y=0} = \frac{U}{\theta} l, \quad \left(\frac{\partial^2 u}{\partial y^2}\right)_{y=0} = \frac{U}{\theta^2} m \qquad (8)$$

these forms being chosen so that the numbers l and m depend only upon the "shape" of the distribution of velocity, and not upon the thickness of the boundary layer or the magnitude of the stream velocity. We are to investigate the relation between l and m and to regard l and also H as functions, $l\,(m)$ and $H\,(m)$, of m. $l\,(m)$ and $H\,(m)$ are therefore computed for all known solutions, exact and approximate, of laminar boundary layer flow which are enumerated and examined in the next Section.

4. EXAMINATION OF KNOWN SOLUTIONS

Brief remarks are now made on all the known solutions of the boundary layer equation. Details of the computations involved in obtaining l, m and H will mostly be omitted, but the results are given in graphical form. Pohlhausen's velocity distributions and the corresponding method of solution are discussed in some detail and fresh light is thrown upon the well-known failure of his method in certain cases.

(*i*) *Pohlhausen's profiles* are given by

$$\frac{u}{U} = 2\eta - 2\eta^3 + \eta^4 + \frac{\lambda}{6}\eta(1-\eta)^3, \quad \eta = \frac{y}{\delta} \quad . \quad . \quad . \quad (9)$$

It is easy to show that

$$\left(\frac{\partial^2 u}{\partial y^2}\right)_{y=0} = -\frac{U\lambda}{\delta^2} \text{ and } \theta = \frac{\delta}{315}\left(37 - \frac{\lambda}{3} - \frac{5\lambda^2}{144}\right).$$

Hence $$m = \frac{\theta^2}{U}\left(\frac{\partial^2 u}{\partial y^2}\right)_{y=0} = -\frac{\lambda}{(315)^2}\left(37 - \frac{\lambda}{3} - \frac{5\lambda^2}{144}\right)^2, \quad . \quad . \quad . \quad . \quad (10)$$

from which

$$\frac{dm}{d\lambda} = -\frac{1}{(315)^2}\left(37 - \frac{\lambda}{3} - \frac{5\lambda^2}{144}\right)\left(37 - \lambda - \frac{25\lambda^2}{144}\right).$$

m therefore has maxima or minima at

$$\lambda = -37.79, \ -17.76, \ +12, \ +28.19,$$

at which m takes the values

$$m = 0, \ 0.183, \ -0.095, \ 0 \quad . \quad . \quad . \quad . \quad (11)$$

respectively.

Now $\left(\frac{\partial u}{\partial y}\right)_{y=0} = \frac{U}{6\delta}(12+\lambda)$, and therefore when $\lambda = -12$, separation of flow occurs, and for $\lambda < -12$, $\left(\frac{\partial u}{\partial y}\right)_{y=0} < 0$. Thus we are only interested in the range of λ, $\lambda \geqslant -12$. As λ increases from -12, m decreases from 0.157 until a minimum in m is reached at $\lambda = +12$. (At this point, it is well known that the solution becomes unworkable.†) Values of m less than -0.095 are therefore unattainable except for much larger values of λ. These larger values of λ however cannot be used since in a solution they would involve a discontinuity in λ from $\lambda = 12$ to $\lambda = 34.79$ (where again $m = -0.095$) and therefore discontinuities in θ and $\left(\frac{\partial u}{\partial y}\right)_{y=0}$, which are

† For example, in *Modern Developments in Fluid Dynamics* (edited by S. Goldstein), Vol. I, p. 161

inadmissible. Furthermore, for values of $\lambda > 12$, the velocity distribution given by (9) has a maximum greater than unity which in itself would make a solution in this region inaccurate. The range of λ available is therefore $-12 \leqslant \lambda \leqslant 12$, for which $0.157 \geqslant m \geqslant -0.095$.

It may be pointed out here that in the problem of the flat plate in a uniform stream, with a constant normal velocity through the plate, the velocity distribution tends, as $x \longrightarrow \infty$, to the asymptotic suction distribution, for which $m = -0.25$†. It is clear therefore that the Pohlhausen distributions are very far from reaching this value of m and could not be expected to give satisfactory results in this problem. (*See*, for example, Ref. 14, Section 1.)

The functions $l(m)$ and $H(m)$ are easily calculated and they are shown in Figs. 1-4.

(*ii*) $U = U_0 (x/c)^k$

Falkner and Skan[3] have obtained exact numerical solutions of this flow which Hartree[7] has amplified on the differential analyser. For any value of k, it is well known that the distributions have a constant "shape." The equation of motion is transformed by writing

$$\left.\begin{aligned} u &= \frac{\partial \psi}{\partial y}, \quad & \psi &= (U\nu x)^{\frac{1}{2}} f(\eta) \\ v &= -\frac{\partial \psi}{\partial x}, \quad & \eta &= \left(\frac{U}{\nu x}\right)^{\frac{1}{2}} y \end{aligned}\right\}$$

into

$$k(f')^2 - \tfrac{1}{2}(k+1) ff'' = k + f''' \quad . \quad . \quad . \quad . \quad . \quad (12)$$

If

$$\left(\tfrac{1}{2}(k+1)\right)^{\frac{1}{2}} \eta = Y, \quad \left(\tfrac{1}{2}(k+1)\right)^{\frac{1}{2}} f(\eta) = F(Y), \quad \beta = \frac{2k}{k+1}$$

(12) becomes

$$F''' + FF'' = \beta\left((F')^2 - 1\right)$$

with the boundary conditions

$$F(0) = 0, \quad F'(0) = 0, \quad F'(Y) \longrightarrow 1 \text{ as } Y \longrightarrow \infty.$$

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† The asymptotic suction distribution can be written as $\frac{u}{U} = 1 - e^{-y/2\theta}$. Therefore $\left(\frac{\partial^2 u}{\partial y^2}\right)_{y=0} = \frac{-U}{4\theta^2}$ whence $m = -0.25$. See the footnote to Section 4 (*v*)

The upper limit for m for $l \geqslant 0$ is 0.068 and in this case $k = -0.0904$ and $l(m) = 0$, *i.e.* the stream velocity decreases in such a way that a separation distribution is just maintained.

It is difficult to determine whether there is a lower limit for m. Calculations have not given values of m less than -0.10, since Hartree pointed out that the practical value of such distributions is not very great. This is true for flow along solid boundaries but in view of the flat plate constant-suction problem for which a value of $m = -0.25$ is required, it is clear that a greater range of solutions of this flow would be useful.

(*iii*) *Linearly decreasing pressure gradient*: ($U = \beta_0 - \beta_1 x$).

This flow has been treated by Howarth[2] using expansion in series, the accuracy of which is fully discussed in the original report. Since $U' < 0$, only positive values of m occur. Separation takes place when $\beta_1 x / \beta_0 = 0.120$ and $m = 0.084$.

(*iv*) *Schubauer's ellipse*: *Hartree's solution*[8]

Hartree has solved the boundary layer equations on his differential analyser for the pressure distribution which Schubauer observed on an ellipse. Near the forward stagnation point a series solution was applied and the analyser solution was started at a distance $0.067c$ from the stagnation point. Hartree actually did not find separation occurring before, or at, the observed point of separation and points out, in a full discussion, that it is meaningless to pursue the solution, using the observed pressure distribution, past the observed point of separation. He also points out how sensitive the solution is near the observed separation point, extremely small changes in pressure gradient being sufficient for the solution to give a separation. Howarth has discussed this experiment in relation to approximate solutions in Ref. 2.

For Hartree's solution, m takes the value 0.0784 at the limit of calculation. In the region $U' > 0$, m takes the value -0.0854 at the stagnation point which corresponds to the value $k = 1$ in (12) and increases steadily to zero at the pressure minimum.

(*v*) *Blasius' equation*: $f''' + \frac{1}{2} f f'' = 0$ . . . . (13)

This is the special case $k = 0$ of equation (12). The boundary conditions are

$$f(0) = K, \quad f'(0) = 0, \quad f'(\infty) = 1$$

and the solution corresponds to the uniform flow past a flat plate through which there is a normal velocity $v_0 = -\frac{K}{2}\left(\frac{U_0 \nu}{x}\right)^{\frac{1}{2}}$. $K > 0$ therefore corresponds to suction and $K=0$ gives the well-known Blasius' distribution for uniform flow past a solid boundary. The equation has been solved for various values of K—Ref. 9 deals with positive values of K and Schlichting[10] gives solutions for positive and negative values of K. This family of velocity distributions is interesting for several reasons, especially for $K>0$. Watson[11] has proved that as $K \longrightarrow +\infty$ the velocity distribution tends to the asymptotic suction distribution.† Thus the negative range for m is $0 \geqslant m \geqslant -0.25$ which covers the range necessary in the flat plate constant-suction problem. For $K<0$, the distributions are curious (physically this corresponds to blowing out of the surface with velocity proportional to $x^{-\frac{1}{2}}$). As K increases m increases but reaches a maximum of about 0.031 and then appears to tend to zero as K tends to $-\infty$. $l(m)$ steadily decreases as K decreases and also tends to zero.

(*vi*) *Iglisch's solution of the flat plate, constant-suction problem.*[12]

This and the next solution described are of considerable interest because they deal with velocity distributions obtained when boundary suction exists. There is reason to suppose that the general type of distributions under conditions of boundary suction may show rather different characteristics from those at solid boundaries, especially in regions where suction is delaying or preventing separation. This solution of Iglisch was obtained by numerical methods, there being no exact solution in finite terms known. The distribution is Blasius' at the leading-edge of the plate[13] and changes progressively until the asymptotic suction distribution is reached. Thus $0 \leqslant m \leqslant -0.25$.

(*vii*) $U=U_0(x/c)$, $v_0 \neq 0$:

This represents the flow from a stagnation point under conditions of boundary suction. The solution is a particular case of Falkner's general solution (equation (12)) with $k=1$ and $f(0)=K$, and has been calculated for various values of K by Schlichting and Bussmann[10]. In this flow, the velocity gradient reinforces the general effect of the suction, and m takes only negative values.

