AA200

Chapter 9 - Viscous flow along a wall

9.1 The no-slip condition

9.2 The equations of motion

9.3 Plane, Compressible Couette Flow (Review)

9.4 The viscous boundary layer on a wall

9.5 The laminar incompressible boundary layer

9.6 Compressible laminar boundary layers

9.7 Mapping a compressible to an incompressible boundary layer

9.8 Turbulent boundary layers

9.9 Falkner-Skan boundary layers

9.10 Transformation between flat plate and curved wall boundary layers

9.11 The Von Karman integral momentum equation

9.12 Thwaites method for integrating the Von Karman equation, discussion of M. Head's method for turbulent boundary layers

9.13 Problems

9.1 The no-slip condition

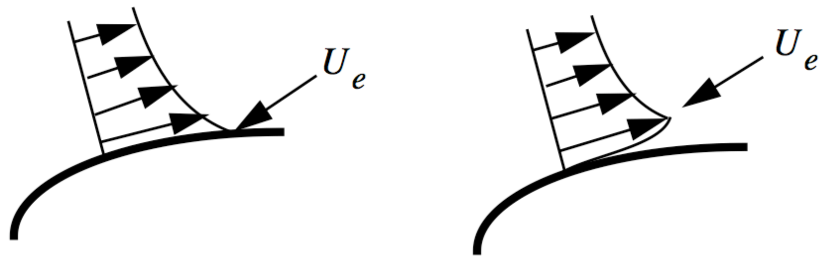


Figure 8.1 Slip versus no-slip flow near a solid surface.

Mean free path in a gas.

$$\lambda = \frac{1}{\sqrt{2}\pi n \sigma^2}$$

Slip velocity.

$$v_{slip} = C\lambda \frac{\partial U}{\partial y}$$

At ordinary temperatures and pressures the mean free path is very small.

9.2 The equations of motion

Steady 2-D flow.

$$\frac{\partial \rho U}{\partial x} + \frac{\partial \rho V}{\partial y} = 0$$

$$\frac{\partial(\rho UU + P - \tau_{xx})}{\partial x} + \frac{\partial(\rho UV - \tau_{xy})}{\partial y} = 0$$

$$\frac{\partial(\rho VU - \tau_{xy})}{\partial x} + \frac{\partial(\rho VV + P - \tau_{yy})}{\partial y} = 0$$

$$\frac{\partial(\rho hU + Q_x)}{\partial x} + \frac{\partial(\rho hV + Q_y)}{\partial y} - \left(U\frac{\partial P}{\partial x} + V\frac{\partial P}{\partial y}\right)$$

$$-\left(\tau_{xx}\frac{\partial U}{\partial x} + \tau_{xy}\frac{\partial U}{\partial y}\right) - \left(\tau_{xy}\frac{\partial V}{\partial x} + \tau_{yy}\frac{\partial V}{\partial y}\right) = 0$$

9.3 Plane, Compressible Couette Flow

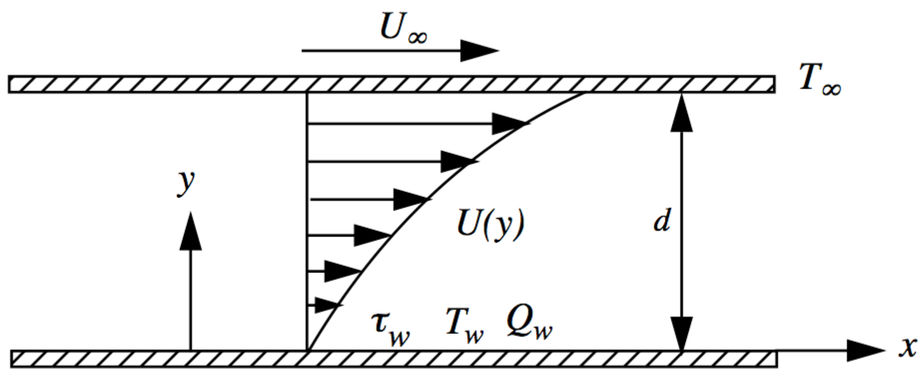


Figure 8.2 Flow produced between two parallel plates in relative motion

The upper wall moves at a velocity U_∞ while the lower wall is at rest. The temperature of the upper wall is T_∞ .

The flow is assumed to be steady and extends to plus and minus infinity in the x-direction. Therefore the velocity and temperature only depend on y.

$$\frac{\partial \rho U}{\partial x} + \frac{\partial \rho V}{\partial y} = 0$$

$$\frac{\partial(\rho UU + P - \tau_{xx})}{\partial x} + \frac{\partial(\rho UV - \tau_{xy})}{\partial y} = 0$$

$$\frac{\partial(\rho VU - \tau_{xy})}{\partial x} + \frac{\partial(\rho VV + P - \tau_{yy})}{\partial y} = 0$$

$$\frac{\partial(\rho hU + Q_x)}{\partial x} + \frac{\partial(\rho hV + Q_y)}{\partial y} - \left(U\frac{\partial P}{\partial x} + V\frac{\partial P}{\partial y}\right)$$

$$-\left(\tau_{xx}\frac{\partial U}{\partial x} + \tau_{xy}\frac{\partial U}{\partial y}\right) - \left(\tau_{xy}\frac{\partial V}{\partial x} + \tau_{yy}\frac{\partial V}{\partial y}\right) = 0$$

The equations of motion reduce to

$$\frac{\partial \tau_{xy}}{\partial y} = 0$$

$$\frac{\partial P}{\partial y} = 0$$

$$\frac{\partial (Q_y - \tau_{xy} U)}{\partial y} = 0$$

Both the pressure and shearing stress are uniform throughout the flow. The shearing stress is related to the velocity through the Newtonian constitutive relation.

$$\tau_{xy} = \mu \frac{dU}{dy} = \tau_w = constant$$

Where τ_w is the shear stress at the lower wall.

The energy equation can be integrated with respect to the velocity.

$$P_r = \frac{C_p \mu}{\kappa}$$

$$C_p T_w = C_p T_\infty + P_r \left(\frac{U_\infty^2}{2} + \frac{Q_w}{\tau_w} U_\infty \right)$$

Energy integral

In terms of the friction coefficient and Reynolds number

$$C_f = \frac{\tau_w}{\frac{1}{2}\rho_\infty U_\infty^2} \qquad R_e = \frac{\rho_\infty U_\infty d}{\mu_\infty}$$

the wall friction coefficient for an adiabatic wall is determined in terms of the Prandtl, Reynolds and Mach numbers.

$$C_f = 2\left(\frac{1 + P_r\left(\frac{\gamma - 1}{3}\right)M_\infty^2}{R_e}\right)$$

$Q_w = 0$

The Reynolds number can be expressed as

$$R_e = \frac{\rho_\infty U_\infty d}{\mu_\infty} = \frac{\frac{1}{2}\rho_\infty U_\infty^2}{\frac{1}{2}\mu_\infty \frac{U_\infty}{d}} = \frac{\textit{dynamic pressure at the upper plate}}{\textit{characteristic shear stress}}$$

9.4 The viscous boundary layer on a wall

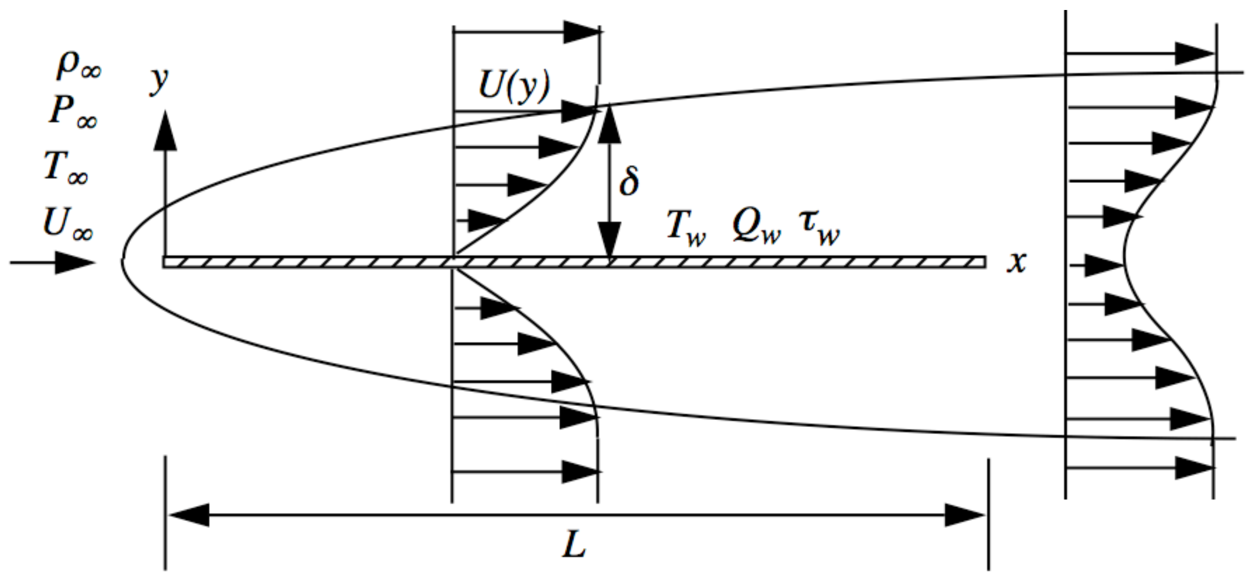


Figure 8.4 Low Reynolds number flow about a thin flat plate of length L.

$$R_{eL} = \frac{\rho_\infty U_\infty L}{\mu_\infty}$$

Reference: *Boundary Layer Theory* by Schlichting

The figure depicts the flow at low Reynolds number less than 100 or so.

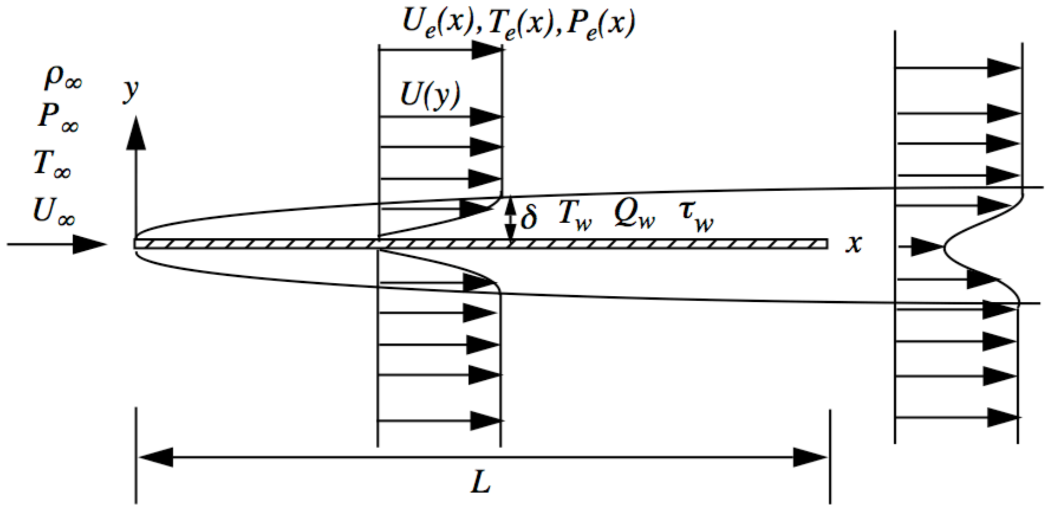


Figure 8.5 High Reynolds number flow developing from the leading edge of a flat plate of length L. R_{eL} *is several hundred or more.*

As the Reynolds number is increased to several hundred or more the velocity profile near the wall becomes quite thin and the guiding effect of the plate leads to a situation where the vertical velocity is small compared to the horizontal velocity.

$$\frac{\delta}{L} \ll 1 \qquad \frac{V}{U} \ll 1 \qquad \frac{\partial(\,)}{\partial x} \ll \frac{\partial(\,)}{\partial y} \qquad U\frac{\partial(\,)}{\partial x} \sim V\frac{\partial(\,)}{\partial y} \qquad \frac{\partial U}{\partial x} \sim -\frac{\partial V}{\partial y}$$

$$\frac{\rho_\infty {U_\infty}^2}{L} \approx \mu\frac{U_\infty}{\delta^2} \Rightarrow \frac{\delta}{L} \approx \frac{1}{(R_{eL})^{1/2}}$$

First consider the y - momentum equation.

$$\frac{\partial(\rho VU - \tau_{xy})}{\partial x} + \frac{\partial(\rho VV + P - \tau_{yy})}{\partial y} = 0$$

Using the approximations just discussed this equation reduces to.

$$\frac{\partial(P - \tau_{yy})}{\partial y} = 0$$

Integrate from the wall to the edge of the boundary layer.

$$P(x, y) = \tau_{yy}(x, y) + P_e(x)$$

Substitute for the pressure in the x - momentum equation.

$$\rho U \frac{\partial U}{\partial x} + \rho V \frac{\partial U}{\partial y} = -\frac{dP_e}{dx} + \frac{\partial}{\partial x}(\tau_{xx} - \tau_{yy}) + \frac{\partial \tau_{xy}}{\partial y}$$

The energy equation

$$\frac{\partial(\rho h U + Q_x)}{\partial x} + \frac{\partial(\rho h V + Q_y)}{\partial y} -$$

$$\left(U\frac{\partial P}{\partial x} + V\frac{\partial P}{\partial y}\right) - \left(\tau_{xx}\frac{\partial U}{\partial x} + \tau_{xy}\frac{\partial U}{\partial y}\right) -$$

$$\left(\tau_{xy}\frac{\partial V}{\partial x} + \tau_{yy}\frac{\partial V}{\partial y}\right) = 0$$

simplifies to

$$\rho U\frac{\partial h}{\partial x} + \rho V\frac{\partial h}{\partial y} + \frac{\partial Q_y}{\partial y} - U\frac{dP_e}{dx} + U\frac{\partial}{\partial x}(\tau_{xx} - \tau_{yy}) -$$

$$\frac{\partial(V\tau_{yy})}{\partial y} - \frac{\partial(U\tau_{xx})}{\partial x} - \tau_{xy}\frac{\partial U}{\partial y} = 0$$

Neglect the normal stress terms.

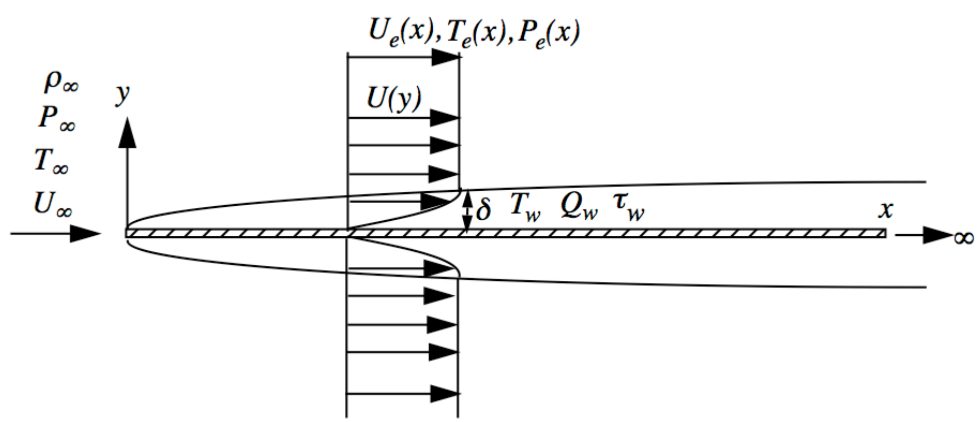
$$\frac{\partial \rho U}{\partial x} + \frac{\partial \rho V}{\partial y} = 0$$

$$\rho U \frac{\partial U}{\partial x} + \rho V \frac{\partial U}{\partial y} = -\frac{dP_e}{dx} + \frac{\partial \tau_{xy}}{\partial y}$$

$$\rho U \frac{\partial h}{\partial x} + \rho V \frac{\partial h}{\partial y} + \frac{\partial Q_y}{\partial y} - U \frac{dP_e}{dx} - \tau_{xy} \frac{\partial U}{\partial y} = 0$$

where

$$\tau_{xy} = \tau_{xy}\Big|_{laminar} + \tau_{xy}\Big|_{turbulent} = \mu \frac{\partial U}{\partial y} + \tau_{xy}\Big|_{turbulent}$$



$$R_{ex} = \frac{\rho_\infty U_\infty x}{\mu_\infty}$$

Figure 8.6 High Reynolds number flow developing from the leading edge of a semi-infinite flat plate.

Newtonian stress.

$$\tau_{ij} = 2\mu S_{ij} - \left(\frac{2}{3}\mu - \mu_v\right)\delta_{ij}S_{kk}$$

$$\tau_{xy} = \mu\left(\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x}\right) \cong \mu\frac{\partial U}{\partial y}$$

Fourier's law.

$$Q_y = -\kappa\frac{\partial T}{\partial y}$$

The laminar boundary layer equations.

$$\frac{\partial \rho U}{\partial x} + \frac{\partial \rho V}{\partial y} = 0$$

$$\rho U\frac{\partial U}{\partial x} + \rho V\frac{\partial U}{\partial y} = -\frac{dP_e}{dx} + \frac{\partial}{\partial y}\left(\mu\frac{\partial U}{\partial y}\right)$$

$$\rho U C_p\frac{\partial T}{\partial x} + \rho V C_p\frac{\partial T}{\partial y} = U\frac{dP_e}{dx} + \frac{\partial}{\partial y}\left(\kappa\frac{\partial T}{\partial y}\right) + \mu\left(\frac{\partial U}{\partial y}\right)^2$$

Measures of boundary layer thickness.

Displacement thickness

$$\delta^* = \int_0^{\delta} \left(1 - \frac{\rho U}{\rho_e U_e}\right) dy$$

Momentum thickness

$$\theta = \int_0^{\delta} \frac{\rho U}{\rho_e U_e} \left(1 - \frac{U}{U_e}\right) dy$$

9.6 The laminar incompressible boundary layer

The equations of motion reduce to

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0$$

$$U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} = -\frac{1}{\rho}\frac{dP_e}{dx} + \nu\left(\frac{\partial^2 U}{\partial y^2}\right)$$

Boundary conditions

$$U(0) = V(0) = 0 \qquad\qquad U(\delta) = U_e$$

The pressure

$$P_t = P_e(x) + \frac{1}{2}\rho U_e(x)^2 \Rightarrow \frac{1}{\rho}\frac{dP_e}{dx} = -U_e\frac{dU_e}{dx}$$

Introduce the stream function

$$U = \frac{\partial \psi}{\partial y} \qquad V = -\frac{\partial \psi}{\partial x}$$

The continuity equation is identically satisfied and the momentum equation becomes:

$$\psi_y \psi_{xy} - \psi_x \psi_{yy} = U_e \frac{dU_e}{dx} + \nu \psi_{yyy}$$

Boundary conditions

$$\psi(x,0) = 0 \qquad \psi_y(x,0) = 0 \qquad \psi_y(x,\infty) = U_e$$

The zero pressure gradient, incompressible boundary layer.

$$\psi_y \psi_{xy} - \psi_x \psi_{yy} = \nu \psi_{yyy}$$

Similarity variables

$$\psi = \left(2\nu U_\infty x\right)^{1/2} F(\alpha) \qquad \alpha = y\left(\frac{U_\infty}{2\nu x}\right)^{1/2}$$

Velocity components

$$\frac{U}{U_\infty} = F_\alpha \qquad \frac{V}{U_\infty} = \left(\frac{\nu}{2U_\infty x}\right)^{1/2}(\alpha F_\alpha - F)$$

Reynolds number is based on distance from the leading edge

$$R_{ex} = \frac{U_\infty x}{\nu}$$

Vorticity

$$\omega = \frac{\partial V}{\partial x} - \frac{\partial U}{\partial y} \cong -U_\infty \left(\frac{U_\infty}{2\nu x}\right)^{1/2} F_{\alpha\alpha}$$

Derivatives

$$\psi_{xy} = -\frac{U_\infty}{2x}\alpha F_{\alpha\alpha}$$

$$\psi_{yy} = U_\infty \left(\frac{U_\infty}{2\nu x}\right)^{1/2} F_{\alpha\alpha}$$

$$\psi_{yyy} = \frac{U_\infty^2}{2\nu x} F_{\alpha\alpha\alpha}$$

Substitute into the stream function equation and simplify

$$U_\infty F_\alpha \left(-\frac{U_\infty}{2x}\alpha F_{\alpha\alpha}\right) - U_\infty \left(\left(\frac{\nu}{2U_\infty x}\right)^{1/2} (\alpha F_\alpha - F)\right) U_\infty \left(\frac{U_\infty}{2\nu x}\right)^{1/2} F_{\alpha\alpha} = \nu \frac{U_\infty^2}{2\nu x} F_{\alpha\alpha\alpha}$$

$$-F_\alpha(\alpha F_{\alpha\alpha}) + (\alpha F_\alpha - F)F_{\alpha\alpha} = F_{\alpha\alpha\alpha}$$

$$-\alpha F_\alpha F_{\alpha\alpha} - FF_{\alpha\alpha} + \alpha F_\alpha F_{\alpha\alpha} - F_{\alpha\alpha\alpha} = 0$$

The Blasius equation

$$F_{\alpha\alpha\alpha} + FF_{\alpha\alpha} = 0$$

Boundary conditions

$$F(0) = 0 \qquad F_{\alpha}(0) = 0 \qquad F_{\alpha}(\infty) = 1$$

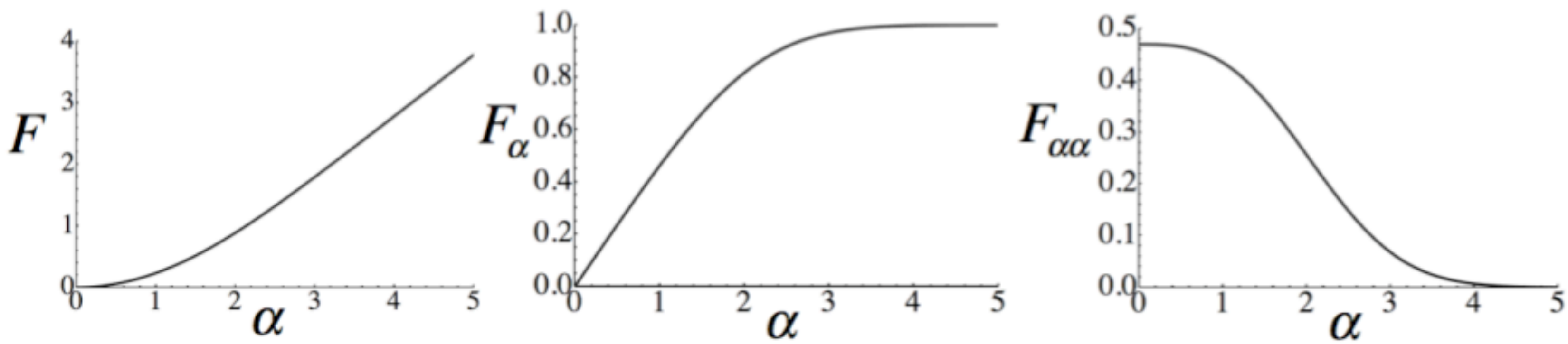


Figure 8.7 Solution of the Blasius equation (8.76) for the streamfunction, velocity and stress (or vorticity) profile in a zero pressure gradient laminar boundary layer.

