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AA210A

Fundamentals of Compressible Flow

Chapter 5 -The conservation equations

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5.1 Leibniz rule for differentiation of integrals

Differentiation under the integral sign. According to the fundamental theorem of calculus if

$$I(x) = \int_{constant}^{x} f(x')dx'$$

then

$$\frac{dI}{dx} = f(x).$$

Similarly if

$$I(x) = \int_{x}^{constant} f(x')dx'$$

then

$$\frac{dI}{dx} = -f(x)$$

Suppose the function depends on two variables

$$I(t) = \int_a^b f(x', t)dx'$$

where the limits of integration are constant.

The derivative of the integral with respect to time is

$$\frac{dI(t)}{dt} = \int_a^b \frac{\partial}{\partial t} f(x', t)dx'$$

But suppose the limits of the integral depend on time.

$$I(t, a(t), b(t)) = \int_{a(t)}^{b(t)} f(x', t)dx'$$

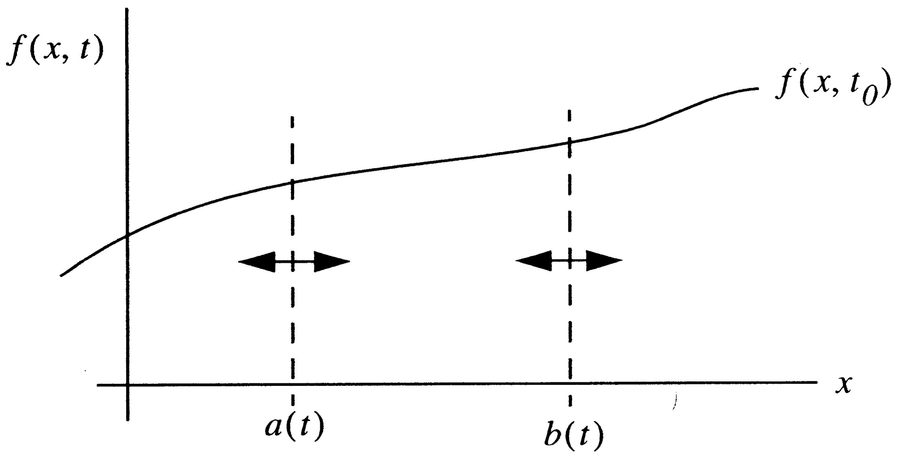


Figure 5.1 Integration with a moving boundary. The function $f(x, t)$ is shown at one instant in time.

From the chain rule.

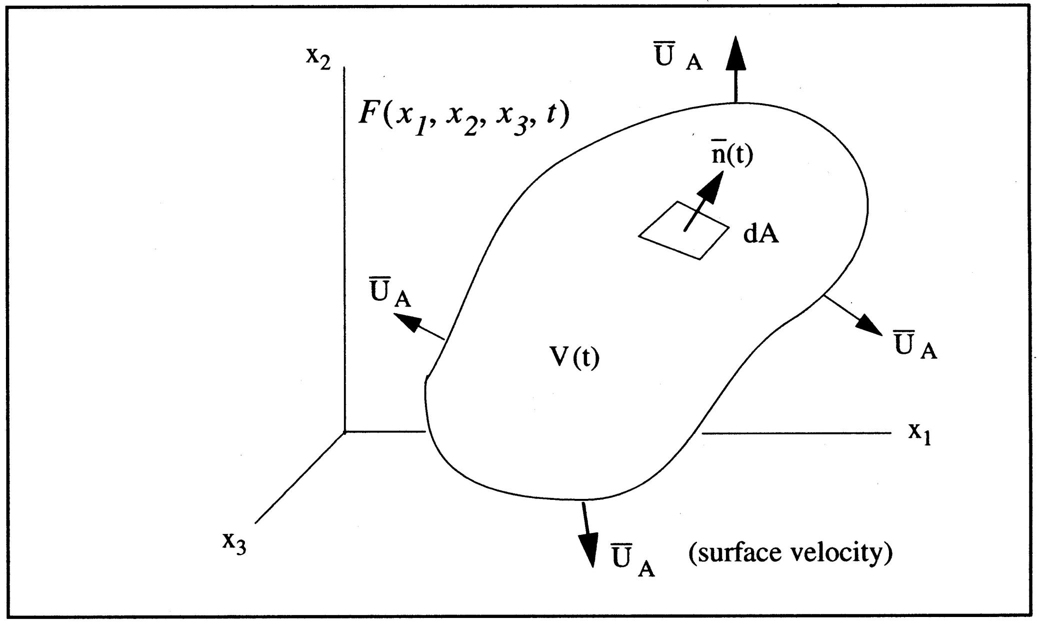
$$\frac{DI}{Dt} = \frac{\partial I}{\partial t} + \frac{\partial I}{\partial a}\frac{da}{dt} + \frac{\partial I}{\partial b}\frac{db}{dt}$$