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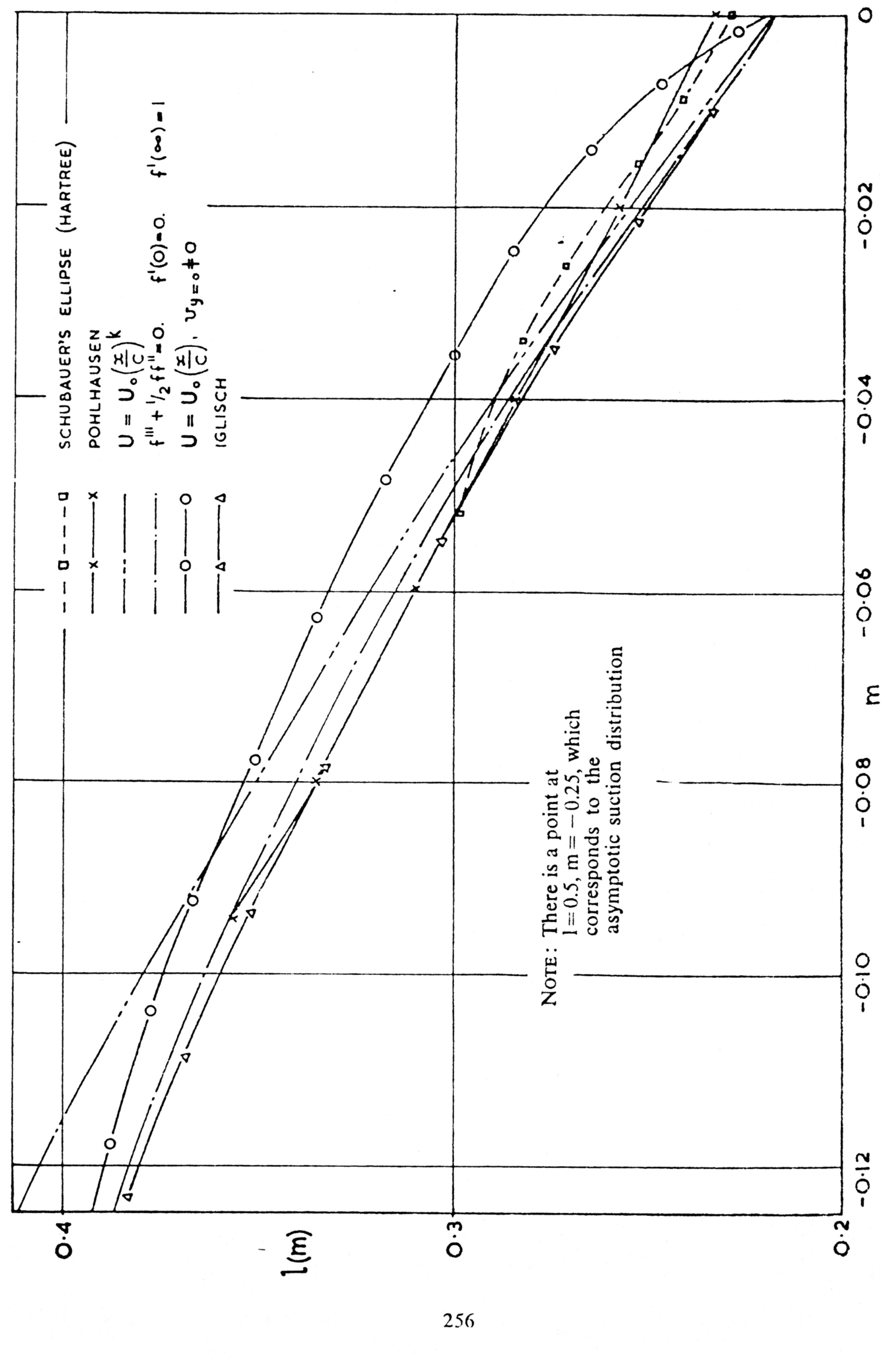
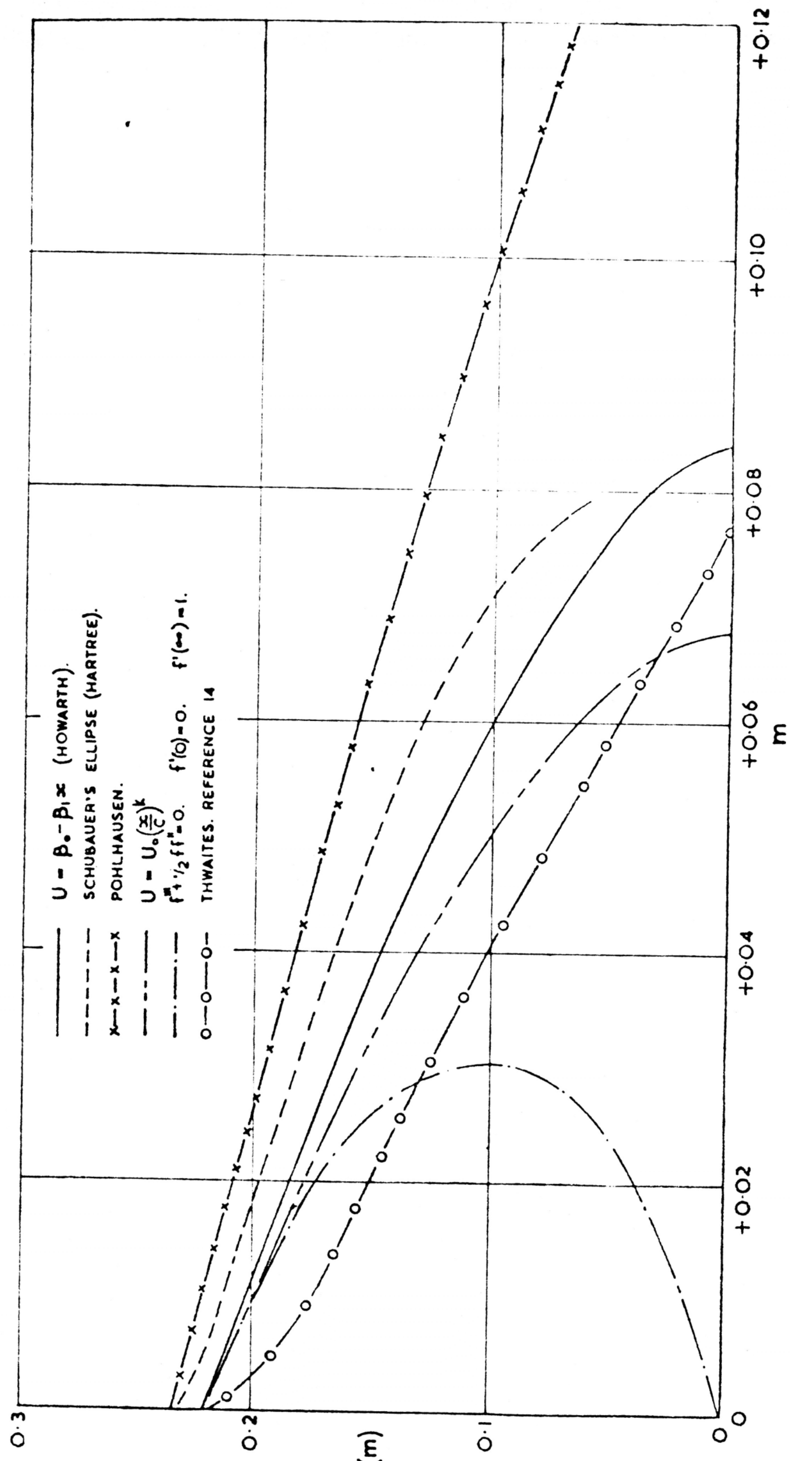
† If $f(\eta)=K+\frac{1}{K}\varphi(K\eta)$, $f'=\varphi'$, $f''=K\varphi''$, $f'''=K^2\varphi'''$, equation (13) becomes $\varphi'''+\frac{1}{2}\varphi'' = \frac{-1}{2K^2}\varphi\varphi''$, with $\varphi'(0)=\varphi(0)=0$, $\varphi'(\infty)=1$. As $K\rightarrow\infty$, the equation tends to $\varphi'''+\frac{1}{2}\varphi''=0$, of which the solution is $\varphi'(t)=1-e^{-\frac{1}{2}t}$ or $\frac{u}{U}=1-e^{v_0 y/\nu}$

If U is not constant a similar result holds, for if $U=o(v_0)$, $u=o(v_0)$ the continuity equation gives $v=v_0+O(U)$, so that the equation of motion is $v_0\frac{\partial u}{\partial y}-\nu\frac{\partial^2 u}{\partial y^2}=o\left(\frac{v_0 U}{y}\right)$ the solution of which as $v_0\rightarrow\infty$ is $\frac{u}{U}=1-e^{v_0 y/\nu}$


SCHUBAUER'S ELLIPSE (HARTREE)
POHLHAUSEN
$U = U_0\left(\frac{x}{c}\right)^k$
$f''' + \frac{1}{2} f f'' = 0. \quad f'(0) = 0. \quad f'(\infty) = 1$
$U = U_0\left(\frac{x}{c}\right), \quad v_{y=0} \neq 0$
IGLISCH
NOTE: There is a point at l = 0.5, m = −0.25, which corresponds to the asymptotic suction distribution
0
−0·02
−0·04
−0·06
−0·08
−0·10
−0·12
m
0·4
0·3
0·2
l(m)
Fig. 1