Friction coefficient

$$C_f = \frac{\tau_w}{(1/2)\rho U_\infty^2} = \frac{0.664}{\sqrt{R_{ex}}}$$

$$C_f = \frac{\sqrt{2}}{\sqrt{R_{ex}}} F_{\alpha\alpha}(0)$$

Normal velocity at the edge of the layer

$$\frac{V_e}{U_\infty} = \frac{0.8604}{\sqrt{R_{ex}}}$$

Boundary layer thickness $\alpha_e = 4.906/\sqrt{2} = 3.469$

$$\frac{\delta_{0.99}}{x} = \frac{4.906}{\sqrt{R_{ex}}} \qquad \frac{\delta^*}{x} = \frac{1.7208}{\sqrt{R_{ex}}} \qquad \frac{\theta}{x} = \frac{0.664}{\sqrt{R_{ex}}}$$

Boundary layer shape factor

$$H = \frac{\delta^*}{\theta} = 2.5916$$

Let $\tau = F_{\alpha\alpha}$

The Blasius equation can be expressed as

$$\frac{d\tau}{\tau} = -F d\alpha$$

$$\frac{\tau}{\tau_w} = e^{-\int_0^\alpha F d\alpha}$$

Let $F(\alpha) = \alpha - G(\alpha)$ Then $\lim_{\alpha \to \infty} G(\alpha) = C_1$

$$\left.\frac{\tau}{\tau_w}\right|_{\alpha > \alpha_e} = e^{-\int_0^\alpha (\alpha - G(\alpha)) d\alpha} = C_2 e^{C_1 \alpha - \frac{\alpha^2}{2}}$$

Vorticity at the edge of the layer decays exponentially with distance from the wall. This support the approach where we divide the flow into separate regions of rotational and irrotational flow.

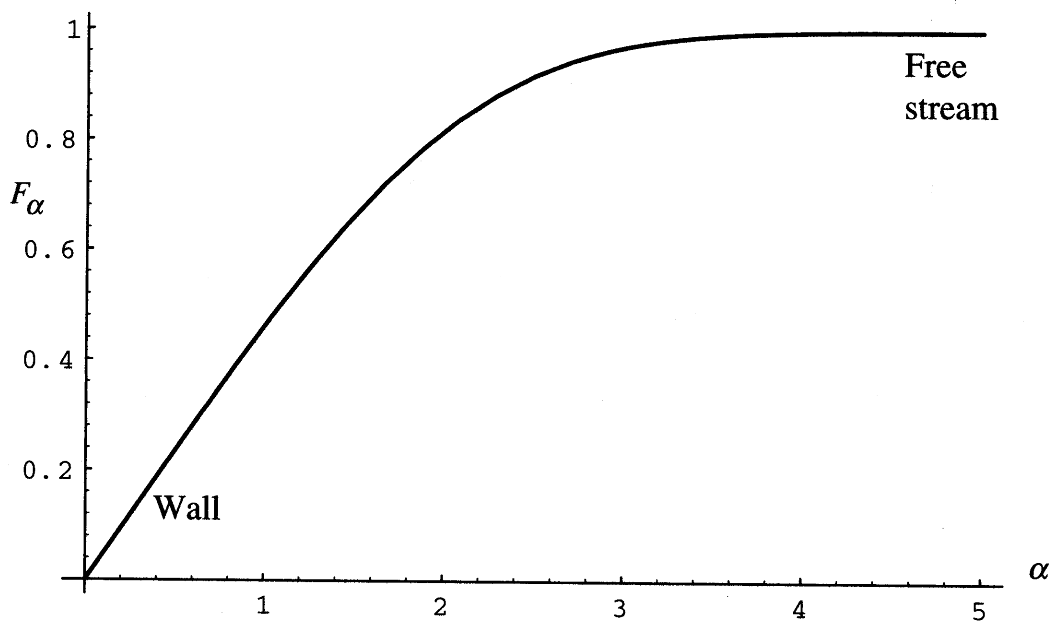


Fig. 10.4. The Blasius velocity profile.

$$C_{f0} = \frac{\tau_0}{\frac{1}{2}\rho U_e^2} = \frac{0.664}{\sqrt{Re}}.$$

Numerical solution

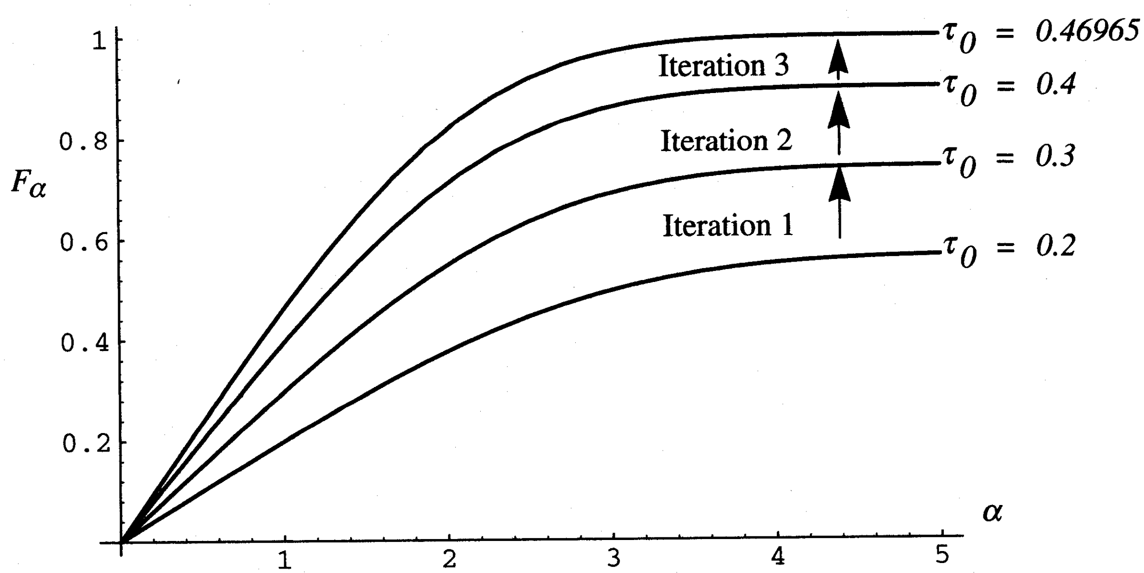


Fig. 10.5. Iteration process leading to the correct match with the free-stream boundary condition $\lim_{\alpha\to\infty} F_\alpha = 1$.

The Blasius equation is invariant under a dilation group and this group can be used to generate the solution in one step!

$$\tilde{\alpha} = e^{b}\alpha,$$
$$\tilde{F} = e^{-b}F,$$
$$\tilde{F}_{\tilde{\alpha}} = e^{-2b}F_{\alpha},$$
$$\tilde{F}_{\tilde{\alpha}\tilde{\alpha}} = e^{-3b}F_{\alpha\alpha}.$$

$$1 = e^{-2b}(0.566067) \quad \Rightarrow \quad b = -0.284557.$$

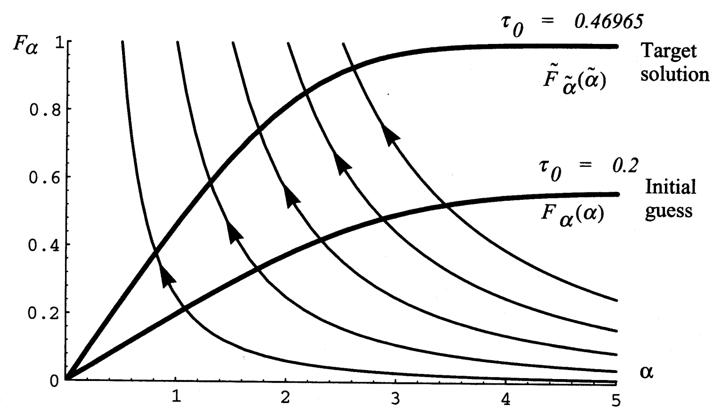


Fig. 10.6. Mapping of an initial guess to the correct solution along the pathlines of the dilation group of the Blasius equation.

$$\tilde{F}_{\tilde{\alpha}\tilde{\alpha}}[0] = e^{-3b}(0.2) \quad \Rightarrow \quad \tilde{F}_{\tilde{\alpha}\tilde{\alpha}}[0] = 0.46965$$

Another perspective: the dilational symmetry of the problem

$$\psi_y \psi_{xy} - \psi_x \psi_{yy} = \nu \psi_{yyy}$$

$$\psi(x,0) = 0 \qquad \psi_y(x,0) = 0 \qquad \psi_y(x,\infty) = U_e$$

Transform the governing equation

$$\tilde{x} = e^a x \qquad \tilde{y} = e^b y \qquad \tilde{\psi} = e^c \psi$$

$$\tilde{\psi}_{\tilde{y}} \tilde{\psi}_{\tilde{x}\tilde{y}} - \tilde{\psi}_{\tilde{x}} \tilde{\psi}_{\tilde{y}\tilde{y}} - \nu \tilde{\psi}_{\tilde{y}\tilde{y}\tilde{y}} = e^{2c-a-2b} \psi_y \psi_{xy} - e^{2c-a-2b} \psi_x \psi_{yy} - \nu e^{c-3b} \psi_{yyy} = 0$$

The equation is invariant if and only if

$$2c - a - 2b = c - 3b.$$

$$\tilde{x} = e^a x \qquad \tilde{y} = e^b y \qquad \tilde{\psi} = e^{a-b} \psi$$

Transform boundary curves and boundary functions

$$\tilde{y} = 0 \quad \Rightarrow \quad e^{b} y = 0 \quad \Rightarrow \quad y = 0.$$

At the wall

$$\tilde{\psi}\left(\tilde{x},0\right) = 0\Big|_{all\ \tilde{x}} \Rightarrow e^{a-b}\psi\left(e^{a}x,0\right) = 0 \Rightarrow \psi\left(x,0\right) = 0\Big|_{all\ x}$$

$$\tilde{\psi}_{\tilde{y}}\left(\tilde{x},0\right) = 0\Big|_{all\ \tilde{x}} \Rightarrow e^{a-2b}\psi_{y}\left(e^{a}x,0\right) = 0 \Rightarrow \psi_{y}\left(x,0\right) = 0\Big|_{all\ x}$$

At $y \longrightarrow \infty$

$$\tilde{\psi}_{\tilde{y}}\left(\tilde{x},\infty\right) = U_{\infty}\Big|_{all\ \tilde{x}} \Rightarrow e^{a-2b}\psi_{y}\left(e^{a}x,\infty\right) = U_{\infty}\Big|_{all\ x}$$

The freestream boundary condition is invariant if and only if $\underline{a = 2b}$

The governing equations and boundary conditions are invariant under the group:

$$\tilde{x} = e^{2b}x \qquad \tilde{y} = e^{b}y \qquad \tilde{\psi} = e^{b}\psi$$

The infinitesimal transformation. Expand near $b = 0$

$$\xi = 2x \qquad \zeta = y \qquad \eta = \psi$$

Characteristic equations

$$\frac{dx}{2x} = \frac{dy}{y} = \frac{d\psi}{\psi}$$

Invariants

$$\alpha = \frac{y}{\sqrt{x}} \qquad F = \frac{\psi}{\sqrt{x}}$$

We can expect the solution to be of the form

$$\psi = \sqrt{x}F(\alpha)$$

Since the governing equations and boundary conditions are invariant under the group we can expect that the solution will also be invariant under the group.

9.7 Falkner-Skan laminar boundary layers

$$\psi_y \psi_{xy} - \psi_x \psi_{yy} - U_e \frac{dU_e}{dx} - \nu \psi_{yyy} = 0 . \tag{9.111}$$

Free stream velocity .

$$U_e = M x^{\beta} \tag{9.112}$$

$$\hat{M} = L^{1-\beta} / T . \tag{9.113}$$

Substitute.

$$\psi_y \psi_{xy} - \psi_x \psi_{yy} - \beta M^2 x^{(2\beta - 1)} - \nu \psi_{yyy} = 0 \tag{9.114}$$

Apply a three-parameter dilation group to the equation.

$$\tilde{x} = e^{a}x \qquad \tilde{y} = e^{b}y \qquad \tilde{\psi} = e^{c}\psi \tag{9.115}$$

$$\begin{gathered}\tilde{\psi}_{\tilde{y}}\tilde{\psi}_{\tilde{x}\tilde{y}} - \tilde{\psi}_{\tilde{x}}\tilde{\psi}_{\tilde{y}\tilde{y}} - \beta M^{2}\tilde{x}^{2\beta-1} - \nu\tilde{\psi}_{\tilde{y}\tilde{y}\tilde{y}} = \\ e^{2c-a-2b}(\psi_{y}\psi_{xy} - \psi_{x}\psi_{yy}) - e^{(2\beta-1)a}(\beta M^{2}x^{2\beta-1}) - e^{c-3b}(\nu\tilde{\psi}_{\tilde{y}\tilde{y}\tilde{y}}) = 0\end{gathered} \quad . \tag{9.116}$$

For invariance we require the parameters to be related as follows

$$2c - a - 2b = c - 3b = (2\beta - 1)a\,. \tag{9.117}$$

Boundary functions and boundary curves must also be invariant.

$$\begin{gathered}\tilde{y} = e^{b}y = 0 \Rightarrow y = 0 \\ \tilde{\psi}(\tilde{x}, 0) = e^{c}\psi(e^{a}x, 0) = 0 \Rightarrow \psi(x, 0) = 0 \\ \tilde{\psi}_{\tilde{y}}(\tilde{x}, 0) = e^{c-b}\psi_{y}(e^{a}x, 0) = 0 \Rightarrow \psi_{y}(x, 0) = 0\end{gathered} \tag{9.118}$$

Free stream boundary condition.

$$\tilde{\psi}_{\tilde{y}}(\tilde{x}, \infty) = e^{c-b}\psi_y(e^a x, \infty) = e^{\beta a} M x^{\beta} \tag{9.119}$$

For invariance

$$c - b = \beta a \tag{9.120}$$

The group that leaves the problem as a whole invariant is

$$\tilde{x} = e^{\frac{2}{1-\beta}b} x \qquad \tilde{y} = e^{b} y \qquad \tilde{\psi} = e^{\frac{1+\beta}{1-\beta}b} \psi \tag{9.121}$$

The solution should be invariant under the same group

$$\frac{\psi}{x^{\left(\frac{1+\beta}{2}\right)}} = F\left(\frac{y}{x^{\left(\frac{1-\beta}{2}\right)}}\right) \tag{9.122}$$

In summary

$$\psi_y \psi_{xy} - \psi_x \psi_{yy} - U_e \frac{dU_e}{dx} - \nu \psi_{yyy} = 0. \tag{9.111}$$

Allow for a virtual origin in x

$$U_e = M(x + x_0)^{\beta}$$

$$\hat{M} = L^{1-\beta}/T .$$

Dimensionless similarity variables

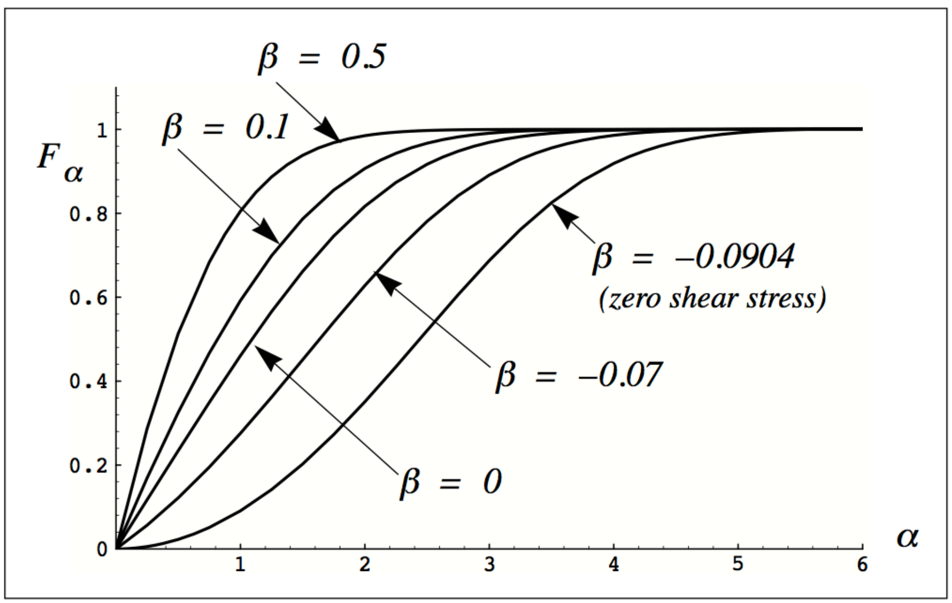
$$\left.\begin{aligned} \alpha &= \left(\frac{M}{2\nu}\right)^{\frac{1}{2}} \frac{y}{(x + x_0)^{(1-\beta)/2}} \\ F &= \frac{\psi}{(x + x_0)^{(1+\beta)/2}(2\nu M)^{1/2}} \end{aligned}\right\}. \tag{9.123}$$

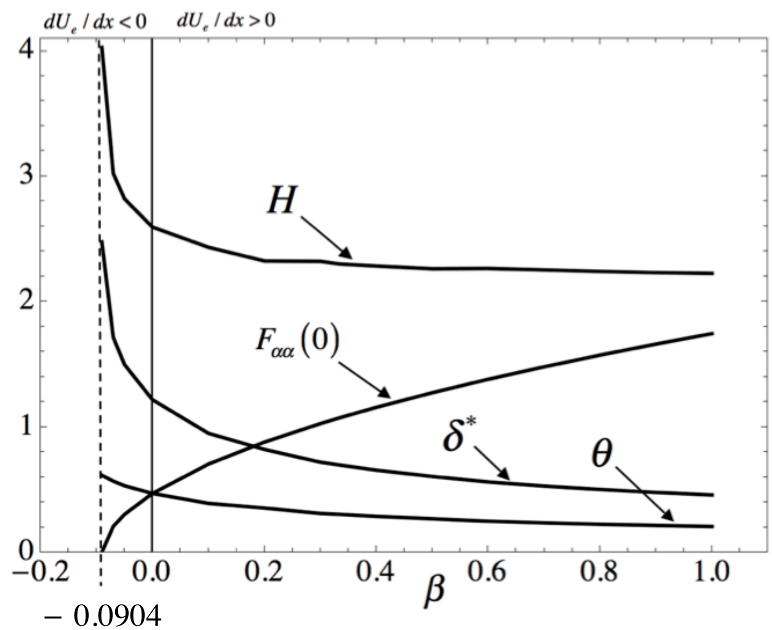
$$(x + x_0)^{2\beta - 1}(F_\alpha((1 + \beta)F - (1 - \beta)\alpha F_\alpha)_\alpha - F_{\alpha\alpha}((1 + \beta)F - (1 - \beta)\alpha F_\alpha) - 2\beta - F_{\alpha\alpha\alpha}) = 0 \tag{9.124}$$

The Falkner-Skan equation.

$$F_{\alpha\alpha\alpha} + (1 + \beta)FF_{\alpha\alpha} - 2\beta(F_\alpha)^2 + 2\beta = 0 \tag{9.125}$$

$$F[0] = 0 \quad ; \qquad F_\alpha[0] = 0 \quad ; \qquad F_\alpha[\infty] = 1 \tag{9.126}$$





Homework 4

AA200A Homework 4 2013 -2014

Due Tuesday April 29

Read Chapter 9

Problem - Consider a zero pressure gradient laminar boundary layer on a flat plate with mass transfer at the wall. Let the vertical component of velocity at the wall be a power law of the form

$$V(x,0) = Mx^{\beta}$$

Identify a value of β that leads to a similarity solution. Numerically solve this case and show how the blowing affects the skin friction coefficient at the wall.