In this case the derivative of the integral with respect to time is

$$\frac{DI}{Dt} = \int_{a(t)}^{b(t)} \frac{\partial}{\partial t} f(x', t)dx' + f(b(t), t)\frac{db}{dt} - f(a(t), t)\frac{da}{dt}.$$

Time rate of change due to movement of the boundaries.

In three dimensions Leibniz' rule describes the time rate of change of the integral of some function of space and time, F , contained inside a control volume V.



$$\frac{D}{Dt}\int_{V(t)} F\,dV = \int_{V(t)} \frac{\partial F}{\partial t}\,dV + \int_{A(t)} F\,\bar{U}_A \bullet \bar{n}\,dA\,.$$

Rate of change of the total amount of F in V

=

Rate due to changes of F within V

+

Rate due to movement of the surface of V

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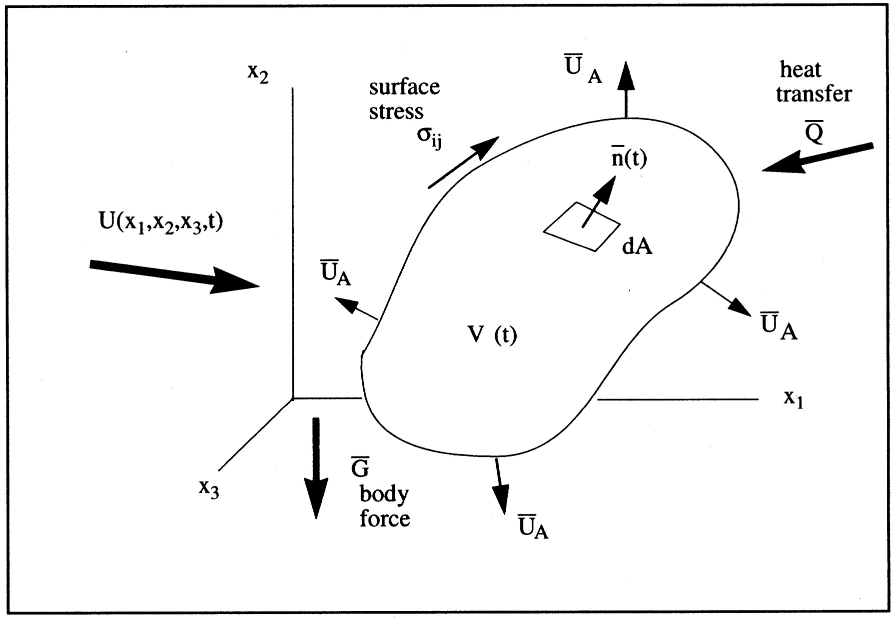
Consider a fluid with the velocity field defined at every point.



Let the velocity of each surface element coincide with the fluid velocity. This is called a Lagrangian control volume.

$$\frac{D}{Dt}\int_{V(t)} F\,dV = \int_{V(t)} \frac{\partial F}{\partial t}dV + \int_{A(t)} F\,\bar{U}\bullet\bar{n}\,dA\,.$$

Use Gauss's theorem to convert the surface integral to a volume integral.

Reynolds transport theorem

$$\frac{D}{Dt}\int_{V(t)} F\,dV = \int_{V(t)} \left(\frac{\partial F}{\partial t} + \nabla\bullet(F\bar{U})\right)dV\,.$$

5.2 Conservation of mass

The Reynolds transport theorem applied to the density is

$$\frac{D}{Dt}\int_{V(t)} \rho dV = \int_{V(t)} \left(\frac{\partial \rho}{\partial t} + \nabla \bullet (\rho \bar{U})\right) dV .$$

Since there are no sources of mass contained in the control volume and the choice of control volume is arbitrary the kernel of the integral must be zero.

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \bullet (\rho \bar{U}) = 0} .$$

This is the general procedure that we will use to derive the differential form of the equations of motion.