U = β₀ − β₁x (HOWARTH).
SCHUBAUER'S ELLIPSE (HARTREE).
POHLHAUSEN.
U = U₀(x/c)^k
f‴ + ½ f f″ = 0. f′(0) = 0. f′(∞) = 1.
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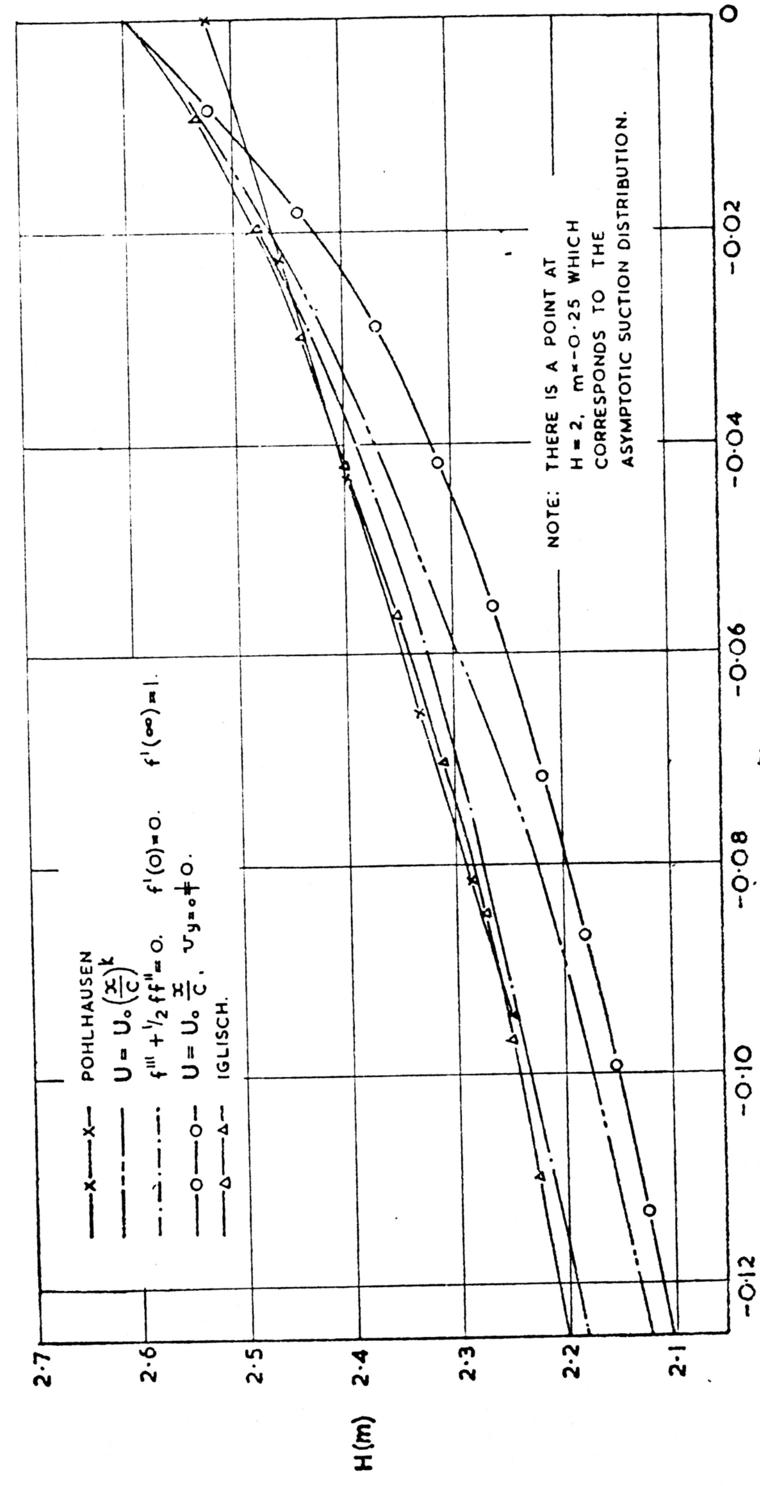
l(m)
0·3
0·2
0·1
0
0
+0·02
+0·04
+0·06
+0·08
+0·10
+0·12
m

Fig. 2



H(m)
2·7
2·6
2·5
2·4
2·3
2·2
2·1
m
0
−0·02
−0·04
−0·06
−0·08
−0·10
−0·12
POHLHAUSEN
$U = U_0 \left(\frac{x}{c}\right)^k$
$f''' + \frac{1}{2} f f'' = 0. \quad f'(0) = 0. \quad f'(\infty) = 1.$
$U = U_0 \frac{x}{c}, \quad v_{y=0} \neq 0.$
IGLISCH.
NOTE: THERE IS A POINT AT H = 2, m = −0·25 WHICH CORRESPONDS TO THE ASYMPTOTIC SUCTION DISTRIBUTION.

Fig. 3

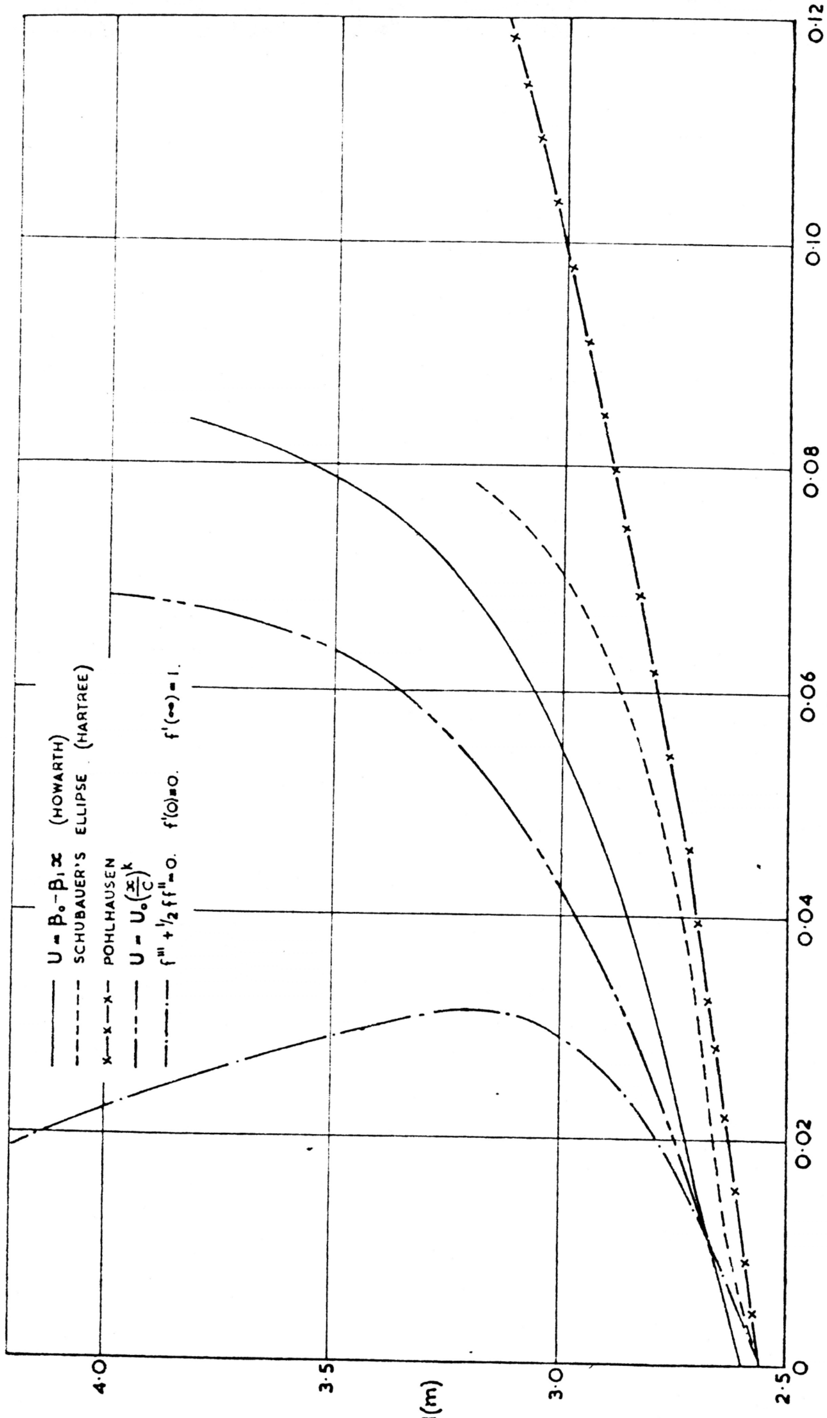

$U = \beta_0 - \beta_1 x$ (HOWARTH)
SCHUBAUER'S ELLIPSE (HARTREE)
POHLHAUSEN
$U = U_0\left(\frac{x}{c}\right)^k$
$f''' + \frac{1}{2}ff'' = 0.$ $f(0) = f'(0) = 0.$ $f'(\infty) = 1.$
H(m)
4·0
3·5
3·0
2·5
0
0·02
0·04
0·06
0·08
0·10
0·12
m

Fig. 4

5. COMMENTS AND COMPARISONS

(*i*) In Section 4, all the known solutions of laminar boundary layer flow have been mentioned. The functions $l(m)$ and $H(m)$ have been drawn for all these solutions and are shown in Figs. 1-4. Each point on these curves represents a certain velocity distribution and therefore in these figures are collected a large range of different distributions, each of which is known to exist.

Values of m divide naturally into the two ranges of positive and negative values, since for solid boundaries positive values of m occur when $U' < 0$, negative values when $U' > 0$.† (For conditions of continuous suction this is no longer true.)

Consider first negative values of m. Fig. 1 shows the function $l(m)$ drawn for the various flows. The range of m shown is $0 \geqslant m \geqslant -0.12$, which is sufficient for all distributions computed except for Iglisch's solution. For this, as has been already pointed out, there is the point for which $l = 0.5$, $m = -0.25$. It is remarkable that the various values of $l(m)$ for any particular m do not differ by more than about 10 per cent., although the distributions are those of widely differing types of flow. The suction distributions for $U = U_0 (x/c)$ differ by about 10 per cent. from the other distributions for $-0.02 \geqslant m \geqslant -0.04$, in which region these distributions differ from each other by only about three per cent. The general comment upon Fig. 1 therefore is one of surprise that the various curves of $l(m)$ lie so closely together. Fig. 3 gives $H(m)$ and here again the values of $H(m)$ do not differ by more than five per cent. $H(m)$ increases steadily with m, and its minimum value appears to be 2.0 at $m = -0.25$.

Figure 2 gives $l(m)$ for $m \geqslant 0$, and it is at once obvious that no such close agreement exists as for $m \leqslant 0$. We can dismiss the curve for $f''' + \frac{1}{2}ff'' = 0$, for not only does this curve correspond to distributions obtained by blowing out through the boundary but also the shape of the curve itself has no practical interest. The three exact solutions are grouped fairly well together, with the solution for $U = \beta_0 - \beta_1 x$ in the middle. The value of m for separation in $U = U_0 (x/c)^{k}$ is 0.068 whereas the value for $U = \beta_0 - \beta_1 x$ is 0.084. Had Hartree predicted separation for Schubauer's ellipse, the value of m would have been near to 0.084. The difference between the Schubauer and the Howarth curves lies in the fact that in the former a boundary layer of some thickness already exists where U' becomes negative, while for the latter the boundary layer has zero thickness at $m = 0$. Finally the Pohlhausen curve lies well away from the other curves. Pohlhausen's $l(m)$ takes values which are too large and this means that in regions of negative velocity gradient the skin friction is given values systematically too high. The value of m at separation is 0.157

† For, from equation (3), $0 = UU' + \frac{\nu U m}{\theta^2}$, or $m = -\frac{U'\theta^2}{\nu}$

The reason for Pohlhausen's method giving inaccurate results in regions of increasing pressure is therefore clear, and it would seem that Falkner's method, which is based on the profiles of $U=U_0\,(x/c)^k$ would give better results. Fig. 4 shows $H\,(m)$ for $m \geqslant 0$. Here also the various curves are fairly widely scattered, although those of the three exact solutions are roughly of the same general shape. Pohlhausen's $H\,(m)$ gives values which are much too low.