Chapter 9, Problem 6

9.5 The von Karman integral equation

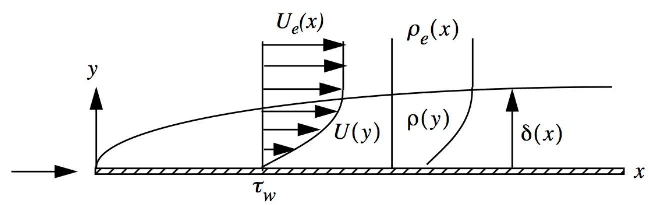


Figure 9.7 Boundary layer velocity and density profiles.

$$\frac{\partial \rho U}{\partial x} + \frac{\partial \rho V}{\partial y} = 0$$

$$\frac{\partial \rho U^2}{\partial x} + \frac{\partial \rho U V}{\partial y} = -\frac{dP_e}{dx} + \frac{\partial \tau_{xy}}{\partial y} \tag{9.64}$$

Integrate the boundary layer equations with respect to y

$$\int_0^{\delta(x)} \left(\frac{\partial \rho U}{\partial x}\right) dy + \int_0^{\delta(x)} \left(\frac{\partial \rho V}{\partial y}\right) dy = 0$$

$$\int_0^{\delta(x)} \left(\frac{\partial \rho U^2}{\partial x}\right) dy + \int_0^{\delta(x)} \left(\frac{\partial \rho U V}{\partial y}\right) dy = -\int_0^{\delta(x)} \left(\frac{dP_e}{dx}\right) dy + \int_0^{\delta(x)} \left(\frac{\partial \tau_{xy}}{\partial y}\right) dy \tag{9.65}$$

$$\frac{d\theta}{dx} + (2\theta + \delta^*)\frac{1}{U_e}\frac{dU_e}{dx} = \frac{C_f}{2} \tag{9.80}$$

Shape factor $$H = \frac{\delta^*}{\theta} \tag{9.81}$$

$$\boxed{\frac{d\theta}{dx} + (2 + H)\frac{\theta}{U_e}\frac{dU_e}{dx} = \frac{C_f}{2}} \tag{9.82}$$

9.8 Thwaites' method for approximate calculation of boundary layer parameters.

From the momentum equation

$$\left.\frac{\partial^2 U}{\partial y^2}\right|_{y=0} = -\frac{U_e}{\nu}\frac{dU_e}{dx} \tag{9.151}$$

From the von Karman equation

$$\left.\frac{\partial U}{\partial y}\right|_{y=0} = (2+H)\theta\frac{U_e}{\nu}\frac{dU_e}{dx} + \frac{U_e^2}{\nu}\frac{d\theta}{dx} \tag{9.152}$$

Nondimensionalize using θ and U_e

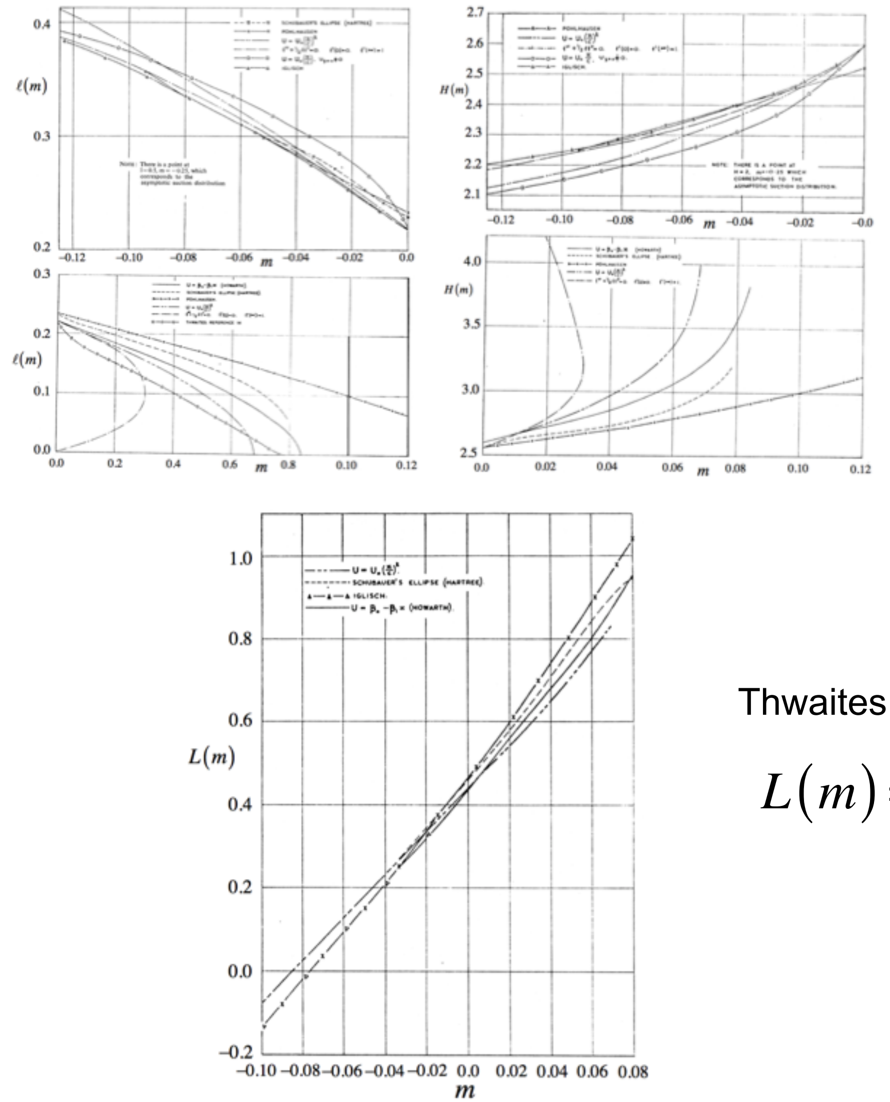
$$\begin{aligned}\left(\frac{\theta^2}{U_e}\right)\left.\frac{\partial^2 U}{\partial y^2}\right|_{y=0} &= -\frac{\theta^2}{\nu}\frac{dU_e}{dx} \\ \left(\frac{\theta}{U_e}\right)\left.\frac{\partial U}{\partial y}\right|_{y=0} &= (2+H)\frac{\theta^2}{\nu}\frac{dU_e}{dx} + \frac{U_e}{2\nu}\frac{d\theta^2}{dx}\end{aligned} \tag{9.153}$$

Define

$$m = \left(\frac{\theta^2}{U_e}\right)\left.\frac{\partial^2 U}{\partial y^2}\right|_{y=0} \qquad l(m) = \left(\frac{\theta}{U_e}\right)\left.\frac{\partial U}{\partial y}\right|_{y=0}. \tag{9.154}$$

Thwaites argued that there should exist a universal function relating m and $l(m)$.

$$\frac{U_e}{\nu}\frac{d\theta^2}{dx} = 2((2+H)m + l(m)) = L(m) \tag{9.155}$$



Thwaites suggests using

$$L(m) = 0.45 + 6m$$

Figure 9.14 Data collected by Thwaites on skin friction, $l(m)$, shape factor $H(m)$ and $L(m)$ for a variety of boundary layer solutions.

Thwaites functions can be calculated explicitly for the Falkner-Skan boundary layers

$$m = F_{\alpha\alpha\alpha}(0)\left(\int_0^{\alpha} F_{\alpha}(1-F_{\alpha})d\alpha\right)^2 = -2\beta\left(\int_0^{\alpha} F_{\alpha}(1-F_{\alpha})d\alpha\right)^2$$

$$l(m) = F_{\alpha\alpha}(0)\int_0^{\alpha} F_{\alpha}(1-F_{\alpha})d\alpha \qquad (9.157)$$

$$H(m) = \frac{\int_0^{\alpha}(1-F_{\alpha})d\alpha}{\int_0^{\alpha} F_{\alpha}(1-F_{\alpha})d\alpha}$$

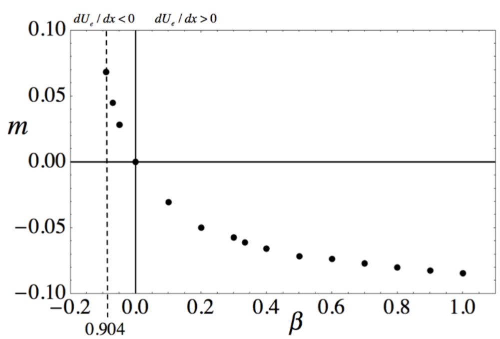


Figure 9.15 The variable m defined in (9.154) versus the free stream velocity exponent β for Falkner-Skan boundary layers.

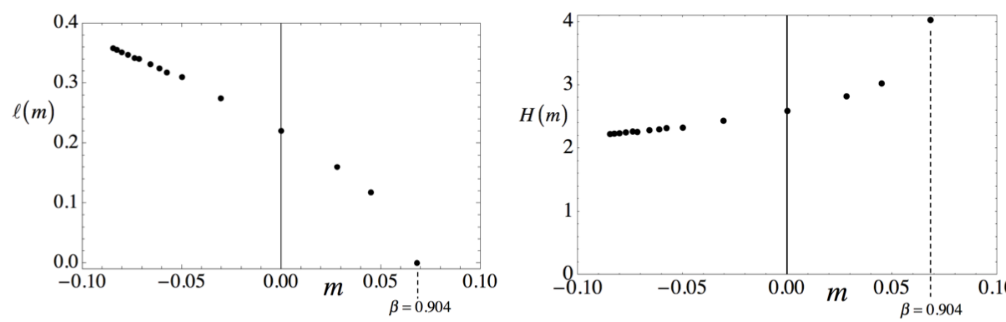


Figure 9.16 Thwaites functions for the Falkner-Skan solutions (9.157).

N. Curle adjusted Thwaites' functions slightly especially near separation.

TABLE 5

Universal functions for Thwaites's method

m	$l(m)$	$H(m)$	m	$l(m)$	$H(m)$
−0·25	0·500	2·00	0·040	0·153	2·81
−0·20	0·463	2·07	0·048	0·138	2·87
−0·14	0·404	2·18	0·056	0·122	2·94
−0·12	0·382	2·23	0·060	0·113	2·99
−0·10	0·359	2·28	0·064	0·104	3·04
−0·080	0·333	2·34	0·068	0·095	3·09
−0·064	0·313	2·39	0·072	0·085	3·15
−0·048	0·291	2·44	0·076	0·072	3·22
−0·032	0·268	2·49	0·080	0·056	3·30
−0·016	0·244	2·55	0·084	0·038	3·39
0	0·220	2·61	0·086	0·027	3·44
+0·016	0·195	2·67	0·088	0·015	3·49
0·032	0·168	2·75	0·090	0	3·55

Figure 9.18 Curle's functions for Thwaites' method.

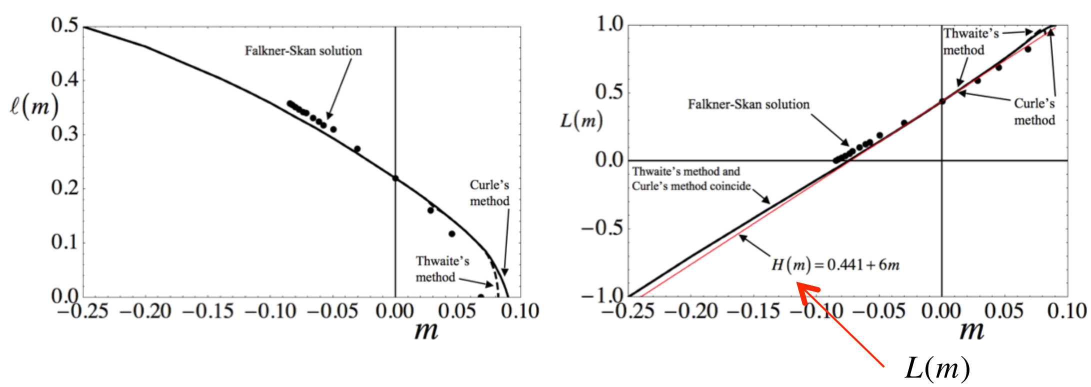


Figure 9.17 Comparison between Curle's functions and Thwaites' functions.

Several researchers of the era suggest using

$$L(m) = 0.441 + 6m\,. \tag{9.158}$$

which is consistent with the friction coefficient for the Blasius case

The von Karman equation becomes

$$U_e \frac{d}{dx}\left(\frac{\theta^2}{\nu}\right) = 0.441 - 6\left(\frac{\theta^2}{\nu}\right)\frac{dU_e}{dx} \tag{9.159}$$

which integrates to

$$\theta^2 = \frac{0.441\nu}{U_e^6}\int_0^x U_e(x')^5 dx' \tag{9.160}$$

The procedure for applying Thwaites' method is as follows.

1) Given $U_e(x)$, use (9.160) to determine $\theta^2(x)$.

At a given x:

2) The parameter m is determined from (9.154) and (9.151).

$$m = -\frac{\theta^2}{\nu}\frac{dU_e}{dx} \tag{9.161}$$

3) The functions $l(m)$ and $H(m)$ are determined from the data in Figure 9.18.

4)The friction coefficient is determined from

$$C_f = \frac{2\nu}{U_e\theta}l(m). \tag{9.162}$$

5) The displacement thickness $\delta^*(m)$ is determined from $H(m)$.

The process is repeated while progressing along the wall to increasing values of x. Separation of the boundary layer is assumed to have occurred if a point is reached where $l(m) = 0$.

The key references used in this section are

1) Thwaites, B. 1948 Approximate calculations of the laminar boundary layer, VII International Congress of Applied Mechanics, London. Also Aeronautical Quarterly Vol. 1, page 245, 1949.

2) Curle, N. 1962 The *Laminar Boundary Layer Equations*, Clarendon Press.

Example - surface velocity from the potential flow about a circular cylinder.

$$\frac{U_e}{U_\infty} = 2Sin\left(\frac{x}{R}\right) \tag{9.163}$$

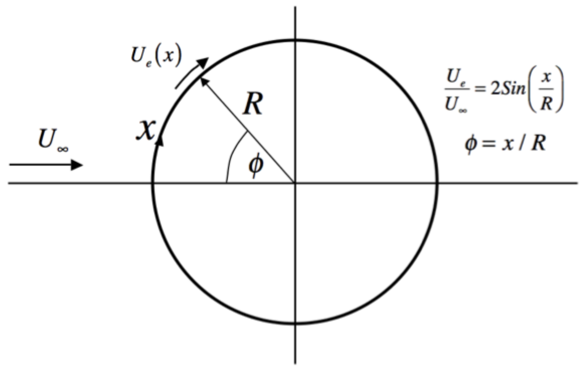


Figure 9.19 Example for Thwaites' method.

$$\left(\frac{\theta}{R}\right)^2 R_e = \frac{0.441}{Sin^6(\phi)}\int_0^{\phi} Sin^5(\phi')d\phi' \tag{9.164}$$

$$R_e = \frac{U_\infty 2R}{\nu} \tag{9.165}$$

Thwaites' method gives a finite momentum thickness at the forward stagnation point. This is useful in a wing leading edge calculation.

$$\lim_{\phi \to 0}\left(\frac{\theta}{R}\right)^2 R_e = \frac{0.441}{\phi^6}\int_0^{\phi} \phi'^5 d\phi' = \frac{0.441}{6} \tag{9.166}$$

The parameter m.

$$m = -\frac{\theta^2}{\nu}\frac{dU_e}{dx} = -\left(\frac{\theta}{R}\right)^2 R_e \frac{d}{d\phi}\left(\frac{U_e}{U_\infty}\right) = \frac{0.882\,Cos(\phi)}{Sin^6(\phi)}\int_0^{\phi} Sin^5(\phi')d\phi' \quad (9.167)$$

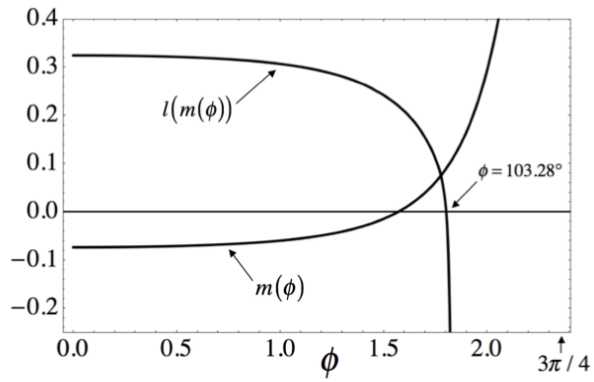


Figure 9.20 Thwaites' functions for the freestream distribution (9.163).

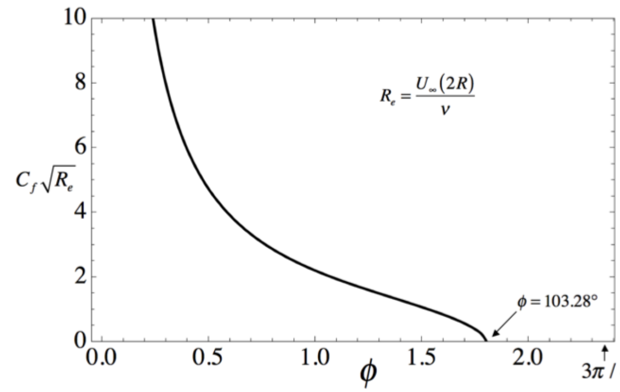


Figure 9.21 Friction coefficient for the freestream distribution (9.163).

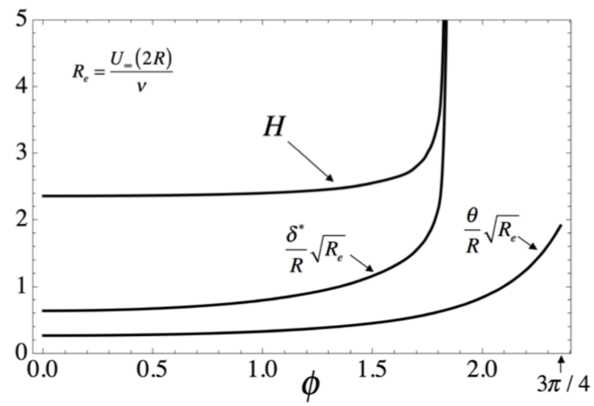


Figure 9.22 Boundary layer thicknesses and shape factor for the freestream distribution (9.163).

9.9 Compressible laminar boundary layers

The boundary layer admits an energy integral very similar to the one for Couette flow.

$$\frac{\partial \rho U}{\partial x} + \frac{\partial \rho V}{\partial y} = 0$$

$$\rho U \frac{\partial U}{\partial x} + \rho V \frac{\partial U}{\partial y} = -\frac{dP_e}{dx} + \frac{\partial}{\partial y}\left(\mu \frac{\partial U}{\partial y}\right)$$

$$\rho U C_p \frac{\partial T}{\partial x} + \rho V C_p \frac{\partial T}{\partial y} = U \frac{dP_e}{dx} + \frac{\partial}{\partial y}\left(\kappa \frac{\partial T}{\partial y}\right) + \mu \left(\frac{\partial U}{\partial y}\right)^2$$

Let $T = T(U)$. Substitute into the energy equation. Use the momentum equation to simplify and introduce the Prandtl number

$$-\frac{dP}{dx}\left(C_p \frac{dT}{dU} + U\right) + \frac{dT}{dU}\left(\frac{P_r - 1}{P_r}\right) \frac{\partial}{\partial y}\left(\mu \frac{\partial U}{\partial y}\right) + \left(\kappa \frac{d^2 T}{dU^2} + \mu\right)\left(\frac{\partial U}{\partial y}\right)^2 = 0$$

Adiabatic wall, Prandtl number equals one.

$$T_{wa} - T = \frac{1}{2C_p}U^2$$

$$T_{wa} = T_e + \frac{1}{2C_p}U_e^{\ 2}$$

$$\frac{T_{wa}}{T_e} = 1 + \left(\frac{\gamma - 1}{2}\right)M_e^{\ 2} = \frac{T_{te}}{T_e}$$

Stagnation temperature is constant through the boundary layer.

Non-adiabatic wall, zero pressure gradient, Prandtl number equals one.

$$T_w = T_\infty + \frac{1}{2C_p}U_\infty^{\ 2} + \frac{Q_w}{\tau_w C_p}U_\infty$$

$$C_f = 2S_t$$

$$S_t = \frac{Q_w}{\rho_\infty U_\infty C_p (T_w - T_{wa})}$$

$$\frac{T - T_w}{T_\infty} = \left(1 - \frac{T_w}{T_\infty}\right)\frac{U}{U_\infty} + \left(\frac{U_\infty^2}{2C_p T_\infty}\right)\frac{U}{U_\infty}\left(1 - \frac{U}{U_\infty}\right)$$

9.10 Mapping a compressible to an incompressible boundary layer

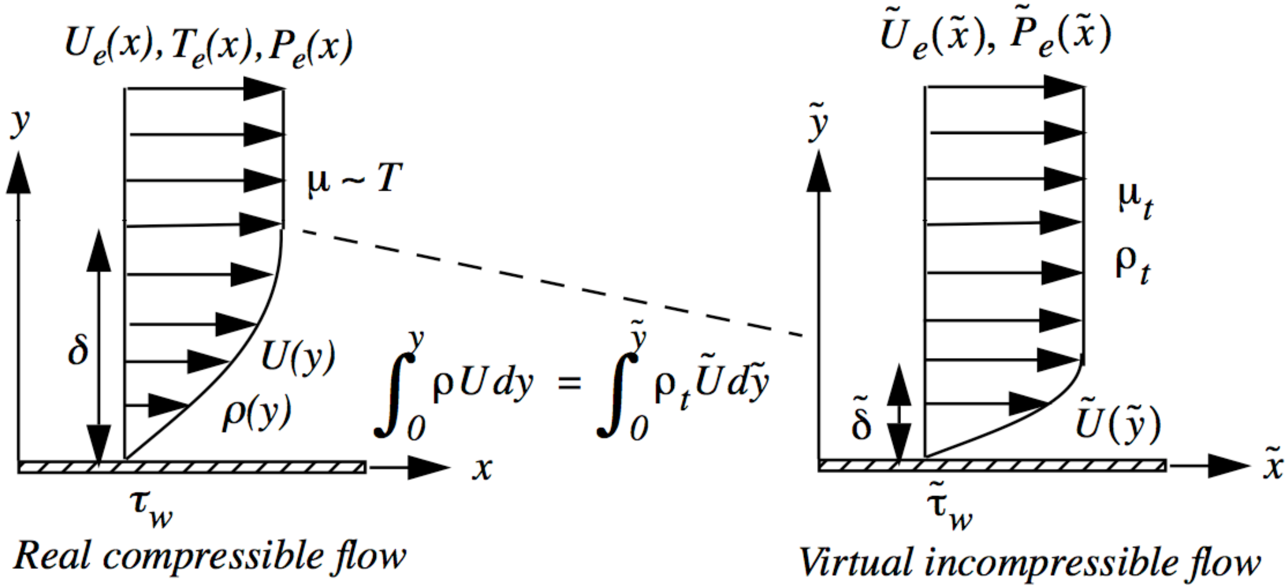


Figure 8.8 Mapping of a compressible flow to an incompressible flow.

Flow at the edge of the compressible boundary layer is isentropic.

$$\rho_t = \rho_e\left(1 + \left(\frac{\gamma - 1}{2}\right)M_e^2\right)^{1/(\gamma - 1)} \qquad \frac{P_t}{P_e} = \left(\frac{T_t}{T_e}\right)^{\gamma/(\gamma - 1)} = \left(\frac{a_t}{a_e}\right)^{(2\gamma)/(\gamma - 1)}$$

Assume viscosity is linearly proportional to temperature

$$\frac{\mu}{\mu_t} = \sigma\frac{T}{T_t}$$

$$\frac{\mu}{\mu_t} = \left(\frac{T}{T_t}\right)^{3/2}\left(\frac{T_t + T_S}{T + T_S}\right)$$

$$\sigma = \left(\frac{T_w}{T_t}\right)^{1/2}\left(\frac{T_t + T_S}{T_w + T_S}\right)$$

If $P_r = 1$ then $\sigma = 1$

Viscosity of the virtual flow is the viscosity of the gas evaluated at the stagnation temperature of the gas.

T_S - Sutherland reference temperature, 110K for Air.

Continuity and momentum equations

$$\frac{\partial \rho U}{\partial x} + \frac{\partial \rho V}{\partial y} = 0$$

$$U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} = -\frac{1}{\rho}\frac{dP_e}{dx} + \frac{1}{\rho}\frac{\partial}{\partial y}\left(\mu\frac{\partial U}{\partial y}\right) + \frac{1}{\rho}\frac{\partial \tau_{xy}}{\partial y}$$

Transformation of coordinates between the real and virtual flow

$$\tilde{x} = \sigma\int_0^x \left(\frac{P_e}{P_t}\left(\frac{a_e}{a_t}\right)\right)dx' = f(x)$$

$$\tilde{y} = \left(\frac{a_e}{a_t}\right)\int_0^y \left(\frac{\rho(x, y')}{\rho_t}\right)dy' = g(x, y)$$

Partial derivatives

$$\frac{\partial \tilde{x}}{\partial x} = f_x = \sigma\left(\frac{P_e}{P_t}\left(\frac{a_e}{a_t}\right)\right)$$

$$\frac{\partial \tilde{x}}{\partial y} = f_y = 0$$

$$\frac{\partial \tilde{y}}{\partial x} = g_x \quad = ??$$

$$\frac{\partial \tilde{y}}{\partial y} = g_y = \left(\frac{a_e}{a_t}\right)\left(\frac{\rho}{\rho_t}\right)$$

Introduce the stream function for steady compressible flow. Let

$$\rho U = \rho_t \frac{\partial \psi}{\partial y} \qquad \rho V = -\rho_t \frac{\partial \psi}{\partial x}$$

Real and virtual stream functions have the same value.

$$\psi(x, y) = \tilde{\psi}(\tilde{x}(x), \tilde{y}(x, y))$$

Partial derivatives of the stream function from the chain rule.

$$\frac{\partial\psi}{\partial x} = \frac{\partial\tilde{\psi}}{\partial\tilde{x}}\frac{\partial\tilde{x}}{\partial x} + \frac{\partial\tilde{\psi}}{\partial\tilde{y}}\frac{\partial\tilde{y}}{\partial x} = \frac{\partial\tilde{\psi}}{\partial\tilde{x}}\frac{d\tilde{x}}{dx} + \frac{\partial\tilde{\psi}}{\partial\tilde{y}}\frac{\partial\tilde{y}}{\partial x}$$

$$\frac{\partial\psi}{\partial y} = \frac{\partial\tilde{\psi}}{\partial\tilde{x}}\frac{\partial\tilde{x}}{\partial y} + \frac{\partial\tilde{\psi}}{\partial\tilde{y}}\frac{\partial\tilde{y}}{\partial y} = \frac{\partial\tilde{\psi}}{\partial\tilde{y}}\frac{\partial\tilde{y}}{\partial y}$$

Velocities

$$U = \frac{\rho_t}{\rho}\frac{\partial\psi}{\partial y} = \left(\frac{a_e}{a_t}\right)\frac{\partial\tilde{\psi}}{\partial\tilde{y}} = \left(\frac{a_e}{a_t}\right)\tilde{U}$$

$$V = -\frac{\rho_t}{\rho}\frac{\partial\psi}{\partial x} = -\sigma\frac{\rho_t}{\rho}\left(\frac{P_e}{P_t}\frac{a_e}{a_t}\right)\frac{\partial\tilde{\psi}}{\partial\tilde{x}} - \frac{\rho_t}{\rho}\frac{\partial\tilde{\psi}}{\partial\tilde{y}}\frac{\partial\tilde{y}}{\partial x}$$

Partial derivatives of U from the chain rule.

$$\frac{\partial U}{\partial x} = \frac{\partial}{\partial\tilde{x}}\left(\left(\frac{a_e}{a_t}\right)\frac{\partial\tilde{\psi}}{\partial\tilde{y}}\right)\frac{d\tilde{x}}{dx} + \frac{\partial}{\partial\tilde{y}}\left(\left(\frac{a_e}{a_t}\right)\frac{\partial\tilde{\psi}}{\partial\tilde{y}}\right)\frac{\partial\tilde{y}}{\partial x} =$$

$$\sigma\left(\frac{P_e}{P_t}\left(\frac{a_e}{a_t}\right)^2\right)\left(\left(\frac{1}{a_e}\right)\frac{\partial a_e}{\partial\tilde{x}}\frac{\partial\tilde{\psi}}{\partial\tilde{y}} + \frac{\partial^2\tilde{\psi}}{\partial\tilde{x}\partial\tilde{y}}\right) + \left(\frac{a_e}{a_t}\right)\frac{\partial^2\tilde{\psi}}{\partial\tilde{y}^2}\frac{\partial\tilde{y}}{\partial x}$$

$$\frac{\partial U}{\partial y} = \frac{\partial}{\partial\tilde{x}}\left(\left(\frac{a_e}{a_t}\right)\frac{\partial\tilde{\psi}}{\partial\tilde{y}}\right)\frac{\partial\tilde{x}}{\partial y} + \frac{\partial}{\partial\tilde{y}}\left(\left(\frac{a_e}{a_t}\right)\frac{\partial\tilde{\psi}}{\partial\tilde{y}}\right)\frac{\partial\tilde{y}}{\partial y} = \left(\frac{a_e}{a_t}\right)^2\left(\frac{\rho}{\rho_t}\right)\left(\frac{\partial^2\tilde{\psi}}{\partial\tilde{y}^2}\right)$$

Convective terms of the momentum equation

$$U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} =$$

$$\sigma\left(\frac{P_e}{P_t}\left(\frac{a_e}{a_t}\right)^3\right)\left(\left(\frac{1}{a_e}\right)\frac{\partial a_e}{\partial \tilde{x}}\left(\frac{\partial\tilde{\psi}}{\partial\tilde{y}}\right)^2 + \frac{\partial\tilde{\psi}}{\partial\tilde{y}}\frac{\partial^2\tilde{\psi}}{\partial\tilde{x}\partial\tilde{y}}\right) + \underline{\left(\frac{a_e}{a_t}\right)^2\frac{\partial\tilde{\psi}}{\partial\tilde{y}}\left(\frac{\partial^2\tilde{\psi}}{\partial\tilde{y}^2}\frac{\partial\tilde{y}}{\partial x}\right)} -$$

$$\left((\sigma)\left(\frac{P_e a_e}{P_t a_t}\right)\left(\frac{a_e}{a_t}\right)^2\left(\frac{\partial^2\tilde{\psi}}{\partial\tilde{y}^2}\right)\frac{\partial\tilde{\psi}}{\partial\tilde{x}} + \underline{\left(\frac{a_e}{a_t}\right)^2\left(\frac{\partial^2\tilde{\psi}}{\partial\tilde{y}^2}\right)\frac{\partial\tilde{\psi}}{\partial\tilde{y}}\frac{\partial\tilde{y}}{\partial x}}\right)$$

Cancel terms

$$U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} =$$

$$\sigma\left(\frac{P_e}{P_t}\left(\frac{a_e}{a_t}\right)^3\right)\left(\left(\frac{1}{a_e}\right)\frac{\partial a_e}{\partial \tilde{x}}\left(\frac{\partial\tilde{\psi}}{\partial\tilde{y}}\right)^2 + \frac{\partial\tilde{\psi}}{\partial\tilde{y}}\frac{\partial^2\tilde{\psi}}{\partial\tilde{x}\partial\tilde{y}} - \left(\frac{\partial^2\tilde{\psi}}{\partial\tilde{y}^2}\right)\frac{\partial\tilde{\psi}}{\partial\tilde{x}}\right)$$

Pressure gradient term

$$\left(\frac{P_e}{P_t}\right) = \left(\frac{a_e}{a_t}\right)^{\frac{2\gamma}{(\gamma-1)}}$$

$$\frac{dP_e}{dx} = \frac{2\gamma P_t}{(\gamma-1)}\left(\frac{a_e}{a_t}\right)^{\frac{\gamma+1}{\gamma-1}}\frac{1}{a_t}\frac{da_e}{d\tilde{x}}\frac{d\tilde{x}}{dx} = \sigma\left(\frac{P_e}{P_t}\left(\frac{a_e}{a_t}\right)\right)\frac{2\gamma P_t}{(\gamma-1)}\left(\frac{a_e}{a_t}\right)^{\frac{\gamma+1}{\gamma-1}}\frac{1}{a_t}\frac{da_e}{d\tilde{x}}$$

$$\frac{dP_e}{dx} = \sigma\left(\frac{P_e}{P_t}\left(\frac{a_e}{a_t}\right)^2\right)\left(\frac{2\gamma P_t}{(\gamma-1)}\left(\frac{a_e}{a_t}\right)^{\frac{\gamma+1}{\gamma-1}}\frac{1}{a_e}\frac{da_e}{d\tilde{x}}\right)$$

Now

$$U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} + \frac{1}{\rho}\frac{dP_e}{dx} =$$

$$\sigma\left(\frac{P_e}{P_t}\left(\frac{a_e}{a_t}\right)^3\right)\left(\frac{\partial\tilde{\psi}}{\partial\tilde{y}}\frac{\partial^2\tilde{\psi}}{\partial\tilde{x}\partial\tilde{y}} - \left(\frac{\partial^2\tilde{\psi}}{\partial\tilde{y}^2}\right)\frac{\partial\tilde{\psi}}{\partial\tilde{x}}\right) +$$

$$\sigma\left(\frac{P_e}{P_t}\left(\frac{a_e}{a_t}\right)^3\right)\left(U^2\left(\frac{a_t}{a_e}\right)^2 + \frac{1}{\rho}\frac{2\gamma P_t}{(\gamma-1)}\left(\frac{a_e}{a_t}\right)^{\frac{2}{\gamma-1}}\right)\left(\left(\frac{1}{a_e}\right)\frac{da_e}{d\tilde{x}}\right)$$

$$\left(\frac{\partial\tilde{\psi}}{\partial\tilde{y}}\right)^2 = U^2\left(\frac{a_t}{a_e}\right)^2$$

At the edge of the boundary layer

$$a_t^2 = a_e^2 + \left(\frac{\gamma - 1}{2}\right) U_e^2 \qquad \frac{1}{a_e}\frac{da_e}{d\tilde{x}} = -\left(\frac{\gamma - 1}{2a_e^2}\right) U_e \frac{dU_e}{d\tilde{x}}$$

Now

$$U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} + \frac{1}{\rho}\frac{dP_e}{dx} =$$

$$\sigma\left(\frac{P_e}{P_t}\left(\frac{a_e}{a_t}\right)^3\right)\left(\frac{\partial\tilde{\psi}}{\partial\tilde{y}}\frac{\partial^2\tilde{\psi}}{\partial\tilde{x}\partial\tilde{y}} - \left(\frac{\partial^2\tilde{\psi}}{\partial\tilde{y}^2}\right)\frac{\partial\tilde{\psi}}{\partial\tilde{x}}\right) -$$

$$\sigma\left(\frac{P_e}{P_t}\left(\frac{a_e}{a_t}\right)^3\right)\left(U^2\frac{(\gamma - 1)}{2a_e^2}\left(\frac{a_t}{a_e}\right)^2 + \frac{1}{\rho}\frac{\gamma P_t}{a_e^2}\left(\frac{a_e}{a_t}\right)^{\frac{2}{\gamma - 1}}\right)\left(U_e\frac{dU_e}{d\tilde{x}}\right)$$

Note that

$$U^2\frac{(\gamma-1)}{2a_e^2}\left(\frac{a_t}{a_e}\right)^2+\frac{1}{\rho}\frac{\gamma P_t}{a_e^2}\left(\frac{a_e}{a_t}\right)^{\frac{2}{\gamma-1}} =$$

$$U^2\frac{(\gamma-1)}{2a_e^2}\left(\frac{a_t}{a_e}\right)^2+\frac{\rho_e}{\rho}\frac{\gamma P_e}{\rho_e}\frac{1}{a_e^2}\left(\frac{a_t}{a_e}\right)^{\frac{2\gamma}{\gamma-1}}\left(\frac{a_e}{a_t}\right)^{\frac{2}{\gamma-1}} =$$

$$\left(\frac{a_t}{a_e}\right)^4\left(\frac{a^2+\frac{(\gamma-1)}{2}U^2}{a_t^2}\right)$$

Where we have used

$$P_t/P_e = (a_t/a_e)^{2\gamma/(\gamma-1)} \qquad \gamma P_e/\rho = (\gamma P)/\rho = a^2$$

Viscous term. Note $\rho T = P/R = P_e/R$

$$\tau_{xy}\big|_{laminar} = \mu\frac{\partial U}{\partial y} = \sigma\mu_t\left(\frac{T}{T_t}\right)\frac{\partial}{\partial y}\left(\left(\frac{a_e}{a_t}\right)\frac{\partial\tilde{\psi}}{\partial\tilde{y}}\right) = \sigma\mu_t\left(\frac{T}{T_t}\right)\left(\frac{a_e}{a_t}\right)\frac{\partial}{\partial y}\left(\frac{\partial\tilde{\psi}}{\partial\tilde{y}}\right) =$$

$$\sigma\mu_t\left(\frac{a_e}{a_t}\right)^2\left(\frac{T}{T_t}\right)\left(\frac{\rho}{\rho_t}\right)\frac{\partial}{\partial\tilde{y}}\left(\frac{\partial\tilde{\psi}}{\partial\tilde{y}}\right) = \sigma\left(\frac{a_e}{a_t}\right)^2\left(\frac{\rho T}{\rho_t T_t}\right)\left(\mu_t\frac{\partial\tilde{U}}{\partial\tilde{y}}\right) = \sigma\left(\frac{a_e}{a_t}\right)^2\left(\frac{P_e}{P_t}\right)\tilde{\tau}_{\tilde{x}\tilde{y}}\big|_{laminar}$$

$$\frac{1}{\rho}\frac{\partial}{\partial y}\left(\mu\frac{\partial U}{\partial y}\right) = \sigma\frac{\mu_t}{\rho_t}\frac{P_e}{P_t}\left(\frac{a_e}{a_t}\right)^3\left(\frac{\partial^3\tilde{\psi}}{\partial\tilde{y}^3}\right)$$

$$\frac{1}{\rho}\frac{\partial}{\partial y}\left(\tau_{xy}\big|_{turbulent}\right) = \sigma\frac{1}{\rho_t}\frac{P_e}{P_t}\left(\frac{a_e}{a_t}\right)^3\frac{\partial}{\partial\tilde{y}}\left(\tilde{\tau}_{xy}\big|_{turbulent}\right)$$

The boundary layer momentum equation becomes

$$U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} + \frac{1}{\rho}\frac{dP_e}{dx} - \frac{1}{\rho}\frac{\partial}{\partial y}\left(\mu\frac{\partial U}{\partial y}\right) - \frac{1}{\rho}\frac{\partial \tau_{xy}}{\partial y} =$$

$$\sigma\left(\frac{P_e}{P_t}\left(\frac{a_e}{a_t}\right)^3\right)\left(\frac{\partial\tilde{\psi}}{\partial\tilde{y}}\frac{\partial^2\tilde{\psi}}{\partial\tilde{x}\partial\tilde{y}} - \left(\frac{\partial^2\tilde{\psi}}{\partial\tilde{y}^2}\right)\frac{\partial\tilde{\psi}}{\partial\tilde{x}}\right) -$$

$$\sigma\left(\frac{P_e}{P_t}\left(\frac{a_e}{a_t}\right)^3\right)\left(\left(\frac{a_t}{a_e}\right)^4\left(\frac{a^2 + \frac{(\gamma - 1)}{2}U^2}{a_t^2}\right)\right)\left(U_e\frac{dU_e}{d\tilde{x}}\right) -$$

$$\sigma\frac{\mu_t}{\rho_t}\frac{P_e}{P_t}\left(\frac{a_e}{a_t}\right)^3\left(\frac{\partial^3\tilde{\psi}}{\partial\tilde{y}^3}\right) - \sigma\frac{1}{\rho_t}\frac{P_e}{P_t}\left(\frac{a_e}{a_t}\right)^3\frac{\partial}{\partial\tilde{y}}\left(\tilde{\tau}_{xy}\Big|_{turbulent}\right) = 0$$

Drop the common multiplying factors

$$\left(\frac{\partial\tilde{\psi}}{\partial\tilde{y}}\frac{\partial^2\tilde{\psi}}{\partial\tilde{x}\partial\tilde{y}} - \left(\frac{\partial^2\tilde{\psi}}{\partial\tilde{y}^2}\right)\frac{\partial\tilde{\psi}}{\partial\tilde{x}}\right) - \left(\frac{a_t}{a_e}\right)^4\left(\frac{a^2 + \frac{(\gamma-1)}{2}U^2}{a_t^2}\right)U_e\frac{dU_e}{d\tilde{x}} -$$

$$\frac{\mu_t}{\rho_t}\left(\frac{\partial^3\tilde{\psi}}{\partial\tilde{y}^3}\right) - \frac{1}{\rho_t}\frac{\partial}{\partial\tilde{y}}\left(\tilde{\tau}_{xy}\big|_{turbulent}\right) = 0$$

$$\tilde{U}_e = \frac{a_t}{a_e}U_e$$

$$\tilde{U}_e\frac{d\tilde{U}_e}{d\tilde{x}} = \left(\frac{a_t}{a_e}\right)^2\left(\frac{1}{a_e^2}\right)\left(a_e^2 + \left(\frac{\gamma-1}{2}\right)U_e^2\right)U_e\frac{dU_e}{d\tilde{x}} =$$

$$\left(\frac{a_t}{a_e}\right)^4 U_e\frac{dU_e}{d\tilde{x}}$$

Now the momentum equation is expressed entirely in tildaed variables.

$$\left(\frac{\partial\tilde{\psi}}{\partial\tilde{y}}\frac{\partial^2\tilde{\psi}}{\partial\tilde{x}\partial\tilde{y}}-\left(\frac{\partial^2\tilde{\psi}}{\partial\tilde{y}^2}\right)\frac{\partial\tilde{\psi}}{\partial\tilde{x}}\right)-\left(\frac{a^2+\frac{(\gamma-1)}{2}U^2}{a_t^2}\right)\tilde{U}_e\frac{d\tilde{U}_e}{d\tilde{x}}-$$

$$\frac{\mu_t}{\rho_t}\left(\frac{\partial^3\tilde{\psi}}{\partial\tilde{y}^3}\right)-\frac{1}{\rho_t}\frac{\partial}{\partial\tilde{y}}\left(\tilde{\tau}_{xy}\big|_{turbulent}\right) = 0$$

$$\tilde{U} = \frac{\partial\tilde{\psi}}{\partial\tilde{y}} \qquad \tilde{V} = -\frac{\partial\tilde{\psi}}{\partial x}$$

$$\left(\tilde{U}\frac{\partial\tilde{U}}{\partial\tilde{x}}+\left(\frac{\partial\tilde{U}}{\partial\tilde{y}}\right)\tilde{V}\right)-\left(\frac{a^2+\frac{(\gamma-1)}{2}U^2}{a_e^{\ 2}+\frac{(\gamma-1)}{2}U_e^{\ 2}}\right)\tilde{U}_e\frac{d\tilde{U}_e}{d\tilde{x}}-$$

$$\nu_t\left(\frac{\partial^2\tilde{U}}{\partial\tilde{y}^2}\right)-\frac{1}{\rho_t}\frac{\partial}{\partial\tilde{y}}\left(\tilde{\tau}_{xy}\big|_{turbulent}\right) = 0$$

For an adiabatic wall, and a Prandtl number of one the factor in brackets is one and the equation maps exactly to the incompressible form.

$$\left(\tilde{U}\frac{\partial\tilde{U}}{\partial\tilde{x}} + \left(\frac{\partial\tilde{U}}{\partial\tilde{y}}\right)\tilde{V}\right) - \tilde{U}_e\frac{d\tilde{U}_e}{d\tilde{x}} - \nu_t\left(\frac{\partial^2\tilde{U}}{\partial\tilde{y}^2}\right) - \frac{1}{\rho_t}\frac{\partial}{\partial\tilde{y}}\left(\tilde{\tau}_{xy}\big|_{turbulent}\right) = 0$$

with boundary conditions

$$\tilde{U}(0) = 0 \qquad \tilde{V}(0) = 0 \qquad \tilde{U}(\tilde{\delta}) = \tilde{U}_e$$

Skin friction

$$\tilde{C}_f = \frac{\tilde{\tau}_w}{(1/2)\rho_t\tilde{U}_e^2} = \frac{\frac{1}{\sigma}\left(\left(\frac{a_t}{a_e}\right)^2\frac{P_t}{P_e}\right)\tau_w}{(1/2)\left(\frac{\rho_t}{\rho_e}\right)\rho_e\left(\frac{a_t}{a_e}\right)^2 U_e^2} = \frac{1}{\sigma}\frac{\rho_e P_t}{\rho_t P_e}\left(\frac{\tau_w}{(1/2)\rho_e U_e^2}\right) = \frac{1}{\sigma}\frac{T_t}{T_e}C_f$$

$$\frac{C_f}{\tilde{C}_f} = \frac{1}{1 + \left(\frac{\gamma - 1}{2}\right)M_e^2}$$

$$\left(\tilde{U}\frac{\partial \tilde{U}}{\partial \tilde{x}} + \left(\frac{\partial \tilde{U}}{\partial \tilde{y}}\right)\tilde{V}\right) - \tilde{U}_e\frac{d\tilde{U}_e}{d\tilde{x}} - \nu_t\left(\frac{\partial^2 \tilde{U}}{\partial \tilde{y}^2}\right) - \frac{1}{\rho_t}\frac{\partial}{\partial \tilde{y}}\left(\tilde{\tau}_{xy}\Big|_{turbulent}\right) = 0 \tag{9.143}$$

with boundary conditions

$$\tilde{U}(0) = 0 \qquad \tilde{V}(0) = 0 \qquad \tilde{U}(\tilde{\delta}) = \tilde{U}_e \tag{9.144}$$

The implication of (9.143) and (9.144) is that the effects of compressibility on the boundary layer can be almost completely accounted for by the scaling of coordinates presented in (9.112) which is driven in the y direction by the decrease in density near the wall due to heating and in the x direction by the isentropic changes in free stream temperature and boundary layer pressure due to flow acceleration or deceleration imposed by the surrounding potential flow.