(*ii*) In general, to any flow along a surface corresponds an $l\,(m)$ curve and an $H\,(m)$ curve. There is an exception in the case of flows with "similar" velocity distributions, when single values of l, m and H describe a flow. Thus in Figs. 1-4 the curves for $U=U_0\,(x/c)^k$ do not represent a single flow but a very large variety of different flows and it is all the more remarkable that the curves lie closely to those of other flows.

It was shown in Section 2 (*ii*) that integration of the momentum equation requires only a relationship between the terms in (7) or the $l\,(m)$ function and also the $H\,(m)$ function. Thus the probable accuracy obtained by using any approximate method upon a certain flow can be assessed by a comparison of the approximate and exact $l\,(m)$ and $H\,(m)$. Thus if an approximate method has the same $l\,(m)$ and $H\,(m)$ as an exact solution, then it will yield exact results. An immediate corollary is that, since no two exact solutions have identical $l\,(m)$ and $H\,(m)$, an approximate method which gives excellent results in one case may not be successful in another. In other words, there is a limit to the degree of accuracy obtainable by an approximate method used on any flow. But, in fact, the accuracy obtainable in any case can be quite sufficient for practical purposes, and the formulæ developed in this paper give what are in effect the best results possible in all cases.

6. THE CONSTRUCTION OF A METHOD

(*i*) It was pointed out in Section 2 (*ii*) that the integration of (2) in conjunction with the boundary conditions (3) requires only the functions $l\,(m)$ and $H\,(m)$, and to construct a method of integration the choice of these functions has to be made.

We refer now to Figs. 1-4, in which $l\,(m)$ and $H\,(m)$ are plotted for all known solutions. The choice of $l\,(m)$ and $H\,(m)$ for use in the present method will be made by taking an average value of these solutions.

When $m \leqslant 0$, the choice is easy, for the curves of Fig. 1 lie closely together. When $m \geqslant 0$ the choice is more difficult. It is first necessary to choose the value for m at separation. (The fact that an arbitrary choice is involved is the fundamental disadvantage of the use of the momentum equation.) For $U=\beta_0-\beta_1 x$, $m=0.084$ at separation and for $U=U_0\,(x/c)^{-0.0904}$, $l=0$ and $m=0.068$. For Hartree's solution of Schubauer's ellipse the value of m at separation, had it been predicted, would appear to be about 0.082. This last value will be taken in the present method to represent separation, since the flow about the ellipse is similar to flow about an aerofoil to which approximate methods of calculation are most applicable.

It is desirable to make $l(m)$ and $H(m)$ take certain values which correspond to particularly common exact solutions. Blasius' distribution occurs at the leading edge of a plate in a stream if there is no stagnation point, and if there is a stagnation point the distribution there is that for $U = U_0 (x/c)$. These two distributions should be included in our solution. The asymptotic suction distribution should also be included, if only because its large negative value of m will avoid the possibility of a breakdown in the method.

Thus we require:

$$\left.\begin{array}{rl} U = U_0: & l(0) = 0.220 \\ & H(0) = 2.591 \\ \text{Asymptotic suction distribution:} & l(-0.25) = 0.5 \\ & H(-0.25) = 2.0 \\ \text{Separation distribution:} & l(0.082) = 0 \\ & H(0.082) = 3.7 \end{array}\right\} \quad . \quad . \quad . \quad (14)$$

$$U = U_0\, x/c: \quad l(-0.0854) = 0{\cdot}359, \quad H(-0.0854) = 2.218$$

Furthermore the function $l(m)$ appears to have an infinite gradient at $l = 0$.

It would be simple to express $l(m)$ by a short analytic form. It is surprising how definite a curve one wants to draw when taking a reasonable mean of the curves in Fig. 1, but in fact it is difficult to choose a function for $l(m)$ which is short, simple and suitable. Therefore we have no hesitation in defining numerically $l(m)$ for the present method, and in Table I $l(m)$ is tabulated. This $l(m)$ includes the points (14) but its value at $m = -0.0854$ (corresponding to a stagnation point) is 0.344 instead of 0.359, which is not a serious inaccuracy.

The function $H(m)$ is found in precisely the same way and is given in Table I. Values of l and H at values of m intermediate to those given can be found by simple interpolation.

With $l(m)$ and $H(m)$ now fixed, the momentum equation can be integrated.

(*ii*) The equations (2) and (3) become, after substituting from (8):—

$$\left.\begin{array}{l} \dfrac{d\theta}{dx} = -(H+2)\dfrac{U'\theta}{U} + \dfrac{\nu l(m)}{U\theta} \\[2ex] 0 = UU' + \dfrac{\nu U m}{\theta^2} \end{array}\right\}$$

On re-arrangement and substitution the first of these becomes

$$\frac{1}{\nu}\frac{U}{2}\frac{d\theta^2}{dx} = (H(m)+2)\,m + l(m)$$

and if

$$L(m) = 2\,[\,(H(m)+2)\,m + l(m)\,] \quad . \quad . \quad . \quad (15)$$

then

$$\frac{U}{\nu}\frac{d\theta^2}{dx} = L(m), \quad m = -\frac{U'\theta^2}{\nu} \quad . \quad . \quad . \quad . \quad (16)$$

The method of solving equations (16) is now obvious. At any point of the system under consideration we know U, U' and θ. These define m and thus $L(m)$ and therefore the gradient of θ^2 is known. A simple step-by-step method can be used and the only requirement for finding the distribution of θ with x is the function $L(m)$ which is tabulated in Table I. For obtaining the other properties of the boundary layer, we require to know $l(m)$ and $H(m)$.

It is clear that the aim to use θ as the principal dependent variable, to avoid algebraic work and to simplify greatly the equation to be integrated, has been attained.

(*iii*) A method is required for the integration of (16) which is simple but which gives a high degree of accuracy.

The equation is of the form

$$\frac{d\theta^2}{dx} = M(\theta^2, U) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

in which θ^2 and U are regarded as independent of one another. Let suffixes 0 and 1 denote values at $x=x_0$, $x=x_1$ respectively, and let $x_1 - x_0 = \delta$. We assume that x, U and partial derivatives of M of any order are $O(1)$, and $\delta = o(1)$.

Then we have $\theta_1^2 = \theta_0^2 + \delta\left(\frac{d\theta^2}{dx}\right)_0 + O(\delta^2)$

or from (17), $\quad \theta_1^2 = \Theta_0^2 + O(\delta^2)$

where $$\Theta_0^2 = \theta_0^2 + \delta M(\theta_0^2, U_0) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

We wish to refine this formula to give θ_1^2 to $O(\delta^3)$.

Now,

$$M(\theta_1^2, U_1) = M(\Theta_0^2, U_1) + O(\theta_1^2 - \Theta_0^2) = M(\Theta_0^2, U_1) + O(\delta^2) \quad . \quad (19)$$

In general,
$$f_1 = f_0 + \delta (f')_0 + \frac{\delta^2}{2} (f'')_0 + O(\delta^3)$$

and
$$(f')_1 = (f')_0 + \delta (f'')_0 + O(\delta^2)$$

and elimination of $(f'')_0$ from these two equations gives

$$f_1 = f_0 + \frac{\delta}{2} ((f')_0 + (f')_1) + O(\delta^3).$$

With $f = \theta^2$ and using (17) and (19) we get

$$\theta_1^2 = \theta_0^2 + \frac{\delta}{2} [M(\theta_0^2, U_0) + M(\Theta_0^2, U_1)] + O(\delta^3)$$

Θ_0^2 being defined in (18).

This formula gives by simple means the value of θ_1^2 correct to $O(\delta^3)$.

(*iv*) The flow away from a stagnation point presents a formal difficulty in the integration of (16) which must be resolved, for if $U = 0$ an infinite number of integral curves exists unless $L(m) = 0$. Thus at a stagnation point we must have $L(m) = 0$ or, from Table I, $m = -0.075$. This value of m immediately fixes the value of θ at the stagnation point, and the value of $\frac{xp}{z\theta p}$ is found as follows.

Near $m = -0.075$, $L(m) = 6(m + 0.075) + a_2 (m + 0.075)^2 + \ldots$ †

Hence from (16), when $U = U_0 (x/c)$,

$$\lim_{x\to 0} \left(\frac{d\theta^2}{dx}\right) = \lim_{x\to 0} \left\{ \frac{6\nu (0.075 - U'\theta^2/\nu)}{U} \right\}$$

$$= -6 \lim_{x\to 0} \left\{ \frac{U''\theta^2 + U' \dfrac{d\theta^2}{dx}}{U'} \right\}$$

$$= -6 \lim_{x\to 0} \left(\frac{d\theta^2}{dx}\right) - 0.45\nu \left(\frac{U''}{(U')^2}\right)_{x=0}$$

Therefore
$$\left(\frac{d\theta^2}{dx}\right)_{x=0} = -\frac{0.45\nu}{7} \left(\frac{U''}{(U')^2}\right)_{x=0}$$

† The gradient of $L(m)$ at $m = -0.075$ from Table I is not exactly 6, but is taken here as 6 for convenience in fitting a later formula

TABLE I†

m	$l\,(m)$	$H\,(m)$	$L\,(m)$
+0.082	0	3.70	0.938
+0.0818	0.011	3.69	0.953
+0.0816	0.016	3.66	0.956
+0.0812	0.024	3.63	0.962
+0.0808	0.030	3.61	0.967
+0.0804	0.035	3.59	0.969
+0.080	0.039	3.58	0.971
+0.079	0.049	3.52	0.970
+0.078	0.055	3.47	0.963
+0.076	0.067	3.38	0.952
+0.074	0.076	3.30	0.936
+0.072	0.083	3.23	0.919
+0.070	0.089	3.17	0.902
+0.068	0.094	3.13	0.886
+0.064	0.104	3.05	0.854
+0.060	0.113	2.99	0.825
+0.056	0.122	2.94	0.797
+0.052	0.130	2.90	0.770
+0.048	0.138	2.87	0.744
+0.040	0.153	2.81	0.691
+0.032	0.168	2.75	0.640
+0.024	0.182	2.71	0.590
+0.016	0.195	2.67	0.539
+0.008	0.208	2.64	0.490
0	0.220	2.61	0.440
−0.016	0.244	2.55	0.342
−0.032	0.268	2.49	0.249
−0.048	0.291	2.44	0.156
−0.064	0.313	2.39	0.064
−0.080	0.333	2.34	−0.028
−0.10	0.359	2.28	−0.138
−0.12	0.382	2.23	−0.251
−0.14	0.404	2.18	−0.362
−0.20	0.463	2.07	−0.702
−0.25	0.500	2.00	−1.000

† The solution of a laminar boundary layer is given on page 280

The values of θ^2 and its derivatives are therefore known and the integration of (16) can proceed from the stagnation point.