In order to solve for the physical velocity profiles we need to determine the temperature in the boundary layer. Look at the case

$$dU_e/dx = 0 \qquad \tau_{xy}\big|_{turbulent} = 0$$

The energy equation was integrated earlier $T = T_t - \frac{1}{2C_p}U^2$

$$\frac{T}{T_e} = 1 + \left(\frac{\gamma - 1}{2}\right)M_e^2\left(1 - \left(\frac{U}{U_e}\right)^2\right)$$

Use $U/U_e = \tilde{U}/\tilde{U}_e$ and $\rho T = \rho_e T_e$

$$\frac{T}{T_e} = 1 + \left(\frac{\gamma - 1}{2}\right)M_e^2\left(1 - \left(\frac{\tilde{U}}{\tilde{U}_e}\right)^2\right) = \frac{\rho_e}{\rho}$$

We need to relate wall normal coordinates in the real and virtual flow

$$dy = \left(\frac{a_t}{a_e}\right)\left(\frac{\rho_t}{\rho_e}\right)\left(\frac{\rho_e}{\rho}\right)d\tilde{y} = \left(\frac{a_t}{a_e}\right)\left(\frac{\rho_t}{\rho_e}\right)\left(1 + \left(\frac{\gamma - 1}{2}\right)M_e^2\left(1 - \left(\frac{\tilde{U}}{\tilde{U}_e}\right)^2\right)\right)d\tilde{y}$$

The spatial similarity variable in the virtual flow is

$$\tilde{\alpha} = \tilde{y}\left(\frac{\tilde{U}_e}{2\nu_t \tilde{x}}\right)^{1/2}$$

$$dy = \left(\frac{a_t}{a_e}\right)\left(\frac{\rho_t}{\rho_e}\right)\left(\frac{2\nu_t \tilde{x}}{\tilde{U}_e}\right)^{1/2}\left(1 + \left(\frac{\gamma - 1}{2}\right)M_e^2\left(1 - \left(\frac{\tilde{U}}{\tilde{U}_e}\right)^2\right)\right) d\left(\tilde{y}\left(\frac{\tilde{U}_e}{2\nu_t \tilde{x}}\right)^{1/2}\right)$$

$$d\left(y\left(\frac{U_e}{2\nu_e x}\right)^{1/2}\right) =$$

$$\left(\frac{a_t}{a_e}\right)\left(\frac{\rho_t}{\rho_e}\right)\left(\frac{2\nu_t \tilde{x}}{\tilde{U}_e}\right)^{1/2}\left(\frac{U_e}{2\nu_e x}\right)^{1/2}\left(1 + \left(\frac{\gamma - 1}{2}\right)M_e^2\left(1 - \left(\frac{\tilde{U}}{\tilde{U}_e}\right)^2\right)\right) d\left(\tilde{y}\left(\frac{\tilde{U}_e}{2\nu_t \tilde{x}}\right)^{1/2}\right)$$

$$\left(\frac{a_t}{a_e}\right)\left(\frac{\rho_t}{\rho_e}\right)\left(\frac{2\nu_t \tilde{x}}{\tilde{U}_e}\frac{U_e}{2\nu_e x}\right)^{1/2} = \left(\frac{a_t}{a_e}\right)\left(\frac{\rho_t}{\rho_e}\right)\left(\frac{\mu_t}{\mu_e}\frac{\rho_e}{\rho_t}\frac{U_e}{\tilde{U}_e}\frac{\tilde{x}}{x}\right)^{1/2} =$$

$$\left(\frac{a_t}{a_e}\right)\left(\frac{\rho_t}{\rho_e}\right)\left(\frac{T_t}{T_e}\frac{\rho_e}{\rho_t}\frac{a_e}{a_t}\frac{P_e}{P_t}\frac{a_e}{a_t}\right)^{1/2} = \left(\frac{\rho_t}{\rho_e}\right)\left(\frac{T_t}{T_e}\frac{\rho_e}{\rho_t}\frac{\rho_e}{\rho_t}\frac{T_e}{T_t}\right)^{1/2} = 1$$

Spatial similarity variables in the two flows are related by

$$d\alpha = \left(1 + \left(\frac{\gamma - 1}{2}\right) M_e^2 \left(1 - \left(\frac{\tilde{U}}{\tilde{U}_e}\right)^2\right)\right) d\tilde{\alpha}$$

$$\alpha(\tilde{\alpha}) = \tilde{\alpha} + \left(\frac{\gamma - 1}{2}\right) M_e^2 \int_0^{\tilde{\alpha}} \left(1 - \left(\frac{\tilde{U}}{\tilde{U}_e}\right)^2\right) d\tilde{\alpha}'$$

$$\tilde{\alpha}_e = 4.906/\sqrt{2} = 3.469$$

$$\alpha_e = \tilde{\alpha}_e + \left(\frac{\gamma - 1}{2}\right) M_e^2 \int_0^{\tilde{\alpha}_e} \left(1 - \left(\frac{\tilde{U}}{\tilde{U}_e}\right)^2\right) d\tilde{\alpha} =$$

$$\alpha_e = 3.469 + 1.67912\left(\frac{\gamma - 1}{2}\right) M_e^2$$

The thickness of the compressible layer increases with Mach number.

Now

$$\frac{T(\alpha(\tilde{\alpha}))}{T_e} = \frac{\rho_e}{\rho(\alpha(\tilde{\alpha}))} = 1 + \left(\frac{\gamma - 1}{2}\right) M_e^2 \left(1 - \left(\frac{\tilde{U}(\tilde{\alpha})}{\tilde{U}_e}\right)^2\right)$$

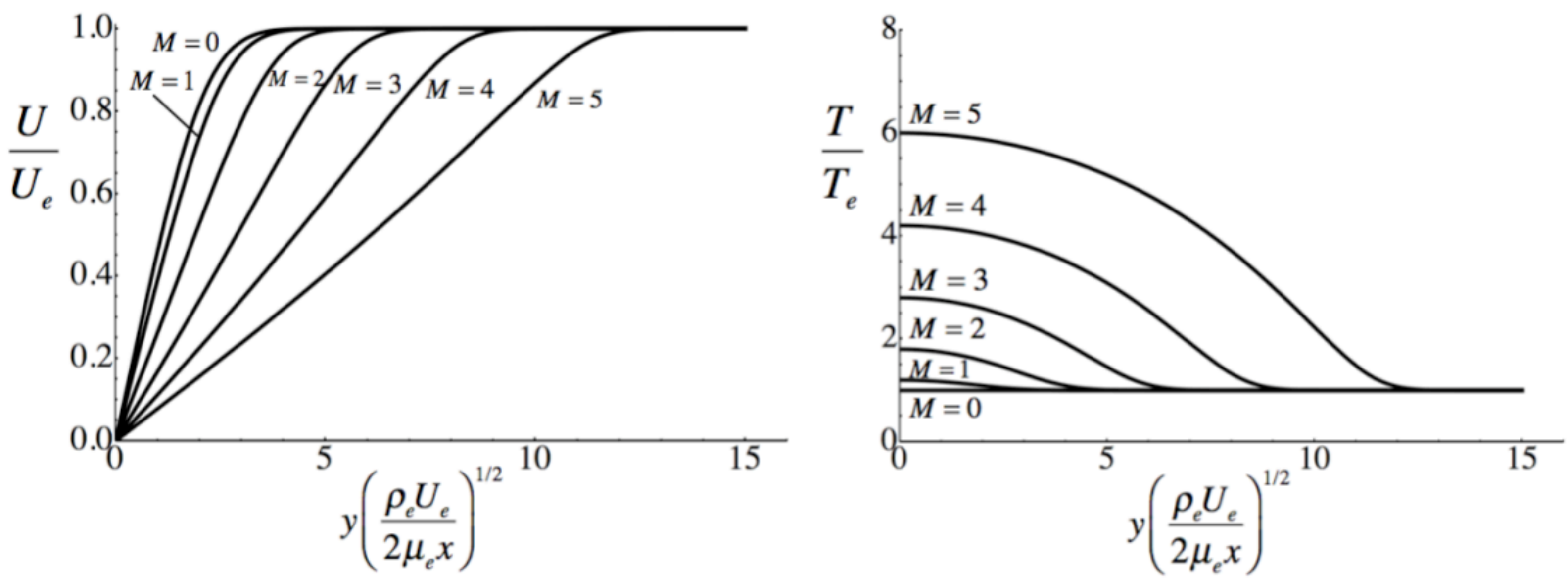


Figure 8.9 Compressible boundary layer profiles on an adiabatic plate for $P_r = 1$, *viscosity exponent* $\omega = 1$, *and* $\gamma = 1.4$.

9.11 Turbulent boundary layers

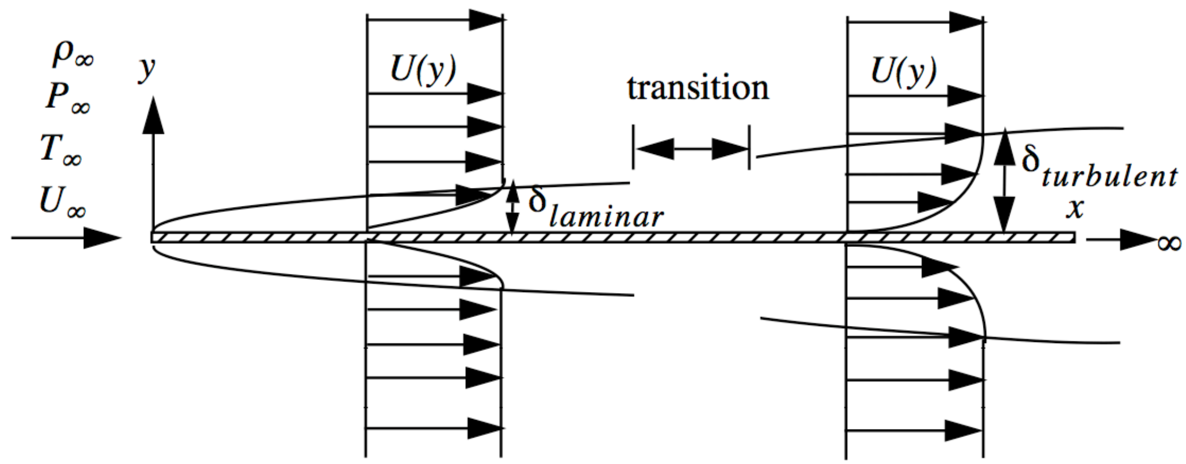


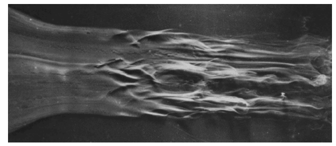
Figure 8.10 Sketch of boundary layer growth in the laminar and turbulent regions.

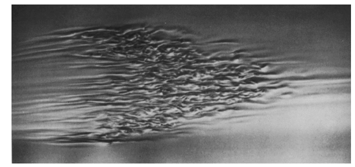
Impirical relations for the thickness of the incompressible case, useful over a limited range of Reynolds number.

$$\frac{\delta}{x} = \frac{0.37}{R_{ex}^{1/5}}$$

Or for a wider range of Reynolds number

$$\frac{\delta}{x} = \frac{0.14}{ln(R_{ex})} G(Ln(R_{ex}))$$





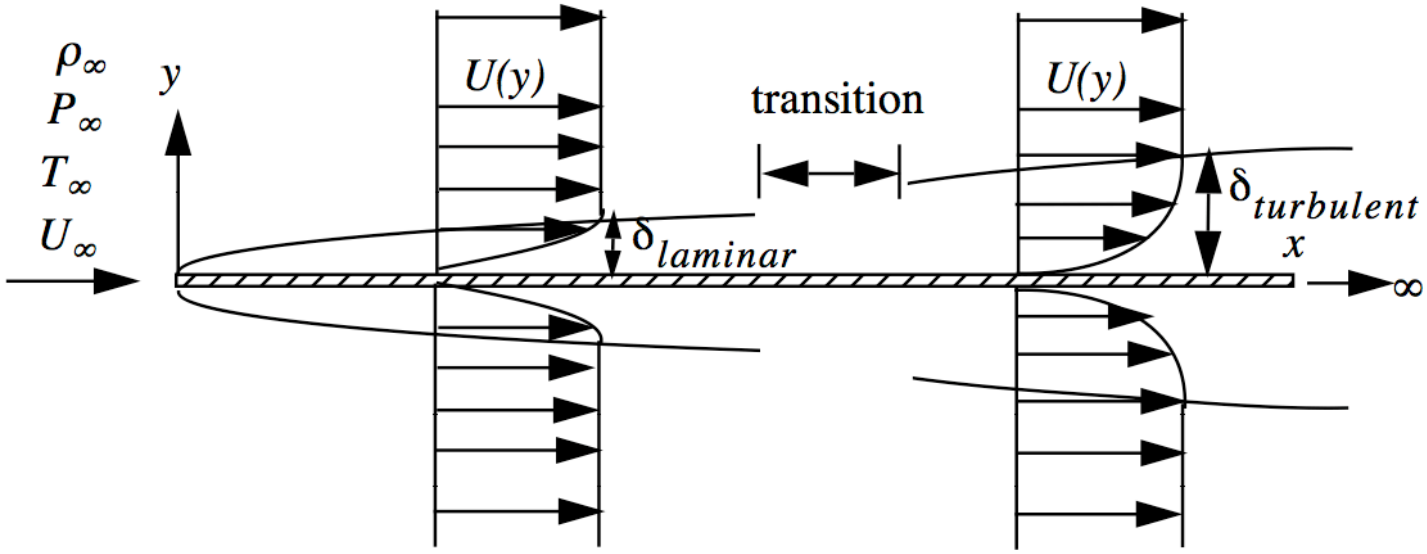
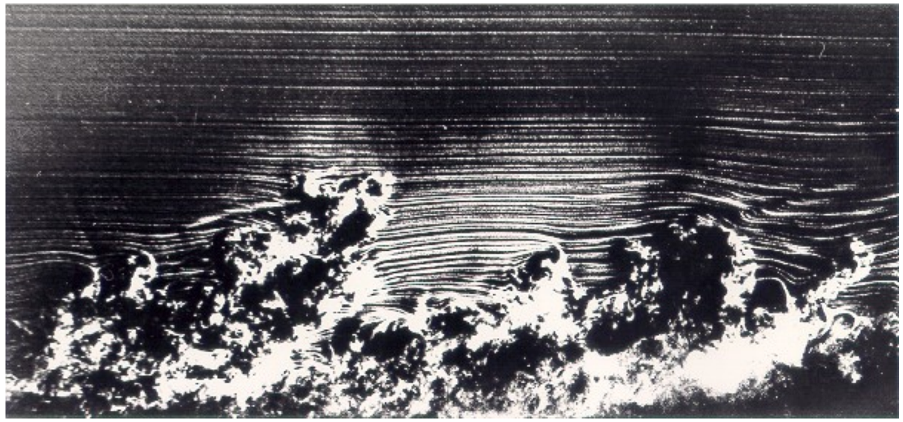
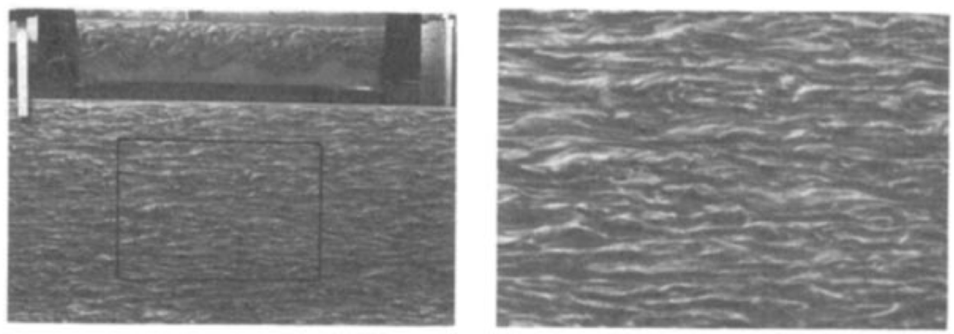
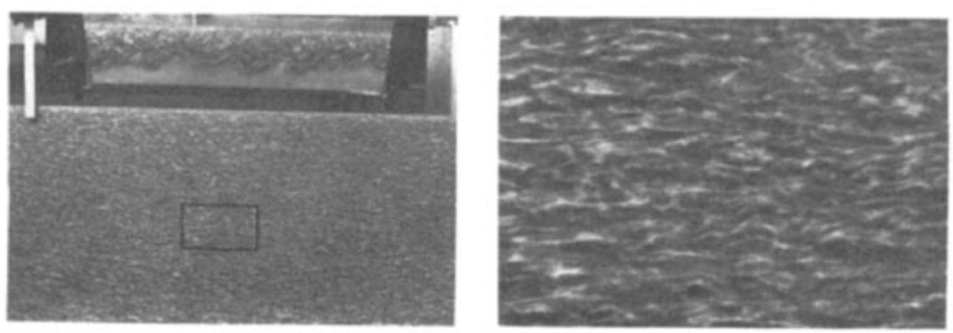


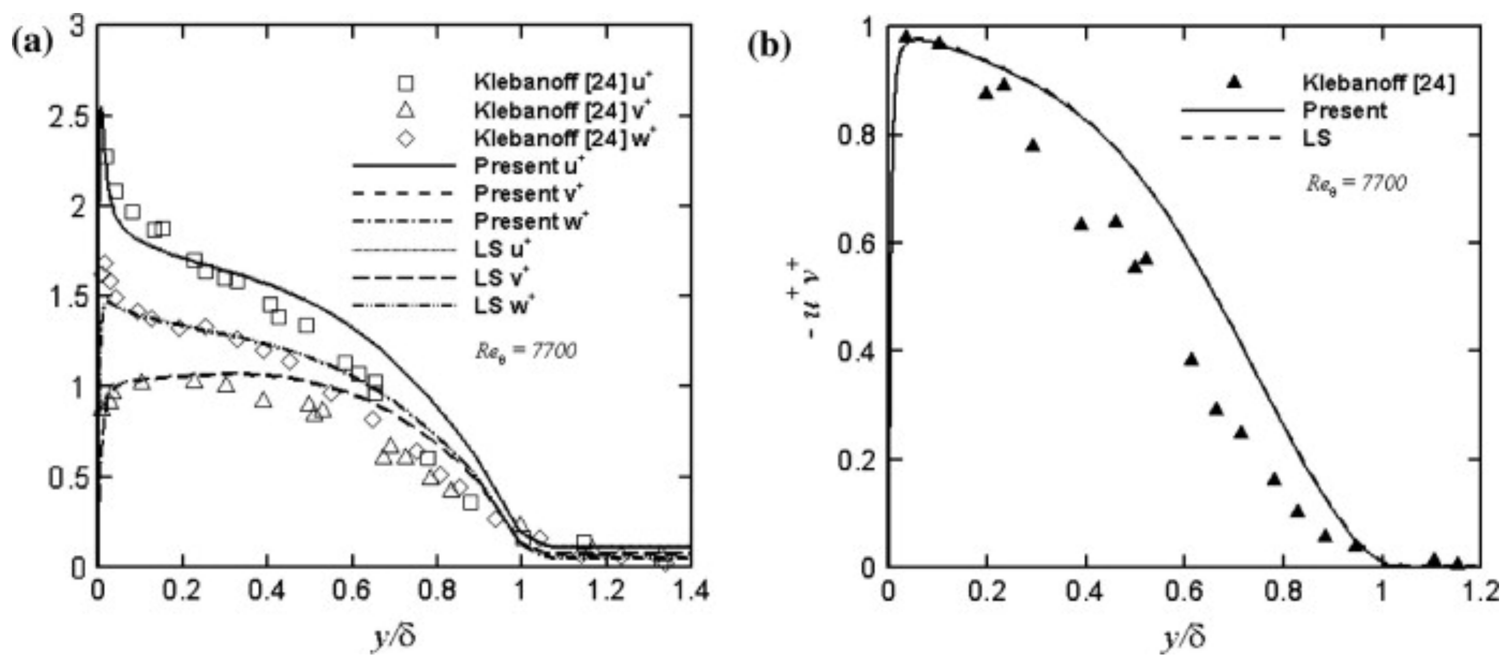
Figure 8.10 Sketch of boundary layer growth in the laminar and turbulent regions.





c




(a)
Klebanoff [24] u+
Klebanoff [24] v+
Klebanoff [24] w+
Present u+
Present v+
Present w+
LS u+
LS v+
LS w+
Re_θ = 7700
y/δ
(b)
Klebanoff [24]
Present
LS
Re_θ = 7700
- u+ v+
y/δ

The incompressible wall friction coefficient

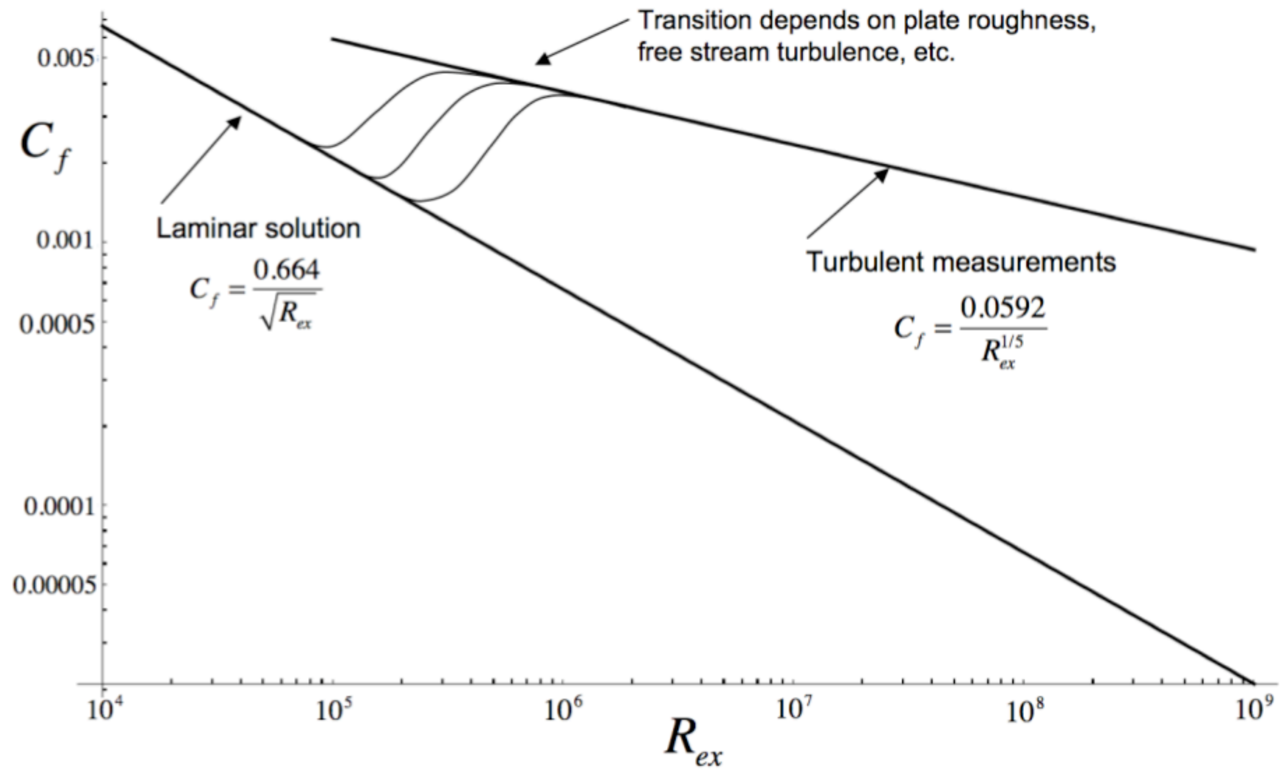


Figure 8.11 Friction coefficient for incompressible flow on a flat plate.

An impirical form of the velocity profile; the so-called 1/7th power law

$$\frac{U}{U_e} = \left(\frac{y}{\delta}\right)^{1/7}$$

The problem with this profile is that it fails to capture one of the most important features of the turbulent boundary layer profile which is that the actual shape of the profile depends on Reynolds number.

A much better, though still impirical, relation is the law of the wake developed by Don Coles at Caltech coupled with the universal law of the wall. In this approach the velocity profile is normalized by the wall friction velocity.

$$u^* = \sqrt{\frac{\tau_w}{\rho}} \qquad \tau_w = \mu \left.\frac{\partial U}{\partial y}\right|_{y=0}$$

Reference: D. Coles, *The Law of the Wake in the Turbulent Boundary Layer*, J. Fluid Mech. Vol 1, 1956

Define dimensionless wall variables

$$y^+ = \frac{yu^*}{\nu} \qquad U^+ = \frac{U}{u^*}$$

The thickness of the boundary layer in wall units is

$$\delta^{+} = \frac{\delta u^{*}}{\nu}$$

and

$$\frac{u^{*}}{U_e} = \left(\frac{\tau_w}{\rho U_e^2}\right)^{1/2} = \left(\frac{C_f}{2}\right)^{1/2} = \left(\frac{0.0592}{2R_{ex}^{1/5}}\right)^{1/2} = \frac{0.172}{R_{ex}^{1/10}}$$

$$\delta^{+} = \frac{\delta}{x}\frac{u^{*}}{U_e}\frac{U_e x}{\nu} = \left(\frac{0.37}{R_{ex}^{1/5}}\right)\left(\frac{0.172}{R_{ex}^{1/10}}\right)R_{ex} = 0.0636R_{ex}^{7/10}$$

Once the Reynolds number is known most of the important properties of the boundary layer are known.

Velocity profile

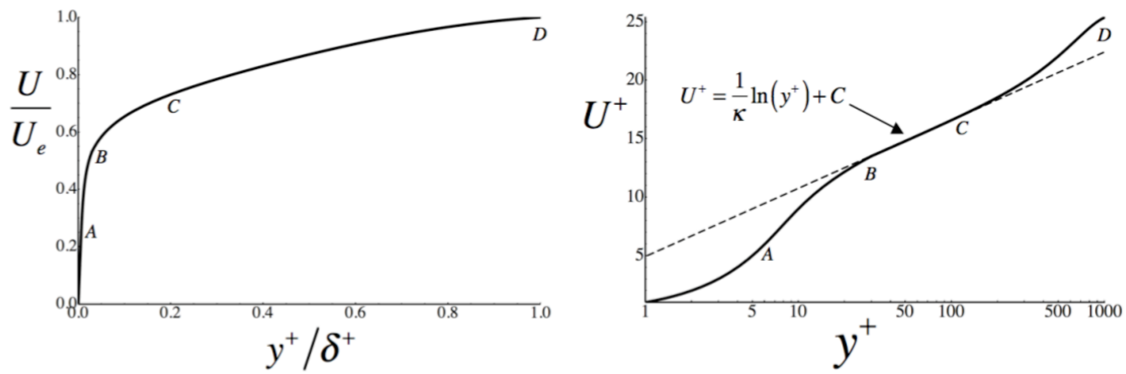


Figure 8.12 Turbulent boundary layer velocity profile in linear and log-linear coordinates. The Reynolds number is $R_{ex} = 10^6$

Viscous sublayer - wall to A $\quad 0 \le y^+ < 7$

$$U^+ = y^+$$

Buffer layer - A to B $\quad 7 \le y^+ < 30$

$$y^+ = U^+ + e^{-\kappa C}\left(e^{\kappa U^+} - 1 - \kappa U^+ - \frac{1}{2}(\kappa U^+)^2 - \frac{1}{6}(\kappa U^+)^3 - \frac{1}{24}(\kappa U^+)^4\right)$$

Logarithmic and outer layer - B to C to D $\qquad dP_e/dx = 0,\ \Pi = 0.62$

$$U^+ = \frac{1}{\kappa}ln(y^+) + C + 2\frac{\Pi(x)}{\kappa}Sin^2\left(\frac{\pi}{2}\frac{y^+}{\delta^+}\right)$$

$$C = 5.1 \qquad \kappa = 0.4$$

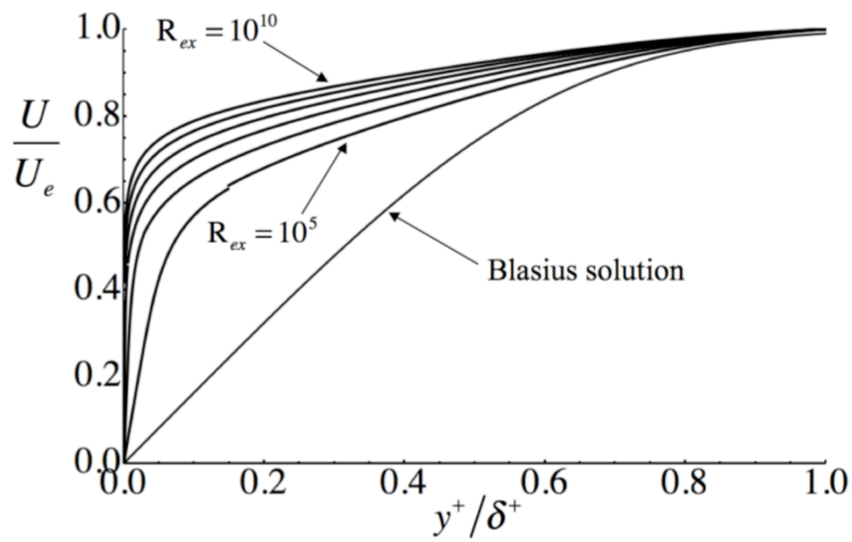


Figure 8.13 Incompressible turbulent boundary layer velocity profiles at several Reynolds numbers compared to the Blasius solution for a laminar boundary layer.

Measurements of velocity in the logarithmic layer can be used to infer the skin friction from the law of the wall.

C increase with increasing roughness Reynolds number

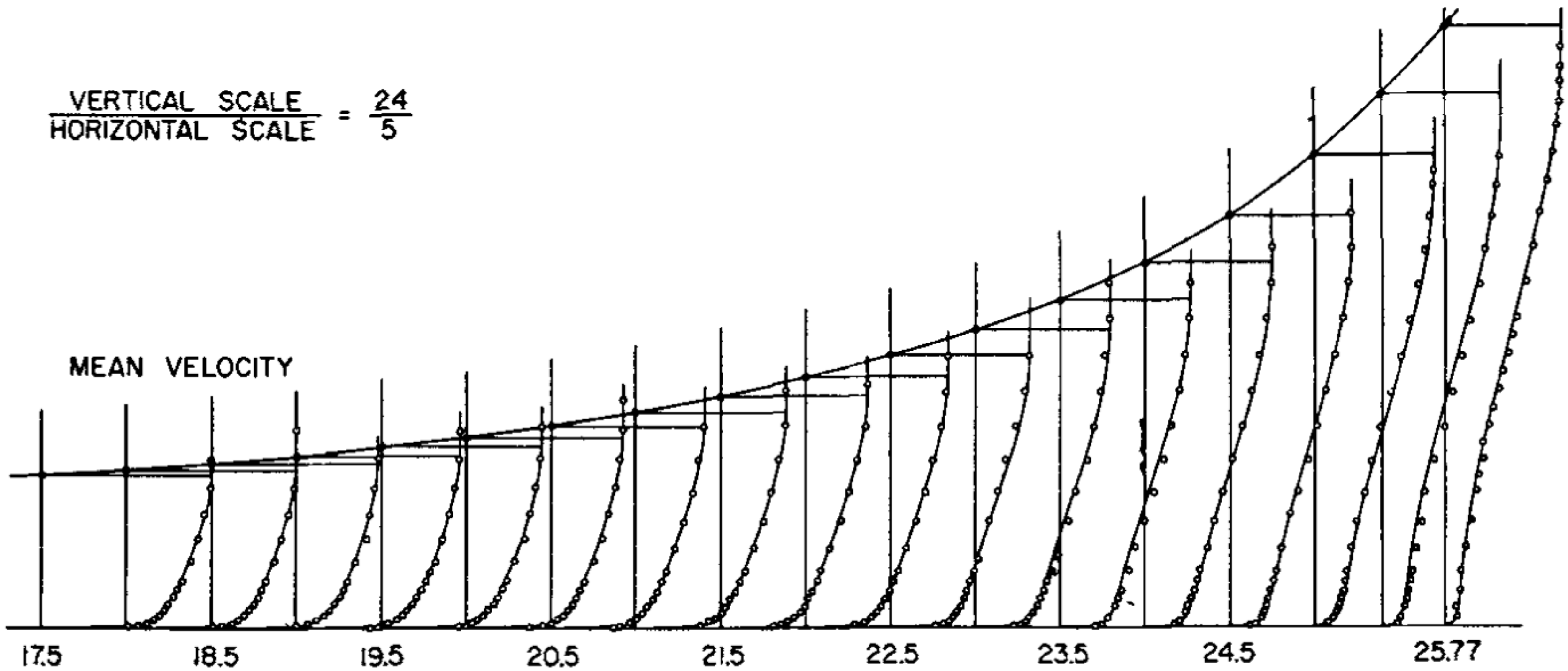
$$\frac{U}{U^*} = \frac{1}{\kappa} ln\left(\left(\frac{yU^*}{\nu}\right) + C\right)$$

k_s Roughness height

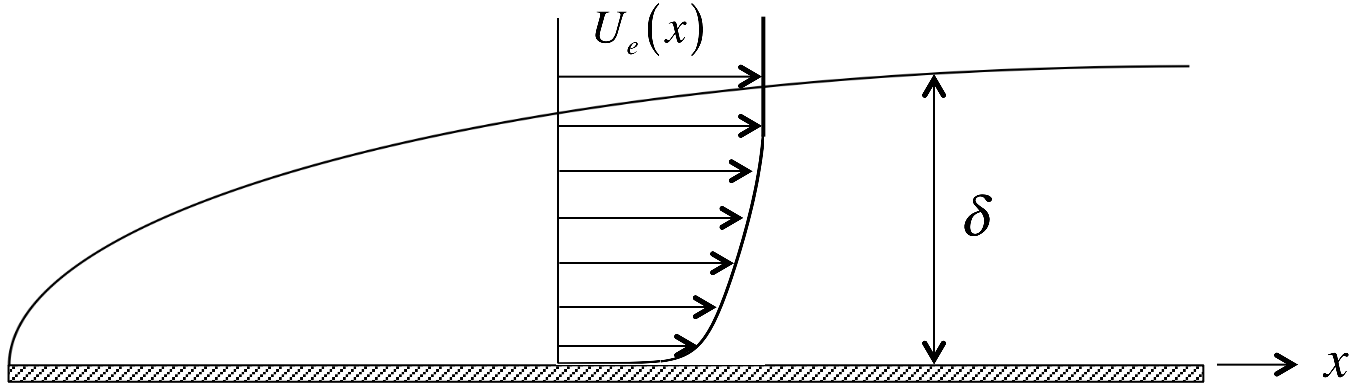
$R_{es} = \frac{k_s u^*}{\nu} < 3$ Hydraulically smooth

$R_{es} = \frac{k_s u^*}{\nu} > 100$ Fully rough

Separating turbulent boundary layer



The method of M. Head 1960 applied to incompressible turbulent boundary layers



At any position x the area flow in the boundary layer is

$$Q = \int_0^{\delta} U\,dy$$

This can be arranged to read

$$Q = \int_0^{\delta} U\,dy = \int_0^{\delta} U_e\,dy - \int_0^{\delta} U_e\left(1 - \frac{U}{U_e}\right)dy = U_e\left(\delta - \delta^*\right)$$

Entrainment velocity

$$V_e = \frac{d}{dx}\left(U_e\left(\delta - \delta^*\right)\right)$$

Reference: M.R. Head, *Entrainment in the Turbulent Boundary Layer*, Aero. Res. Council. R&M 3152, 1960

Head defined the boundary layer shape factor

$$H_1 = \frac{\left(\delta - \delta^*\right)}{\theta}$$

His model consists of two assumptions:

1) Assume

$$\frac{V_e}{U_e} = \frac{1}{U_e}\frac{d}{dx}\left(U_e\left(\delta - \delta^*\right)\right) = F\left(H_1\right)$$

2) Assume

$$H_1 = G\left(H\right) \qquad H = \frac{\delta^*}{\theta}$$

In addition he assumed that the skin friction followed the impirical formula due to Ludweig and Tillman

$$C_f = \frac{0.246}{10^{0.678H} R_\theta^{\ 0.268}} \qquad R_\theta = \frac{U_e\theta}{\nu}$$

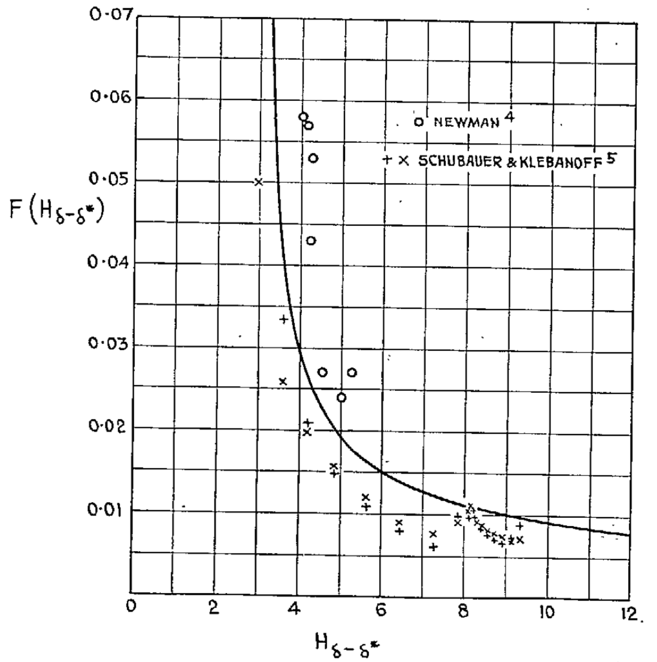


FIG. 1. $\frac{1}{U}\frac{d}{dx}[U(\delta - \delta^*)]$ as a function of $H_{\delta-\delta^*}$.

$H_1 = H_{\delta\text{-}\delta^*}$

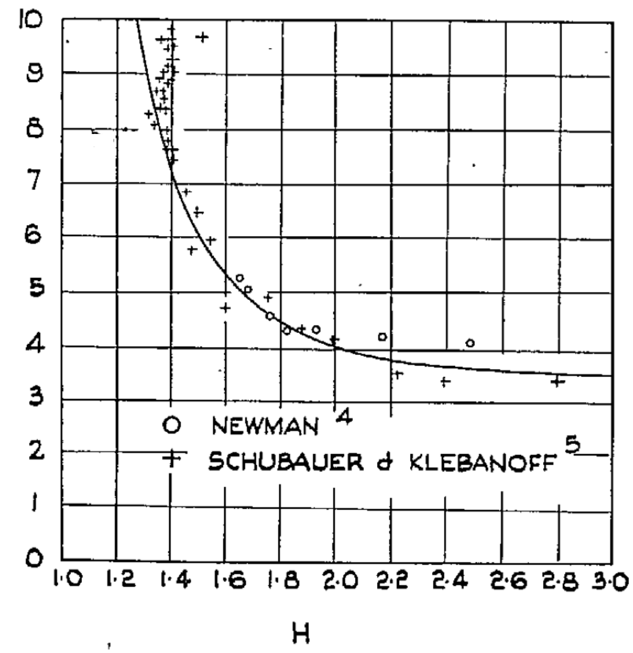


FIG. 2. Variation of $H_{\delta-\delta^*}$ with conventional form parameter H.