(v) The distribution of l, m and H with x having been obtained, it may be required to construct the velocity distribution at any point.

If the distribution is expressed in the usual way, $u/U=f(y/\delta)$, it is very tedious to find expressions for the coefficients implied in this expression in terms of l, m and H.

If however the distribution is expressed as

$$\frac{y}{\theta}=F\left(\frac{u}{U}\right)=F(t)$$

the work becomes a great deal easier. This form of the velocity distribution is considered in detail in Ref. 14. It can be verified that:

$$\frac{\theta}{U}\left(\frac{\partial u}{\partial y}\right)_{y=0}=l=\frac{1}{F'(0)}, \quad \frac{\theta^2}{U}\left(\frac{\partial^2 u}{\partial y^2}\right)_{y=0}=m=-\frac{F''(0)}{[F'(0)]^3} \quad \text{and } H=\int_0^1 F(t)\,dt,$$

whence

$$F'(0)=\frac{1}{l}, \quad F''(0)=-\frac{m}{l^3}, \quad H=\int_0^1 F(t)\,dt. \qquad (20)$$

Suppose that $\dfrac{y}{\theta}=F(t)=a_1t+a_2t^2+a_3t^3, \quad t\leqslant 1.$

Then (20) gives at once $a_1=\dfrac{1}{l}, \quad 2a_2=-\dfrac{m}{l^3}, \quad \dfrac{a_1}{2}+\dfrac{a_2}{3}+\dfrac{a_3}{4}=H,$

whence

$$\frac{y}{\theta}=\frac{1}{l}\,\frac{u}{U}-\frac{m}{2l^3}\left(\frac{u}{U}\right)^2+\left(4H+\frac{2m}{3l^3}-\frac{2}{l}\right)\left(\frac{u}{U}\right)^3.$$

The velocity distribution can be calculated from this form, and the simplification resulting from the use of the form $\dfrac{y}{\theta}=F\left(\dfrac{u}{U}\right)$ is clear.

When a separation distribution is required, the form taken above for $F(t)$ is insufficient, for then $l=0$ and the a's take infinite values. This can be remedied by the use of the form $F(t)=b_0t^{\frac{1}{2}}+b_1t$, for which it is easy to obtain

$$b_0=\sqrt{(-2/m)}, \quad b_1=2H-(4/3)\sqrt{(-2/m)}.$$

An example of the use of this construction of distribution is given in a later paragraph.

Better approximations to the velocity distribution can be obtained by using more of the boundary conditions (for example $\left(\frac{\partial^3 u}{\partial y^3}\right)_{y=0}=0$) and also by using a function F which tends to infinity as $(u/U) \longrightarrow 1$. However, since the approximate method gives only three pieces of information about the velocity distribution, it is doubtful whether better representations than those above could be obtained without very great complication. Even then, it is doubtful whether the approximate distribution would be sufficiently good for calculations such as stability calculations to be performed.

(*vi*) It is worth while to remark here upon a test which is commonly applied to methods using the momentum equation but which has no significance, as will now be shown. In the flow $U=U_0\,(x/c)^k$, a separation distribution is maintained for the value $k=-0.0904$.

The test applied to approximate methods is to assume a stream velocity $U=U_0\,(x/c)^k$ and find the value of k for which a separation distribution is maintained. The closeness of the value of k so obtained to -0.0904 is taken as an indication of the accuracy of the method.

However, the value of k so given depends only upon the value of H of the distribution which is assumed by the method to be a separation distribution.

For suppose a distribution of constant "shape" is maintained for which $\left(\frac{\partial u}{\partial y}\right)_{y=0}=0$, and so $l=0$.

Then since the "shape" is constant, m has a constant value. Under these conditions, the momentum equation together with the boundary condition become, from (16):

$$2\,(H+2)m=\frac{U}{\nu}\,\frac{d\theta^2}{dx}=-U\frac{d}{dx}\left(\frac{m}{U'}\right)=-Um\frac{d}{dx}\left(\frac{1}{U'}\right),\ \text{whence}\ \frac{UU''}{(U')^2}=2\,(H+2).$$

On the assumption of $U=U_0\,(x/c)^k$,

$$k\,(k-1)/k^2=2\,(H+2)\ \text{or}\ k=-1/(2H+3).$$

Thus the value of k obtained depends only upon the value of H assumed at separation, and clearly there is little significance in testing an approximate method in this way.

7. THE SIMPLEST METHOD

(*i*) With the values assumed in Table I for $l(m)$ and $H(m)$, $L(m)$ is found to be nearly a linear function of m, especially in the most common region of m, say $m > -0.1$. This fact suggests a comparison of the $L(m)$ curves for all the solutions known. Using the curves in Figs. 1-4 it is simple to calculate $L(m)$ for all the flows considered and in Fig. 5 the comparison is shown graphically.

Two features of these curves are surprising. Firstly, it is curious that although the flows considered are of very different characters, the $L(m)$ curves do not differ greatly from each other, and they differ by much less than the separate $l(m)$ and $H(m)$ curves, especially towards a point of separation. Secondly, it is curious that, at least in the range of m which is most common, *i.e.* $m > -0.1$, the curves are almost linear.

These facts suggest strongly that a good approximation to all types of flow will be obtained by assuming a linear form for $L(m)$ (or at most a quadratic form) in (16). By taking a reasonable mean of the curves in Fig. 5 we may assume

$$L(m) = 0.45 + 6m \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

It may be remarked that the possible range of choice for the two coefficients in (21), expressed as a percentage of their values, is about two per cent., which gives an indication of the error which may arise by the use of the form (21).

A slightly more satisfactory form of $L(m)$ would include the asymptotic suction distribution. For this, $m = -0.25$, $H = 2.0$, $l = 0.5$ and hence $L(-0.25) = -1.0$. The value of $L(m)$ given by (21) at $m = -0.25$ is -1.05. A quadratic form of $L(m)$ to satisfy the relations: $L(-0.25) = -1.0$, $L(0) = 0.455$, $L(0.075) = 0.925$, is

$$L(m) = 0.455 + 6.16m + 1.37m^2.$$

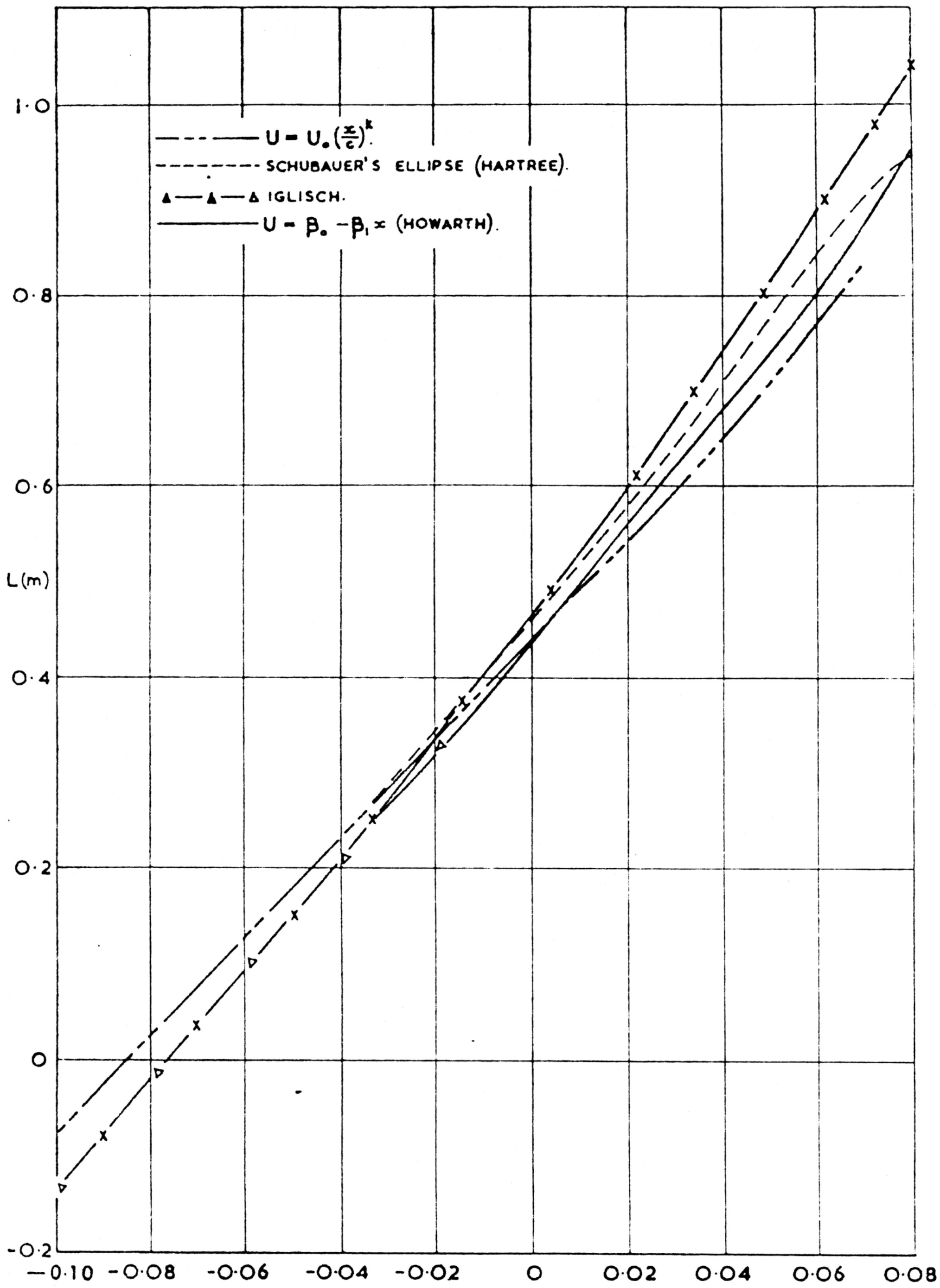
In the next section the consequences of the assumption (21) will be demonstrated, and similar consideration could be given to the quadratic assumption. The author feels that (21) is perfectly adequate.