We will use $F(H_1) = \dfrac{0.0306}{(H_1 - 3.0)^{0.6169}}$

$$G(H) = 3.0445 + \frac{0.8702}{(H - 1.1)^{1.2721}}$$

Several classical references recommend different functions for F and G

Calculation of Separation Points in Incompressible Turbulent Flows
T. CEBECI, G. J. MOSINSKIS, AND A. M. O. SMITH
Douglas Aircraft Company, Long Beach, Calif.
J. AIRCRAFT VOL. 9, NO. 9

Also

Boundary Layer Theory *H. Schlichting*

Recommend

Schlichting uses 0.0306

Entrainment Relation

$$(1/u_e)(d/dx)(u_e\theta H_1) = 0.0299(H_1 - 3.0)^{-0.6169} \qquad (5)$$

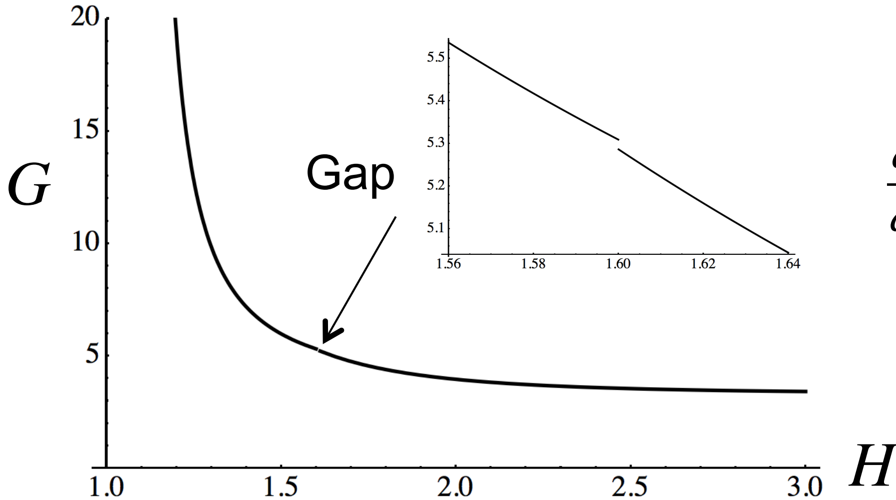
Shape Factor Relation

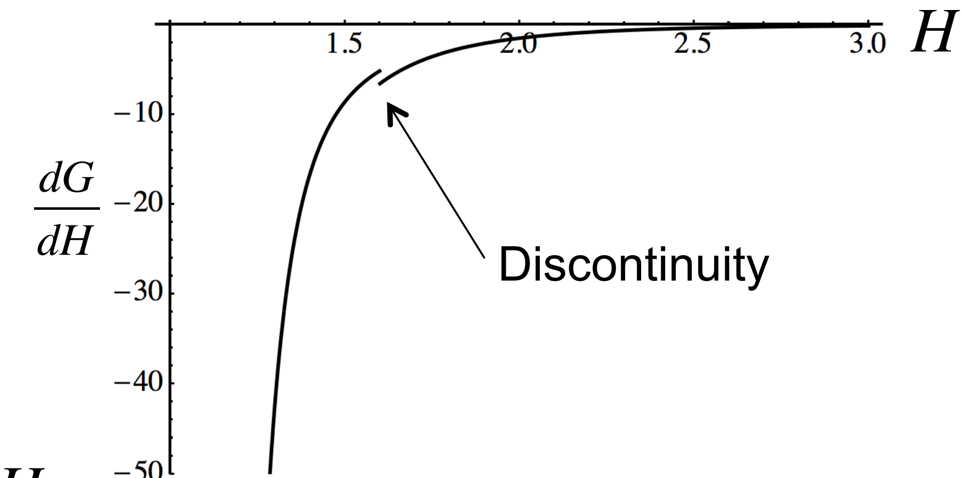
+3.3 is missing

$H_1 = G(H)$ where

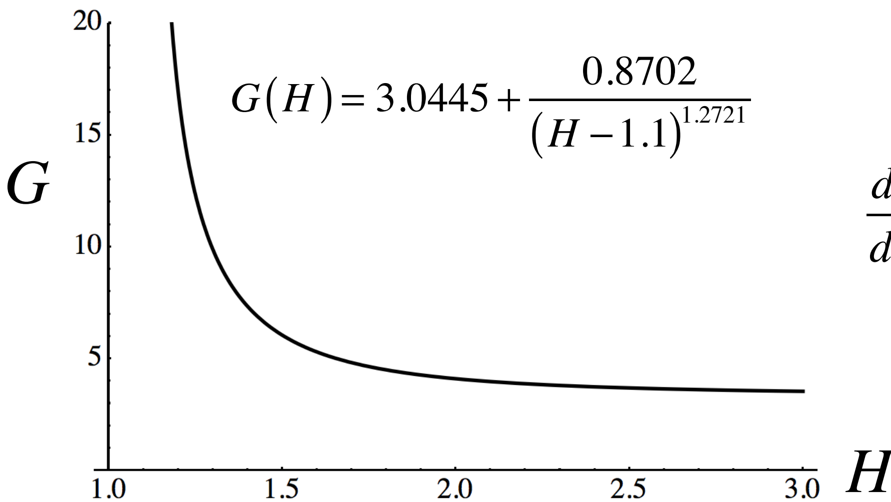
$$G(H) = \begin{cases} 0.8234(H - 1.1)^{-1.287} & H \leq 1.6 \\ 1.5501(H - 0.6778)^{-3.064} + 3.3 & H \geq 1.6 \end{cases} \qquad (6)$$

Cebeci - Schlichting

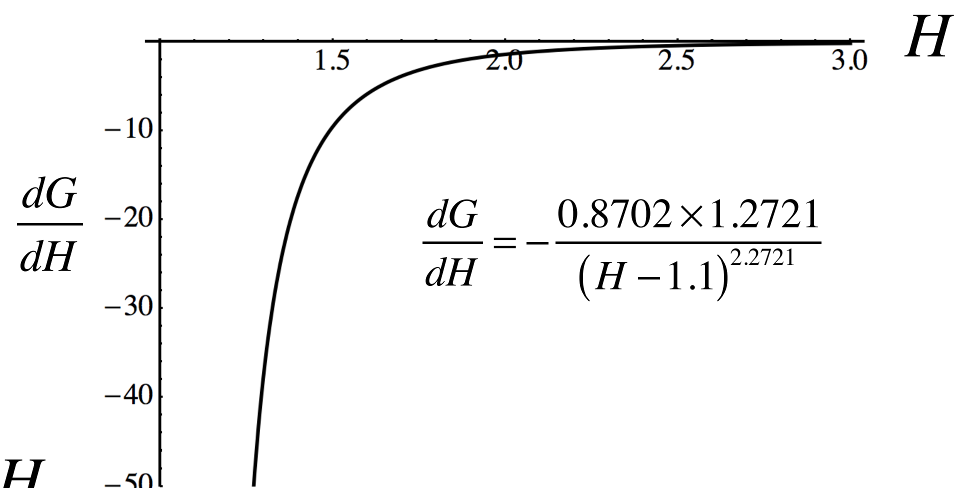




I prefer a single smooth function

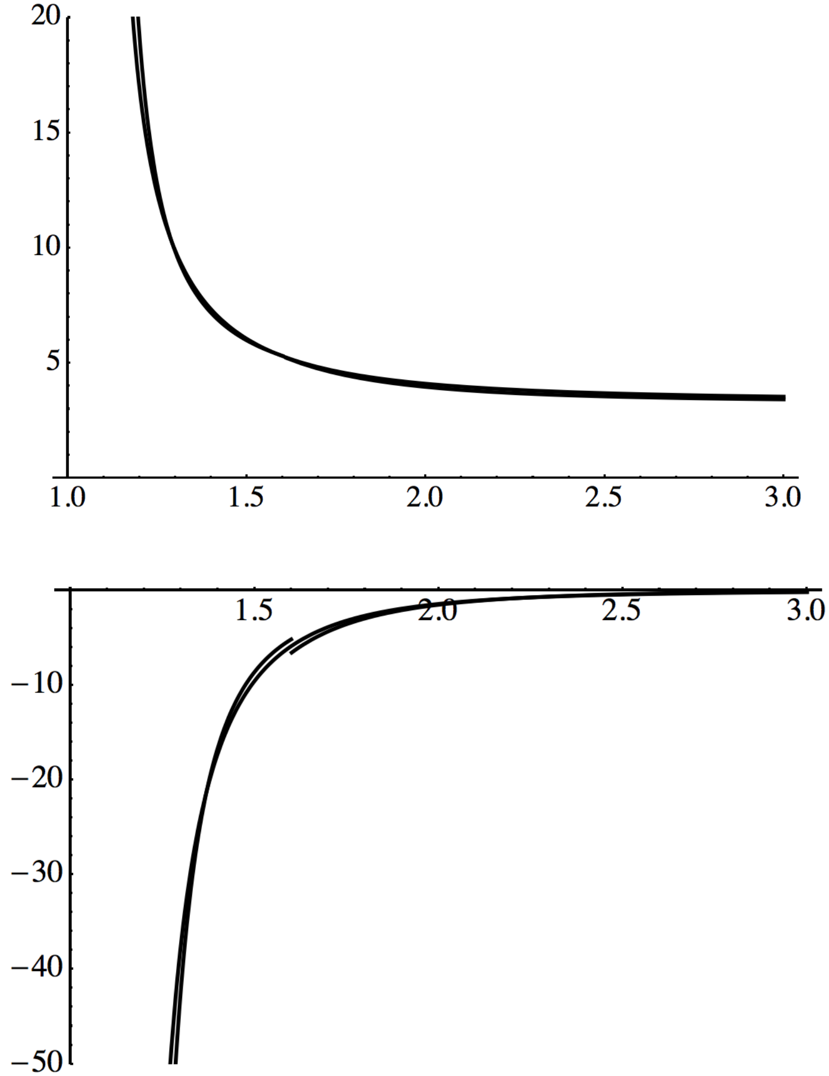


$$G(H) = 3.0445 + \frac{0.8702}{(H-1.1)^{1.2721}}$$



$$\frac{dG}{dH} = -\frac{0.8702 \times 1.2721}{(H-1.1)^{2.2721}}$$

Comparison



From Head's paper

2.3. *Determination of Functions F and G.* For this purpose the experimental data of Newman[2] and of Schubauer and Klebanoff[3] have been used†. In each case values of δ were obtained from tables of the measured profiles, δ being arbitrarily defined as the value of y for which $u/U = 0{\cdot}995$. From the values of δ and the corresponding values of H, θ, U and x, the quantities $\frac{1}{U}\frac{d}{dx}[U(\delta - \delta^*)]$ and $H_{\delta-\delta^*}$ were obtained and are shown plotted in Figs. 1 and 2. If the assumptions made in the previous Sections had been correct, and if both the analysis and the experimental data had been entirely free from error then, of course, the points obtained from the two sets of results should have coincided with common curves defining the two functions. In fact, however, as will be seen from the Figures there is considerable scatter of the points, and in Fig. 1 there is a fairly marked and consistent discrepancy between the two sets of results which makes the drawing of a hypothetical common curve, representing the function $F(H_{\delta-\delta^*})$, a somewhat arbitrary procedure. However, such a curve has been drawn, its justification being found *a posteriori*, in the accuracy with which it has enabled the form-parameter development to be predicted in the cases considered below. The curve relating $H_{\delta-\delta^*}$ to the normal form parameter H is rather more accurately defined, although here also there is some discrepancy between the two sets of results, and the values of H given by Schubauer and Klebanoff for the region where the pressure gradient was favourable appear somewhat high.

Typical range of H vs R_{ex} for turbulent boundary layers

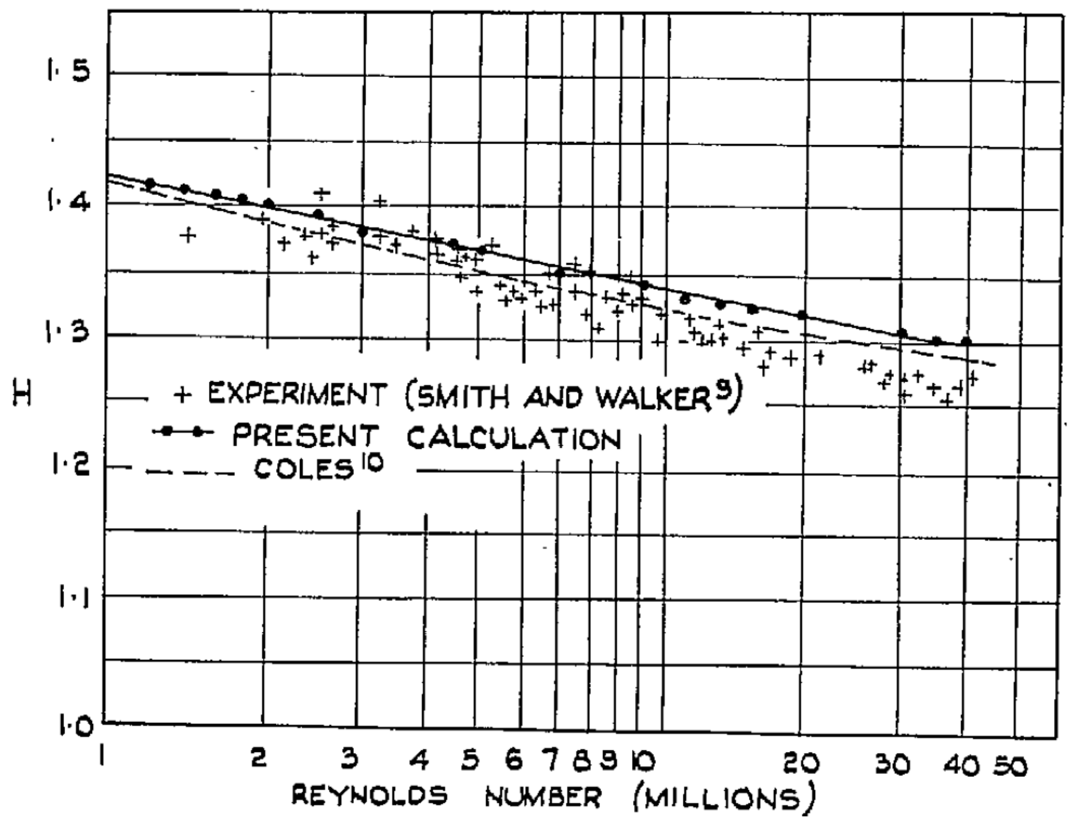


Fig. 3. Flat-plate results compared with experiment.

Recall the von Karman integral momentum equation

$$\boxed{\frac{d\theta}{dx} + (2 + H)\frac{\theta}{U_e}\frac{dU_e}{dx} = \frac{C_f}{2}} \tag{9.82}$$

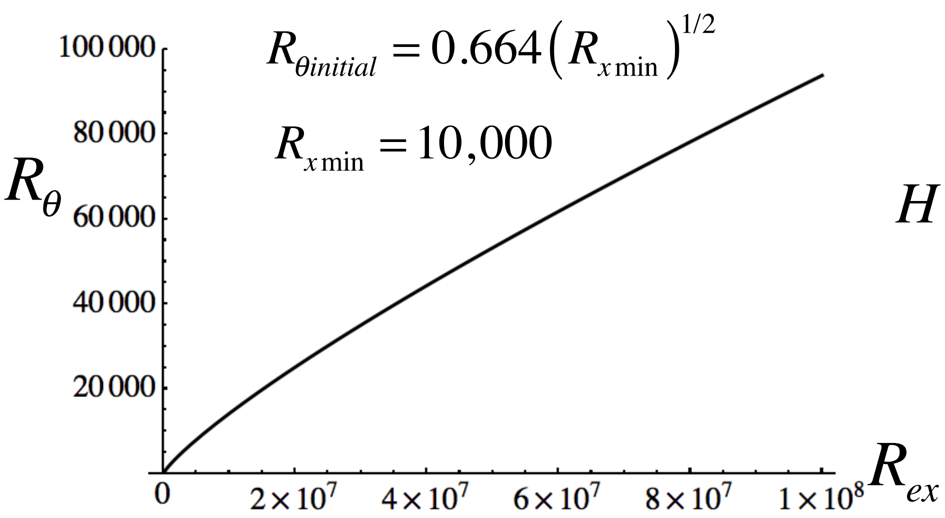
For given initial conditions on theta and H and known free stream velocity distribution $U_e(x)$ this equation is solved along with the auxiliary equations

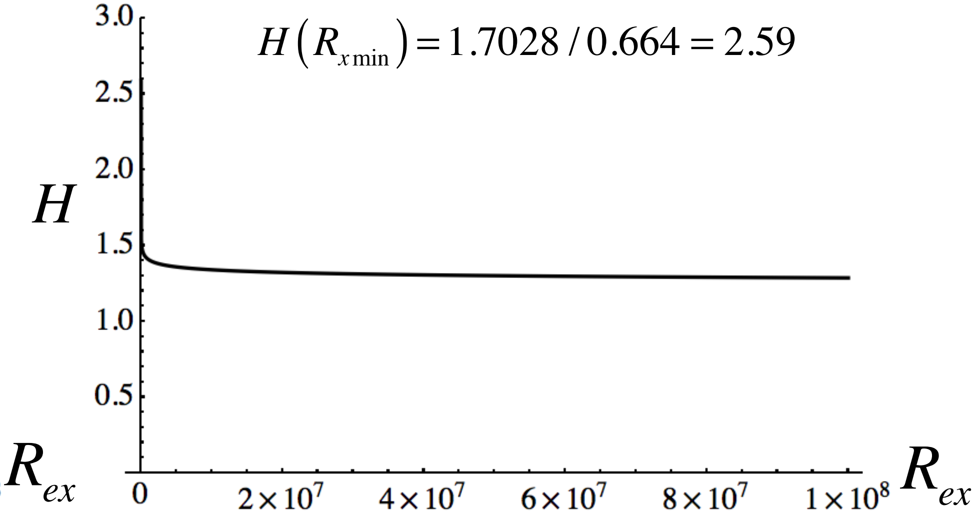
$$C_f = \frac{0.246}{10^{0.678H} R_\theta^{\;0.268}} \qquad R_\theta = \frac{U_e\theta}{\nu}$$

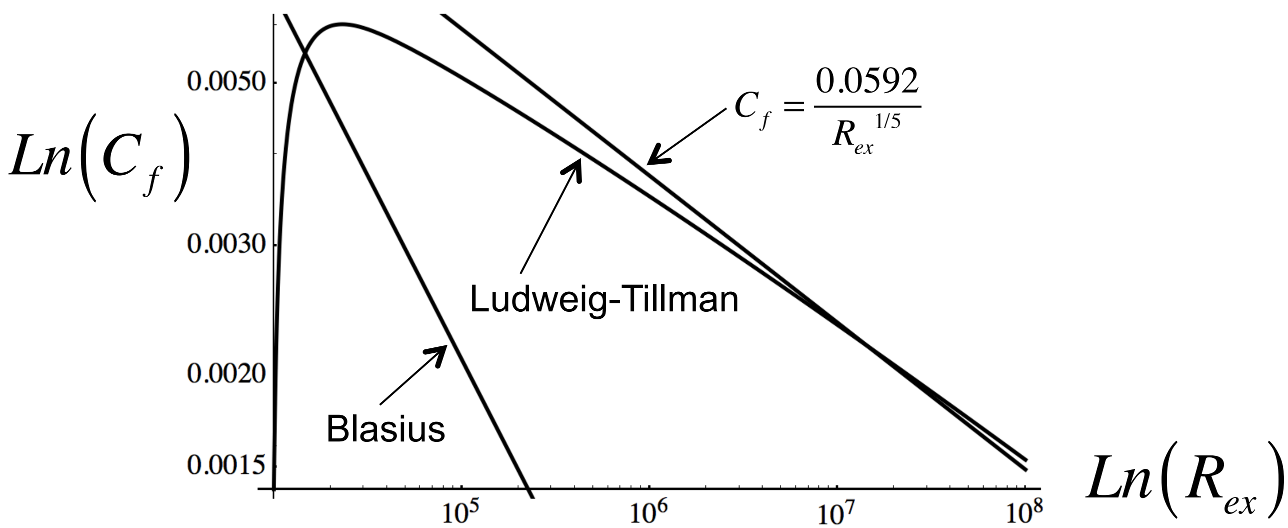
$$\frac{1}{U_e}\frac{d}{dx}\left(U_e\theta H_1\right) = F\left(H_1\right) = \frac{0.0306}{\left(H_1 - 3.0\right)^{0.6169}}$$

$$H_1 = G\left(H\right) = 3.0445 + \frac{0.8702}{\left(H - 1.1\right)^{1.2721}}$$

Zero pressure gradient turbulent boundary layer $C_p = 0$







Potential flow about a circular cylinder

$$\frac{U_e}{U_\infty} = 2Sin\left(\frac{x}{R}\right) \tag{9.163}$$

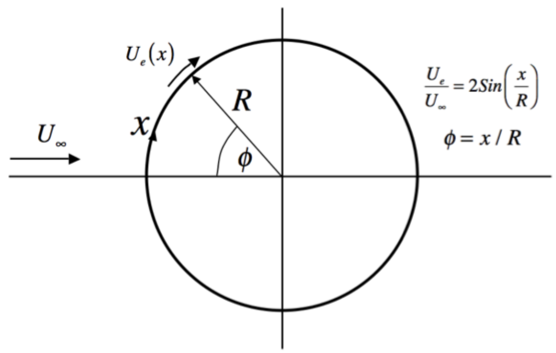


Figure 9.19 Example for Thwaites' method.

$$\left(\frac{\theta}{R}\right)^2 R_e = \frac{0.441}{Sin^6(\phi)}\int_0^\phi Sin^5(\phi')d\phi' \tag{9.164}$$

$$R_e = \frac{U_\infty 2R}{\nu} \tag{9.165}$$

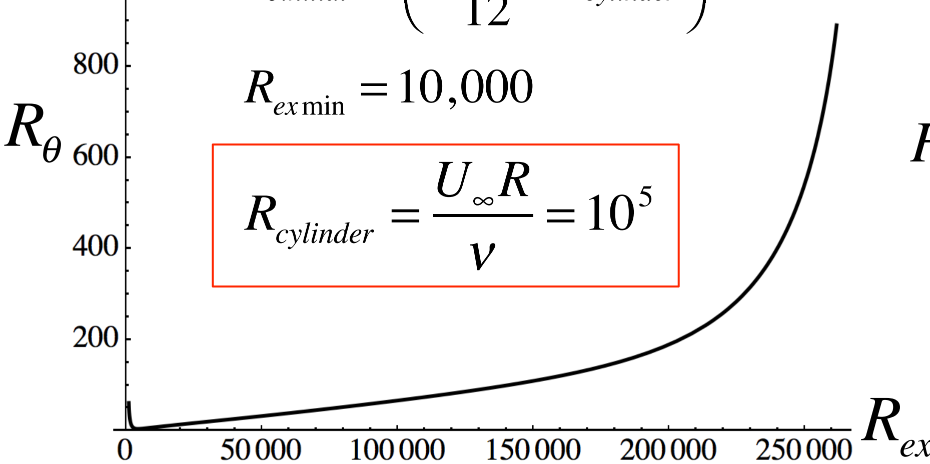
Thwaites' method gives a finite momentum thickness at the forward stagnation point. This is useful in a wing leading edge calculation.