(*ii*) Substituting $L(m) = 0.45 + 6m$ in (16), we get

$$\frac{U}{\nu}\frac{d\theta^2}{dx} = 0.45 - \frac{6U'\theta^2}{\nu}$$

which can be integrated to give

$$\theta^2 = 0.45\nu U^{-6}\int_0^x U^5 dx \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$



$U = U_0 (\frac{x}{c})^k$
SCHUBAUER'S ELLIPSE (HARTREE).
IGLISCH.
$U = \beta_0 - \beta_1 x$ (HOWARTH).
1·0
0·8
0·6
L(m)
0·4
0·2
0
-0·2
−0·10
-0·08
-0·06
-0·04
-0·02
0
0·02
0·04
0·06
0·08
m

Fig. 5

Thus (22) gives at once the distribution of θ with x. At a stagnation point, it can easily be verified that (22) gives the values of θ^2 and $\left(\frac{d\theta^2}{dx}\right)$ which have already been given in Section 6 (*iv*).

The distribution of m with x is known from that of θ^2 and then l and H can be calculated from Table I. Thus the local coefficient of skin-friction, τ_0, given by $\frac{\tau_0}{\rho U^2} = \frac{\nu}{U\theta}\, l$ and δ^* are obtained.

Notice that although (22) was derived by a consideration of the momentum equation, that equation is not now required. The functional relationship between $\frac{U}{\nu}\frac{d\theta^2}{dx}$ and $-\frac{U'\theta^2}{\nu}$ leads at once to (22) and then only the $l\,(m)$ and $H\,(m)$ functions are required to obtain the skin-friction and displacement thickness. Now $l\,(m)$ and $H\,(m)$ assumed in Table I do not give a linear form for $L\,(m)$ while (22) does, and so (22) in conjunction with Table I does not satisfy the momentum equation exactly. This is not necessarily a defect of the method, and indeed it may even give a greater " total " of accuracy on all τ_0, θ and δ^*. If for example a method gives inexact values for τ_0 and θ, then satisfaction of the momentum equation will only lead to inexact values of δ^*, whereas a different choice of $H\,(m)$, while violating the momentum equation, might nevertheless give better values of δ^*. We have therefore quite exhausted the possibilities of the momentum equation as a basis for approximate methods, and its prime and most important use must now remain as a test of the accuracy of exact and numerical solutions of the equation of motion.

(*iii*) Young and Winterbottom[5] obtained a similar equation by making an approximation to Pohlhausen's method, their equation being (in our notation) $\theta^2 = 0.47\nu U^{-6.28}\int_0^x U^{+5.28}dx$. It is clear from the foregoing analysis that this is equivalent to choosing $L\,(m) = 0.47 + 6.28m$. The gradient of this is rather too large, for when $m > 0$ this $L\,(m)$ is rather greater than that for Schubauer's ellipse, and when $m < 0$ its value becomes rather too small.

Falkner[4] also derived an equation which in our notation becomes $\theta^2 = c\nu U^{-13}\left[\int_0^x U^6 dx\right]^2$ by varying the power of U occurring in the integral until a best agreement was reached with the family of distributions given by $U = U_0\,(x/c)^k$. It is not possible to derive an equivalent function $L\,(m)$ from this, the accuracy of which it is difficult to estimate.

Tetervin[15] has given a general method of integration of the momentum equation using the assumption that H is constant and $\frac{\tau_0}{\rho U^2} = \frac{\nu}{U^2}\left(\frac{\partial u}{\partial y}\right)_{y=0} = kR_\theta^n$ in which $R_\theta = \frac{U\theta}{\nu}$ and k and n are constants. In the laminar flow case, $n=1$ (*see* Section 7 (*ii*)), and Tetervin's assumptions are equivalent to assuming $H(m)$ and $l(m)$ to be constant, which is clearly insufficient for our purposes. From (15) it is clear that this assumption gives a linear form for $L(m)$ so that the momentum equation is integrable to give a form for θ similar to (22).

8. THE DIFFICULTY OF CONTINUOUS SUCTION

This section is properly out of place in a paper which deals only with solid boundaries, but it is thought as well to explain the occurrence of a difficulty when there is a normal velocity at the boundary. Few exact solutions of flow exist with such continuous suction, and it certainly is desirable that an approximate method should be devised. An extremely brief discussion of the difficulties is given below.

Putting $y=0$ in (1) and using the forms (8) we get

$$\frac{\theta^2 U'}{\nu} = -m + \frac{\theta v_0}{\nu} l(m), \quad v_0 \text{ being the suction velocity.}$$

In Fig. 6 are drawn curves of $+m-(v_0\theta/\nu)/l(m)$ against $l(m)$, Table I being used in the calculation. (For the sake of simplicity in the argument, let us assume v_0 varies inversely as θ and that $v_0\theta/\nu = -0.3$. The argument does not lose generality by this.) Each point on the curve $\theta v_0/\nu = -0.3$ in Fig. 6 corresponds to a point on a boundary in a region of retarded flow in which separation takes place. Assume for simplicity that U' is constant. Then for separation $l(m)=0$ and $-\theta^2 U'/\nu$ must equal $+0.082$. This is the first disadvantage of the present method, for separation is given by a constant value of $-\theta^2 U'/\nu$ for *all* values of v_0, which cannot even be approximately true. Now as x increases, θ increases and also $-U'\theta^2/\nu$. Thus x increases "up" the curve $v_0\theta/\nu = -0.3$ in Fig. 6 from right to left. Eventually the maximum of this curve is reached, at $l(m)=0.08$, and $-\theta^2 U'/\nu$ must then decrease to reach the separation point. Thus the momentum equation must be so arranged that $-\theta^2 U'/\nu$ can decrease just at the right moment. Unfortunately this does not happen, and in any example a maximum value for $-U'\theta^2/\nu$ is reached beyond which the solution cannot be taken. Thus the separation value of $U'\theta^2/\nu$ is never reached and the method is unworkable.

The real reason for the breakdown is the maximum in the curve of

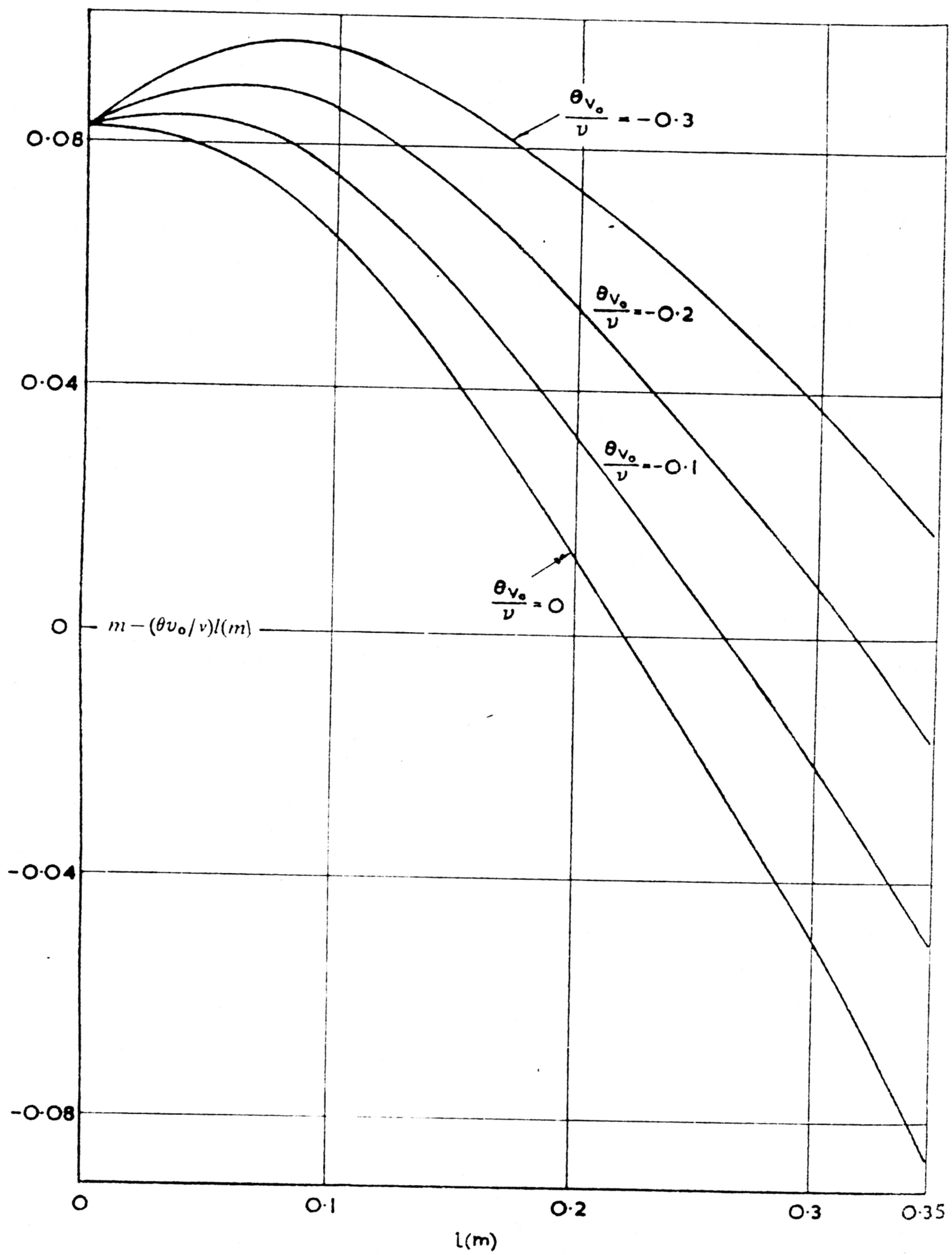

0·08
0·04
0
-0·04
-0·08
$m-(\theta v_0/\nu)l(m)$
$\frac{\theta v_0}{\nu}=-0·3$
$\frac{\theta v_0}{\nu}=-0·2$
$\frac{\theta v_0}{\nu}=-0·1$
$\frac{\theta v_0}{\nu}=0$
0
0·1
0·2
0·3
0·35
l(m)

Fig. 6

$m-(\theta v_0/\nu)\,l(m)$ against $l(m)$, which in turn is due to the zero gradient of the $(m,\ l(m))$ curve at $l(m)=0$. For a method to be applied to continuous suction, the $(m,\ l(m))$ curve must have a negative gradient at $l(m)=0$ and the solution will be valid only as long as $\theta v_0/\nu$ is such that there is no maximum of $m-(\theta v_0/\nu)\,l(m)$ against $l(m)$. Families of distributions which satisfy these conditions occur in the solution of $U=U_0\,(x/c)^k$, $m<-0.0904$ in which $\frac{v_0}{U}$ varies as $\left(\frac{x}{c}\right)^{\frac{1}{2}(k-1)}$ and a paper is being prepared to show that these distributions do, in fact, resolve the difficulty.