$$\lim_{\phi \to 0}\left(\frac{\theta}{R}\right)^2 R_e = \frac{0.441}{\phi^6}\int_0^\phi \phi'^5 d\phi' = \frac{0.441}{6} \tag{9.166}$$

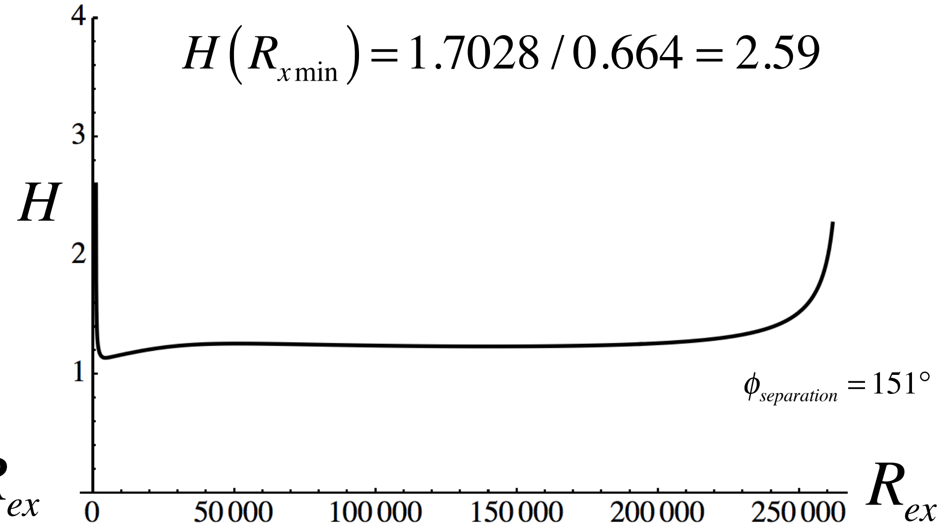
$$R_{\theta initial} = \left(\frac{0.664}{12} R_{cylinder} \right)^{1/2}$$

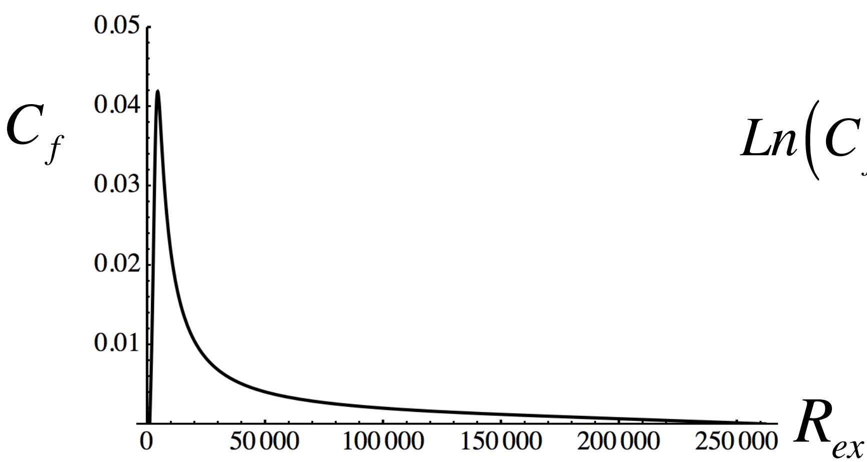
$$R_{ex\min} = 10{,}000$$

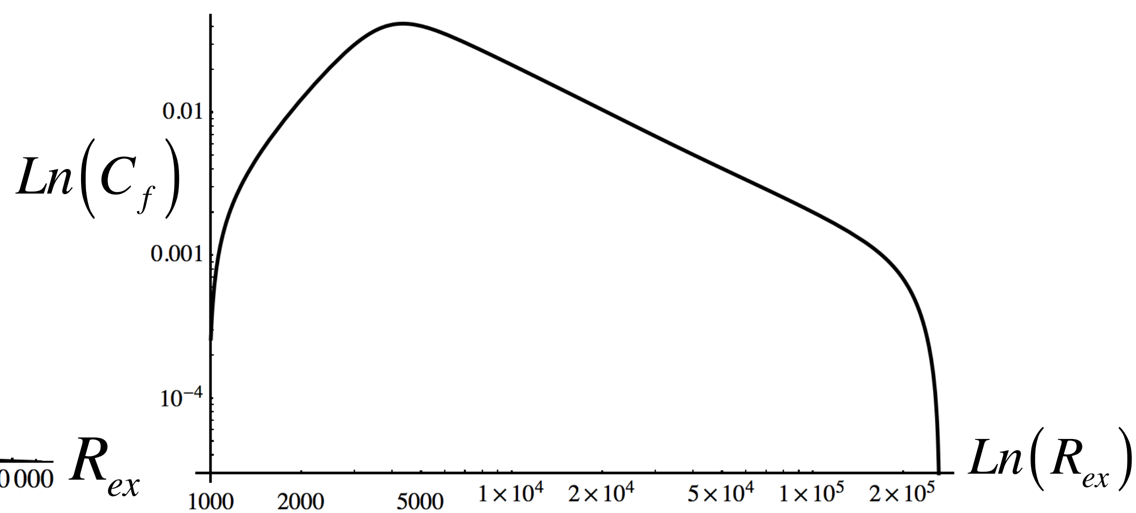
$$\boxed{R_{cylinder} = \frac{U_\infty R}{\nu} = 10^5}$$



$$H\left(R_{x\min}\right) = 1.7028 / 0.664 = 2.59$$



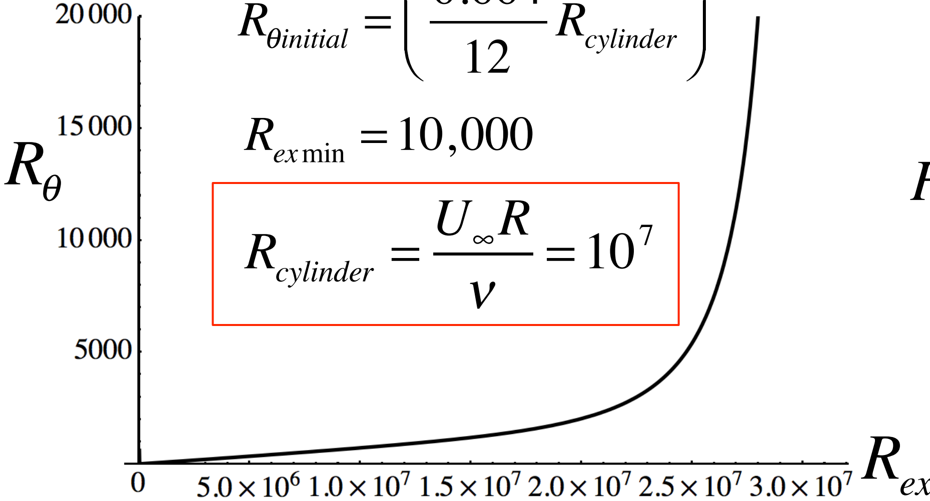




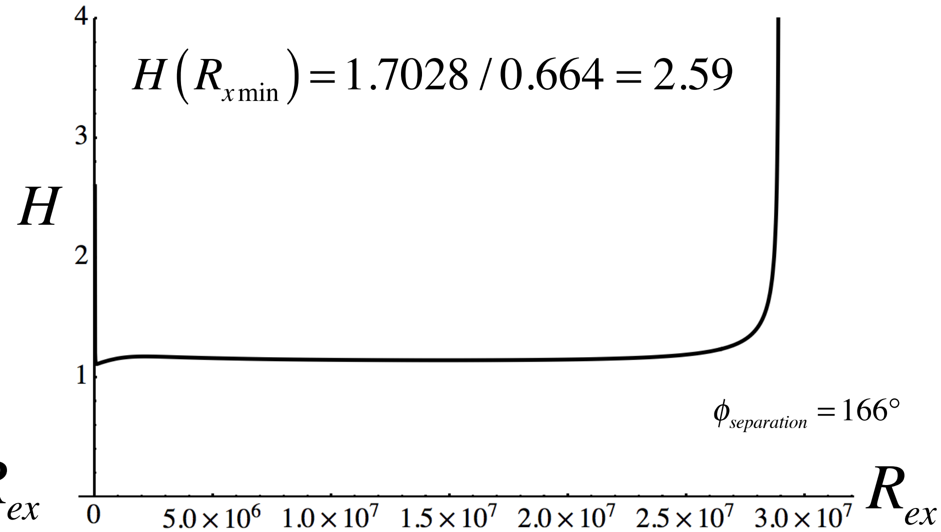
$$R_{\theta initial} = \left(\frac{0.664}{12} R_{cylinder} \right)^{1/2}$$

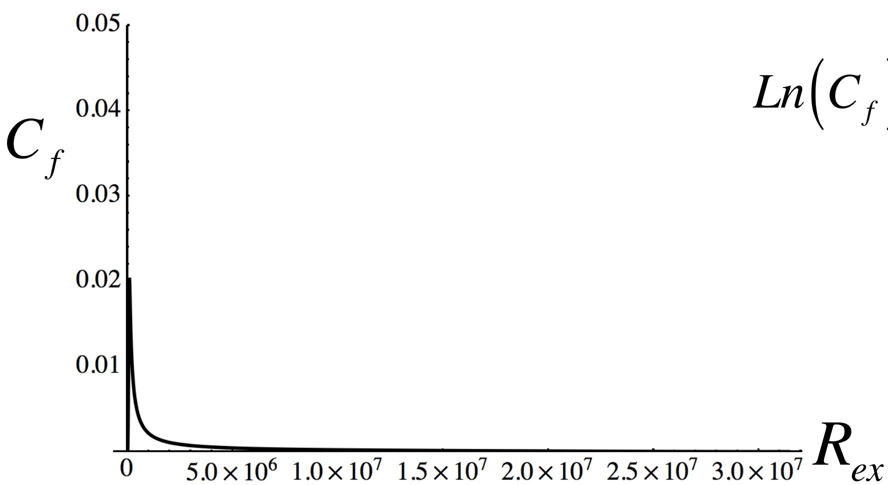
$$R_{ex\min} = 10{,}000$$

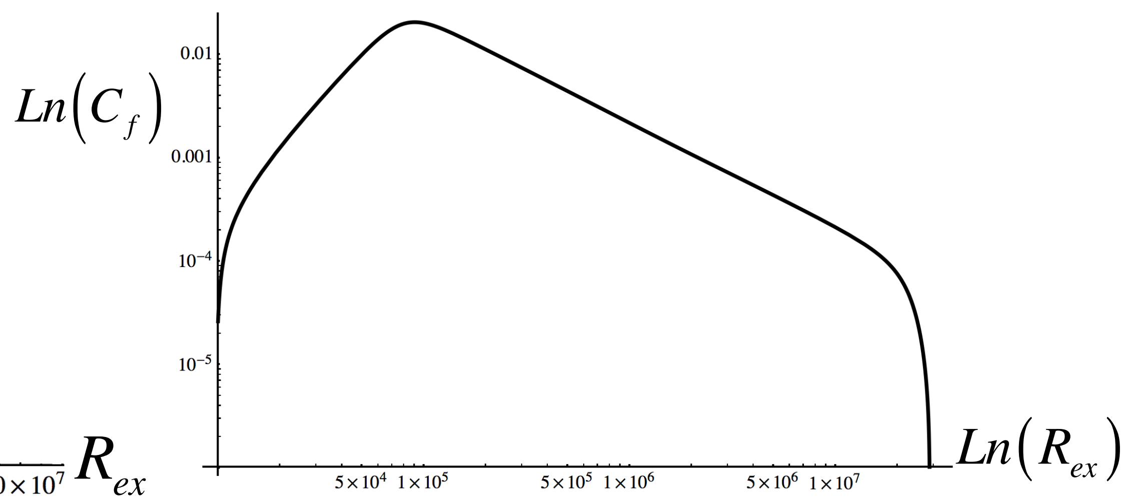
$$\boxed{R_{cylinder} = \frac{U_\infty R}{\nu} = 10^7}$$



$$H\left(R_{x\min}\right) = 1.7028/0.664 = 2.59$$







9.12 Transformation between flat plate and curved wall boundary layers

Boundary layer equations

$$\frac{\partial \rho U}{\partial x} + \frac{\partial \rho V}{\partial y} = 0$$

$$\rho U \frac{\partial U}{\partial x} + \rho V \frac{\partial U}{\partial y} + \frac{dP_e}{dx} - \frac{\partial \tau_{xy}}{\partial y} = 0$$

$$\rho U C_p \frac{\partial T}{\partial x} + \rho V C_p \frac{\partial T}{\partial y} - U \frac{dP_e}{dx} + \frac{\partial Q_y}{\partial y} - \tau_{xy} \frac{\partial U}{\partial y} = 0$$

Transform variables by adding an arbitrary function of x to the y coordinate

$$\tilde{x} = x$$
$$\tilde{y} = y + g(x)$$
$$\tilde{U}(\tilde{x}, \tilde{y}) = U(x, y)$$
$$\tilde{V}(\tilde{x}, \tilde{y}) = V(x, y) + U(x, y)\frac{dg(x)}{dx}$$
$$\tilde{\rho}(\tilde{x}, \tilde{y}) = \rho(x, y)$$
$$\tilde{\tau}_{xy}(\tilde{x}, \tilde{y}) = \tau_{xy}(x, y)$$
$$\tilde{Q}_y(\tilde{x}, \tilde{y}) = Q_y(x, y)$$
$$P_e(\tilde{x}) = P_e(x)$$

$$\frac{\partial \tilde{U}}{\partial \tilde{x}} = \frac{\partial U}{\partial x} - \frac{dg}{dx}\frac{\partial U}{\partial y}$$
$$\frac{\partial \tilde{U}}{\partial \tilde{y}} = \frac{\partial U}{\partial y}$$
$$\frac{\partial^2 \tilde{U}}{\partial \tilde{y}^2} = \frac{\partial^2 U}{\partial y^2}$$
$$\frac{\partial \tilde{V}}{\partial \tilde{y}} = \frac{\partial V}{\partial y} + \frac{dg}{dx}\frac{\partial U}{\partial y}$$
$$\frac{\partial \tilde{\rho}}{\partial \tilde{x}} = \frac{\partial \rho}{\partial x} - \frac{dg}{dx}\frac{\partial \rho}{\partial y}$$

$$\frac{\partial \tilde{\rho}}{\partial \tilde{y}} = \frac{\partial \rho}{\partial y}$$
$$\frac{\partial \tilde{T}}{\partial \tilde{x}} = \frac{\partial T}{\partial x} - \frac{dg}{dx}\frac{\partial T}{\partial y}$$
$$\frac{\partial \tilde{T}}{\partial \tilde{y}} = \frac{\partial T}{\partial y}$$
$$\frac{\partial \tilde{Q}_y}{\partial \tilde{y}} = \frac{\partial Q_y}{\partial y}$$
$$\frac{\partial \tilde{\tau}_{xy}(\tilde{x}, \tilde{y})}{\partial \tilde{y}} = \frac{\partial \tau_{xy}(x, y)}{\partial y}$$

$$\tilde{\rho}\tilde{U}\frac{\partial \tilde{U}}{\partial \tilde{x}} + \tilde{\rho}\tilde{V}\frac{\partial \tilde{U}}{\partial \tilde{y}} + \frac{\partial \tilde{P}_e}{\partial \tilde{x}} - \frac{\partial \tilde{\tau}_{\tilde{x}\tilde{y}}}{\partial \tilde{y}} = \rho U\left(\frac{\partial U}{\partial x} - \cancel{\frac{dg}{dx}\frac{\partial U}{\partial y}}\right) + \rho\left(V + \cancel{U\frac{dg}{dx}}\right)\frac{\partial U}{\partial y} + \frac{\partial P_e}{\partial x} - \frac{\partial \tau_{xy}}{\partial y} = \rho U\frac{\partial U}{\partial x} + \rho V\frac{\partial U}{\partial y} + \frac{\partial P_e}{\partial x} - \frac{\partial \tau_{xy}}{\partial y}$$

Insert the transformations of variables and derivatives into the equations of motion. The result is that the equations are mapped to themselves.

$$\frac{\partial\tilde{\rho}\tilde{U}}{\partial\tilde{x}} + \frac{\partial\tilde{\rho}\tilde{V}}{\partial\tilde{y}} = \frac{\partial\rho U}{\partial x} + \frac{\partial\rho V}{\partial y} = 0$$

$$\tilde{\rho}\tilde{U}\frac{\partial\tilde{U}}{\partial\tilde{x}} + \tilde{\rho}\tilde{V}\frac{\partial\tilde{U}}{\partial\tilde{y}} + \frac{d\tilde{P}_e}{d\tilde{x}} - \frac{\partial\tilde{\tau}_{xy}}{\partial\tilde{y}} = \rho U\frac{\partial U}{\partial x} + \rho V\frac{\partial U}{\partial y} + \frac{dP_e}{dx} - \frac{\partial\tau_{xy}}{\partial y} = 0$$

$$\tilde{\rho}\tilde{U}C_p\frac{\partial\tilde{T}}{\partial\tilde{x}} + \tilde{\rho}\tilde{V}C_p\frac{\partial\tilde{T}}{\partial\tilde{y}} - \tilde{U}\frac{d\tilde{P}_e}{d\tilde{x}} + \frac{\partial\tilde{Q}_y}{\partial\tilde{y}} - \tilde{\tau}_{xy}\frac{\partial\tilde{U}}{\partial\tilde{y}} =$$

$$\rho UC_p\frac{\partial T}{\partial x} + \rho VC_p\frac{\partial T}{\partial y} - U\frac{dP_e}{dx} + \frac{\partial Q_y}{\partial y} - \tau_{xy}\frac{\partial U}{\partial\tilde{y}} = 0$$

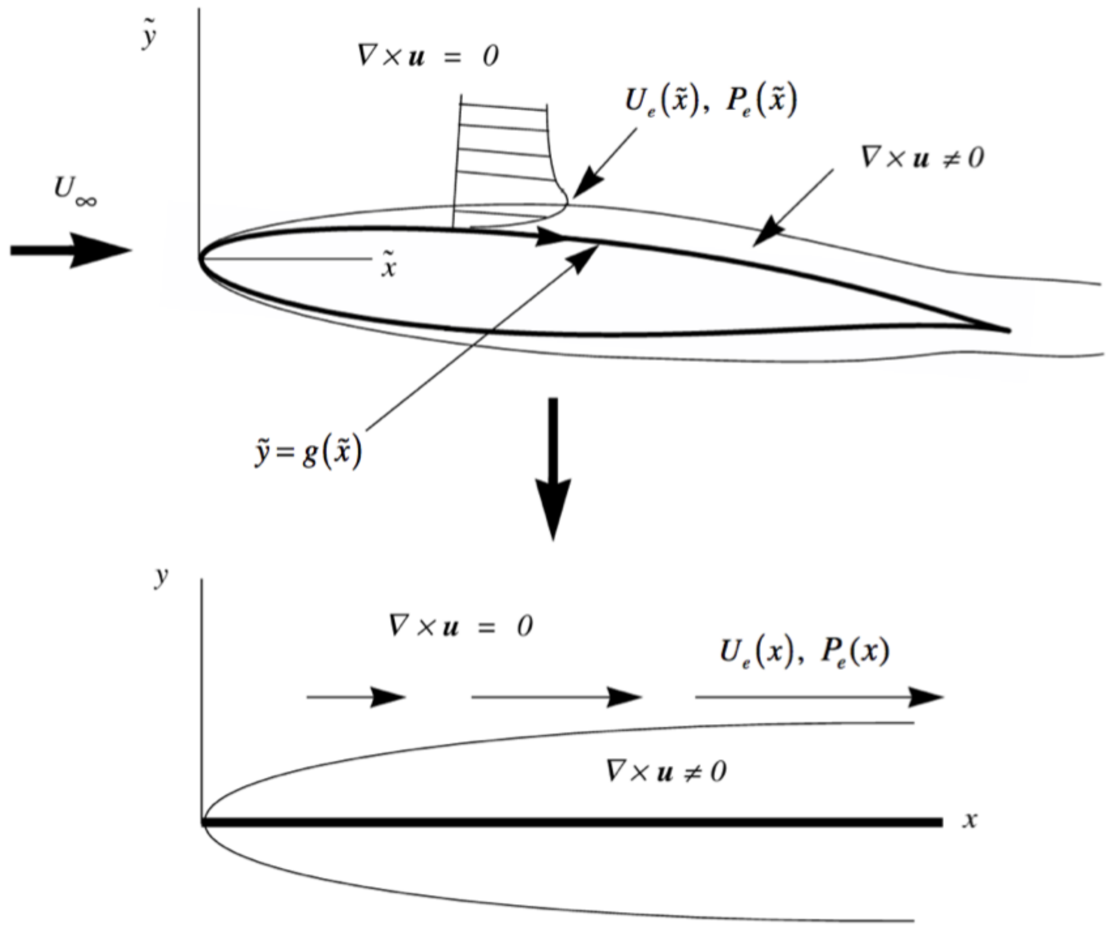


Figure 9.29 Mapping of the boundary layer developing over an airfoil to the boundary layer on a flat plate with a pressure gradient.

Viscous-inciscid interaction algorithm

An iterative algorithm can be used to determine the viscous flow over a complex shape such as the airfoil shown in Figure 9.29

1) Solve for the potential flow over the airfoil.

2) Use the potential flow velocity at the airfoil surface as the $U_e(x)$ for a boundary layer calculation beginning at the leading edge.

3) Determine the displacement thickness of the boundary layer and use the data to define a new airfoil shape. Repeat the potential flow calculation using the new airfoil shape to determine a new $U_e(x)$.

4) Using the new $U_e(x)$ repeat the boundary layer calculation.

A few iterations of this viscous-inviscid interaction procedure will converge to an accurate solution for the viscous, compressible flow over the airfoil.