It is, however, very necessary to be able to estimate the accuracy of any approximate method, and for this reason an exact solution of a flow in which continuous suction is insufficient to prevent separation is urgently required. Watson[16] has used his asymptotic theory to estimate the amount of suction required to produce a constant separation profile for the flow $U=U_0\,(x/c)^k$ but it is most necessary to have not only a solution for small velocities of suction, but also for some other flows. For example, a full solution of flow for $U=\beta_0-\beta_1 x$ for various values of v_0 would be of inestimable use.

9. EXAMPLES

Two examples will be given.

(*i*) $U=\beta_0-\beta_1 x$, which has been solved by Howarth[2].

(22) gives immediately $\frac{\beta_1\theta^2}{\nu}=0.075\,\left((1-\xi)^{-6}-1\right),\ \xi=\frac{\beta_1 x}{\beta_0}$.

Also, $\frac{\beta_1}{\nu}\,\theta^2=m$ in this case, and $l(m)$ and $H(m)$ can be immediately evaluated from Table I. Table II gives the various properties of the boundary layer for several positions of x. The whole work took less than an hour.

In Fig. 7 are plotted the values of $\sqrt{\frac{\beta_1}{\nu}}\,\theta$, $\sqrt{\frac{\beta_1}{\nu}}\,\delta^*$ and $\frac{1}{U}\sqrt{\frac{\nu}{\beta_1}}\left(\frac{\partial u}{\partial y}\right)_{y=0}$ against ξ. These quantities are non-dimensional. Also on Fig. 7 are shown the exact values obtained by Howarth. It is clear that the approximate method gives good results. The separation point given by $m=0.082$ is at $\xi=0.117$, whereas the exact result gives $\xi=0.120$. Thus the agreement is good, and could obviously be made better by assigning a slightly greater value to m at separation. This arbitrariness is at once a virtue and a vice of such an approximate method.

TABLE II

ξ	$\frac{\beta_1 \theta^2}{\nu} = m$	$\sqrt{\frac{\beta_1}{\nu}}\theta$	$l(m)$	$\frac{1}{U}\sqrt{\frac{\nu}{\beta_1}}\left(\frac{\partial u}{\partial y}\right)_{y=0}$	$H(m)$	$\sqrt{\frac{\beta_1}{\nu}}\delta^*$	Velocity distribution at $\xi = 0.05$			
							Exact		Approximate	
							$\frac{u}{U}$	$\frac{y}{\theta}$	$\frac{u}{U}$	$\frac{y}{\theta}$
0	0	0	0.224	∞	2.57	0	0	0	0	0
0.0125	0.0059	0.077	0.214	2.75	2.60	0.200	0.099	0.552	0.1	0.541
0.0250	0.0123	0.111	0.203	1.78	2.63	0.292	0.205	1.104	0.2	1.051
0.0375	0.0193	0.139	0.191	1.32	2.67	0.371	0.317	1.656	0.3	1.547
0.050	0.0270	0.164	0.178	1.03	2.71	0.444	0.430	2.208	0.4	2.045
0.0625	0.0355	0.188	0.162	0.808	2.77	0.521	0.541	2.760	0.5	2.562
0.0750	0.0447	0.211	0.144	0.630	2.84	0.599	0.645	3.313	0.6	3.116
0.0875	0.0549	0.234	0.124	0.487	2.93	0.686	0.738	3.865	0.7	3.723
0.10	0.0661	0.257	0.099	0.335	3.09	0.794	0.815	4.416	0.8	4.400
0.1125	0.0785	0.280	0.052	0.163	3.48	0.974	0.923	5.521	0.9	5.163
0.120	0.0865	0.294					0.987	7.177	1.0	6.030
0.117	0.0821	0.288	0	0	3.70	1.065	1.0	8.834		

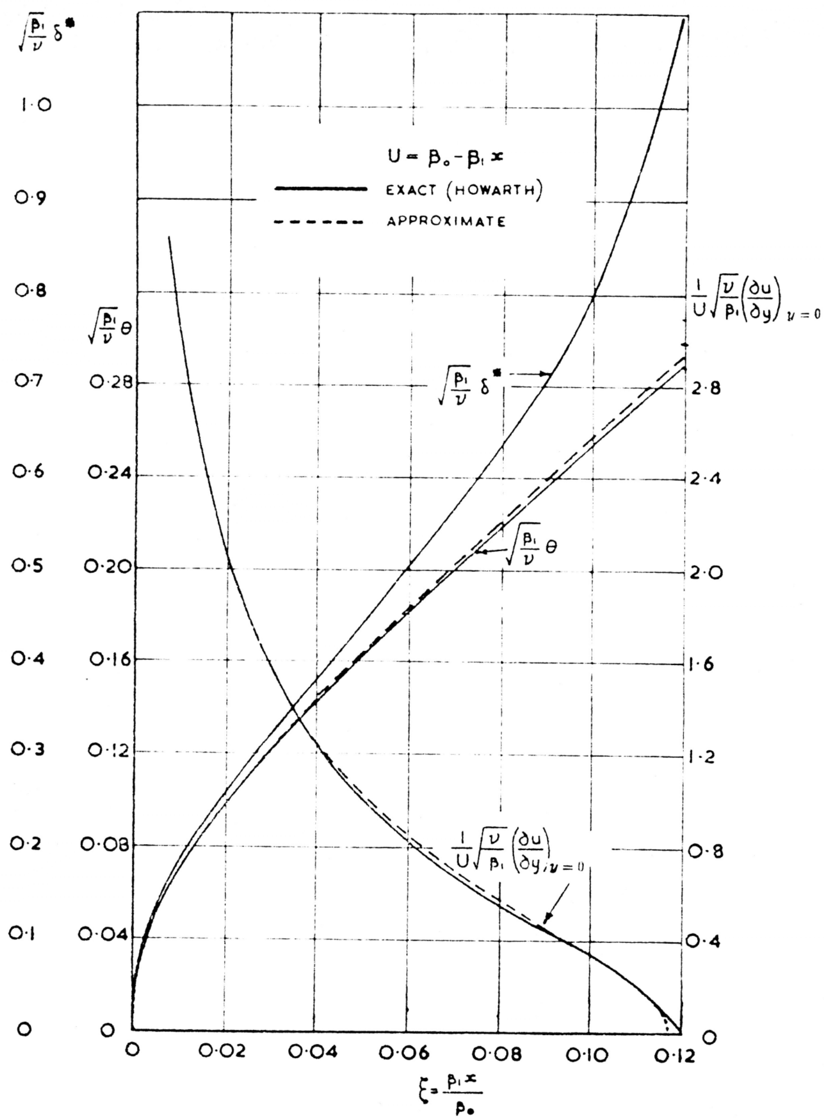


Fig. 7

As a test of the accuracy of the construction of distributions given in (ν) of Section 6, let us construct by that method the distribution at $\xi=0.05$ in this example. At this point, $l(m)=0.178$, $m=0.0270$ and $H(m)=2.71$. Hence from (20) $a_1=5.62$, $a_2=-2.39$, $a_3=2.8$ and the distribution can be quickly calculated. Table II tabulates some values, and also gives the exact distribution. Fig. 8 demonstrates the results

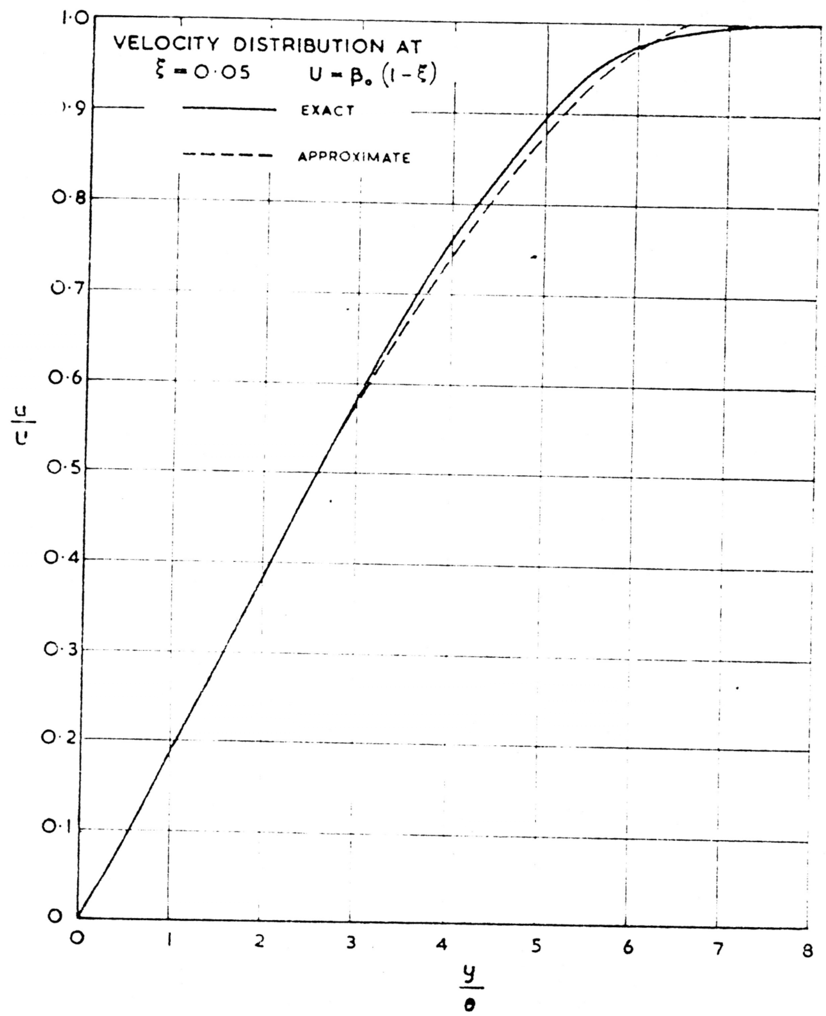


Fig. 8

and it is seen that the agreement is good, even though the approximate method gives a finite value of y/θ at $u/U = 1$. It is doubtful, however, whether this distribution is sufficiently accurate for the purpose of, for example, stability calculations.

(ii) Schubauer's observed pressure distribution

The distribution of velocity over an ellipse which Schubauer observed has been used as a test of many approximate methods. The validity of Schubauer's results has been discussed by several authors and need not concern us here since we are primarily interested in the accuracy of an approximate method of solving the boundary layer equation. Hartree[8] has solved the equation on the differential analyser, using a distribution of velocity U which he obtained from Schubauer's results; this may be regarded as an exact solution with which comparison is profitable.

The complete solution according to the simplest method of this paper is shown in Table III. The values of $\frac{U}{U_0}$, $\left(\frac{U}{U_0}\right)^2$, $\frac{2c}{U_0^2}UU'$ which are not denoted thus † are taken directly from Hartree's tables. The values of $\left(\frac{U}{U_0}\right)^2$ denoted thus † are obtained by integration of $\frac{2c}{U_0^2}UU'$ and they in turn gave the values of U/U_0 denoted thus †. In the columns marked S_1 and S_2 are tabulated the summations occurring in the use of Simpson's rule to integrate $\left(\frac{U}{U_0}\right)^5$. The approximate solution is started at $x/c=0.2$ since at that point Hartree started the analyser solution. A series solution was applied for $\frac{x}{c}<0.2$. The value of $\int_0^{0.2}\left(\frac{U}{U_0}\right)^5 d\left(\frac{x}{c}\right)$ is obtained from (22) and is given by

$$\int_0^{0.2}\left(\frac{U}{U_0}\right)^5\frac{dx}{c}=2.2\left(\frac{\theta^2U^6}{c\nu U_0^5}\right)_{x=0.2}$$

in which θ and U take the values given by Hartree. Thus the integral $\int_0^x\left(\frac{U}{U_0}\right)^5\frac{dx}{c}$ can be continued properly for $x>0.2$. From this $U_0\theta^2/(\nu c)$ is obtained, and thence m, $l(m)$ and $H(m)$ from which $\frac{c}{U_0}\sqrt{\frac{\nu}{U_0c}}\left(\frac{\partial u}{\partial y}\right)_{y=0}$ and $\sqrt{\frac{U_0c}{\nu}}\frac{\delta^*}{c}$ can be directly obtained. The whole work as set out in Table III took almost one hour. For an arbitrarily given distribution of U/U_0 whose gradient must be found, the work would take considerably longer.

In Fig. 9, these approximate results are compared graphically with the exact results. The error in θ is less than 0.5 per cent. for $x/c<1.75$. As x/c increases from 1.75, the error increases to a final value of about two per cent. at $x=1.95$. This degree of accuracy in an approximate method is very good and justifies to a remarkable extent the use of the simple form (21). For $x/c<1.5$, the error in δ^* is less than 0.5 per cent., but for $x/c>1.5$ the value of $H(m)$ is systematically too high (as can be seen at once from Fig. 4) and thus in this range the error in δ^* increases to a final value of about five per cent. The error in the skin friction,

$$\frac{c}{U_0}\sqrt{\frac{\nu}{U_0c}}\left(\frac{\partial u}{\partial y}\right)_{y=0}$$

TABLE III

$\frac{x}{c}$	$\frac{U}{U_0}$	$\left(\frac{U}{U_0}\right)^2$	$\frac{2c}{U_0^2}UU'$	$\left(\frac{U}{U_0}\right)^5$	S_1	S_2	$\frac{1}{cU_0^5}\int_0^x U^5 dx$	$\frac{U_0\theta^2}{c\nu}$	m	$\sqrt{\frac{U_0c}{\nu}}\cdot\frac{\theta}{c}$	$l(m)$	$H(m)$	$\frac{c}{U_0}\sqrt{\frac{\nu}{U_0c}}\cdot\left(\frac{\partial u}{\partial y}\right)_{y=0}$	$\sqrt{\frac{U_0c}{\nu}}\cdot\frac{\delta^*}{c}$
0.15	0.846	0.716	5.05	0.434										
0.2	0.966	0.933	3.72	0.841	0		0.0445	0.0246	−0.0474	0.157	0.295	2.37	1.82	0.372
0.25	1.045	1.092	2.72	1.246										
0.3	1.100	1.210	2.03	1.611	7.436		0.1684	0.0428	−0.0395	0.207	0.284	2.40	1.51	0.497
0.35	1.140	1.300	1.57	1.927										
0.4	1.171	1.370	1.25	2.198	8.953		0.3604	0.0630	−0.0336	0.251	0.276	2.42	1.29	0.607
0.45	1.194	1.427	1.01_5	2.431										
0.5	1.213	1.472	0.83_5	2.628	33.503		0.6029	0.0851	−0.0293	0.292	0.269	2.44	1.12	0.712
0.55	1.229	1.510	0.70	2.802										
0.6	1.242	1.542	0.59_5	2.953	50.292	0	0.8827	0.1083	−0.0259	0.329	0.263	2.45	0.99	0.806
0.7	1.262	1.592	0.41_5	3.198										
0.8	1.275	1.627	0.28	3.375		19.120	1.5200	0.1590	−0.0175	0.399	0.252	2.49	0.81	0.994
0.9	1.284	1.650	0.18	3.496										
1.0	1.290	1.664	0.10_5	3.572		40.051	2.2177	0.2166	−0.0088	0.465	0.238	2.53	0.66	1.176
1.1	1.292†	1.670†	0.06	3.603										
1.2	1.295†	1.677	0.03	3.642		61.677	2.9386	0.2804	−0.0033	0.529	0.229	2.55	0.56	1.349
1.3	1.295†	1.678†	-0.00_5	3.646										
1.4	1.294†	1.675	-0.05_5	3.630		83.533	3.6671	0.3513	0.0075	0.593	0.211	2.60	0.46	1.542
1.5	1.291†	1.666†	−0.12	3.583										
1.6	1.285†	1.650	-0.19_5	3.496		104.991	4.3824	0.4390	0.0333	0.663	0.166	2.75	0.32	1.823
1.7	1.276†	1.628†	-0.25_5	3.382										
1.8	1.265†	1.600	-0.29_5	3.238	0	125.253	5.0578	0.5557	0.0648	0.745	0.102	3.06	0.17	2.280
1.85	1.258†	1.585	−0.30	3.160	9.593		5.2177	0.5906	0.0704	0.768	0.088	3.18	0.14_5	2.442
1.90	1.253†	1.570	-0.29_5	3.089	18.967		5.3739	0.6248	0.0736	0.790	0.078	3.29	0.12_5	2.600
1.95	1.247†	1.556	-0.28_5	3.019	28.128		5.5266	0.6606	0.0754	0.813	0.069	3.36	0.10_5	2.732
2.0	1.242†	1.542	-0.26_5	2.953	37.085		5.6759	0.6961	0.0743	0.334				
2.1	1.232†	1.518	−0.21	2.839										

The first three columns of $\frac{U}{U_0}$, $\left(\frac{U}{U_0}\right)^2$, $\frac{2c}{U_0^2}UU'$ are taken directly from Hartree's report

The values denoted thus † are not given by Hartree and have been deduced from his other values

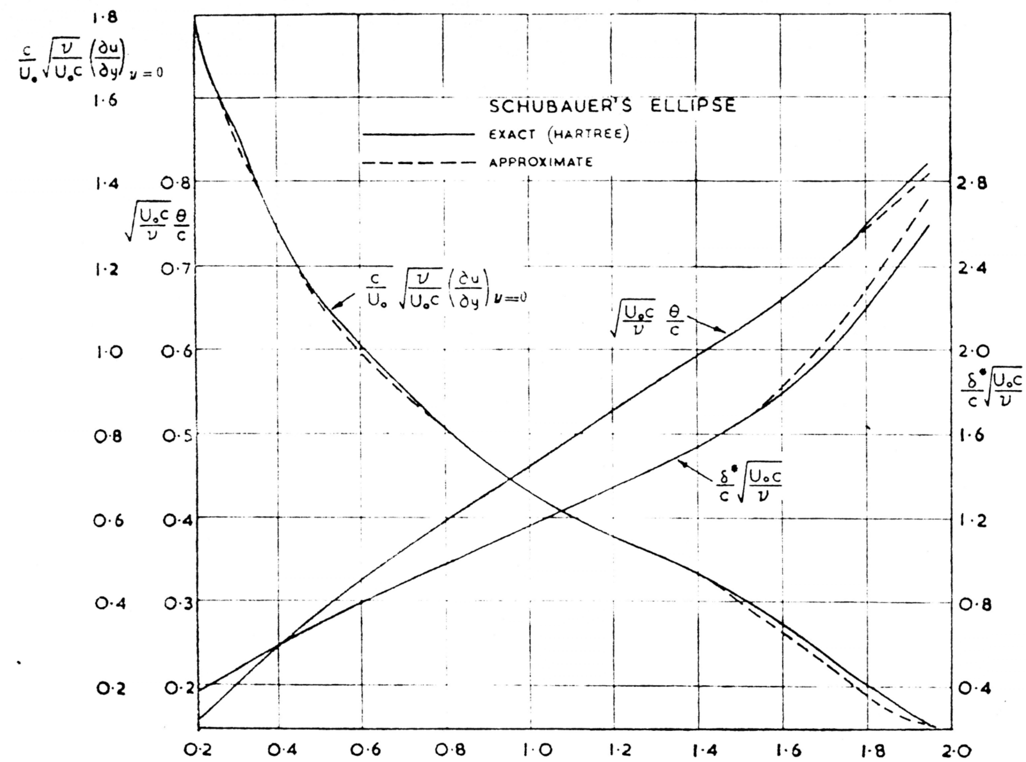


Fig. 9

is less than one per cent. for $x/c < 1.4$, but thereafter increases to a maximum of 15 per cent. at $x/c = 1.8$. At $x/c = 1.95$ however the skin friction given by the approximate method takes the exact value.

CONCLUSIONS

A method has been derived in this paper for determining approximately, but with a good degree of accuracy, the principal characteristics of the laminar boundary layer. The method is extremely simple in use and has none of the disadvantages common to nearly all other approximate methods.

Much of the paper discusses the uses of the momentum equation and a certain comparison of all known solutions of laminar boundary layer flow shows clearly the reasons for success or otherwise of an approximate method. It is also shown that no advantage is gained by making more complicated initial assumptions when using the momentum equation, the possibilities of which as the basis of an approximate method are now exhausted.

The solution of a laminar boundary layer (*see* Table I, page 265) is given by:

$$\theta^2 = 0.45U^{-6}\nu \int_0^x U^5 dx \qquad (i)$$

$$m = -\frac{U'\theta^2}{\nu} \qquad (ii)$$

$$\left(\frac{\partial u}{\partial y}\right)_{y=0} = \frac{U}{\theta} l(m) \qquad (iii)$$

$$\delta^* = \theta H(m) \qquad (iv)$$

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