Incompressible flow

Expand the continuity equation.

$$\frac{\partial\rho}{\partial t} + \overline{U} \bullet \nabla\rho + \rho\nabla \bullet \overline{U} = 0.$$

If the density is constant then the continuity equation reduces to

$$\rho\nabla \bullet \overline{U} = 0.$$

5.3 Conservation of momentum

The stress tensor in a fluid is composed of two parts; an isotropic part due to the pressure and a symmetric part due to viscous friction.

$$\sigma_{ij} = -P\delta_{ij} + \tau_{ij}$$

where

$$\bar{\bar{I}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{cases} \delta_{ij} = 1 \; ; & i = j \\ \delta_{ij} = 0 \; ; & i \neq j \end{cases}$$

We deal only with Newtonian fluids for which the stress is linearly related to the rate-of-strain.

$$\tau_{ij} = 2\mu S_{ij} - \left(\frac{2}{3}\mu - \mu_v\right)\delta_{ij} S_{kk}$$

where

$$S_{ij} = (1/2)(\partial U_i/\partial x_j + \partial U_j/\partial x_i).$$

Notice that viscous forces contribute to the normal stresses through the non-zero diagonal terms in the stress tensor.

$$\tau_{ij} = 2\mu S_{ij} - \left(\frac{2}{3}\mu - \mu_v\right)\delta_{ij}S_{kk}$$

Sum the diagonal terms to generate the mean normal stress

$$\sigma_{mean} = (1/3)\sigma_{ii} = -P + \mu_v S_{kk}.$$

The “bulk viscosity” that appears here is often assumed to be zero. This is the so-called Stokes hypothesis. In general the bulk viscosity is not zero except for monatomic gases but the Stokes hypothesis is often invoked anyway.

The rate of change of the total amount of momentum inside the control volume is determined by the external forces that act on the control volume surface.

$$\frac{D}{Dt}\int_{V(t)} \rho \bar{U} dV = \int_{A(t)} (-P\bar{\bar{I}} + \bar{\bar{\tau}}) \bullet \bar{n} dA + \int_{V(t)} \rho \bar{G} dV .$$

Use the Reynolds transport theorem to replace the left-hand-side and Gauss's theorem to replace the surface integrals.

$$\int_{V(t)} \left(\frac{\partial \rho \bar{U}}{\partial t} + \nabla \bullet (\rho \bar{U}\bar{U} + P\bar{\bar{I}} - \bar{\bar{\tau}}) - \rho \bar{G} \right) dV = 0 .$$

Since there are no sources of momentum inside the control volume and the choice of control volume is arbitrary, the kernel must be zero.

$$\boxed{\frac{\partial \rho \bar{U}}{\partial t} + \nabla \bullet (\rho \bar{U}\bar{U} + P\bar{\bar{I}} - \bar{\bar{\tau}}) - \rho \bar{G} = 0} .$$

5.4 Conservation of energy

The rate of change of the total energy inside the control volume is determined by the rate at which the external forces do work on the control volume plus the rate of heat transfer across the control volume surface.

$$\frac{D}{Dt}\int_{V(t)} \rho(e+k)dV = \int_{A(t)} \left(\left(-P\bar{\bar{I}} + \bar{\bar{\tau}}\right) \bullet \bar{U} - \bar{Q}\right) \bullet \bar{n}dA + \int_{V(t)} \left(\rho\bar{G} \bullet \bar{U}\right)dV .$$

In a linear heat conducting medium

$$Q_i = -\kappa(\partial T / \partial x_i)$$

Again, use the Reynolds transport theorem to replace the left-hand-side and Gauss's theorem to replace the surface integrals.

$$\int_{V(t)} \left(\frac{\partial\rho(e+k)}{\partial t} + \nabla \bullet \left(\rho\bar{U}\left(e + \frac{P}{\rho} + k\right) - \bar{\bar{\tau}} \bullet \bar{U} + \bar{Q}\right) - \rho\bar{G} \bullet \bar{U}\right)dV = 0 .$$

We make the usual argument.

Since there are no sources of energy inside the control volume and the choice of control volume is arbitrary, the kernel must be zero.

$$\frac{\partial \rho(e+k)}{\partial t} + \nabla \bullet \left(\rho \overline{U} \left(e + \frac{P}{\rho} + k \right) - \bar{\bar{\tau}} \bullet \overline{U} + \overline{Q} \right) - \rho \overline{G} \bullet \overline{U} = 0$$

Stagnation enthalpy

$$h_t = e + \frac{P}{\rho} + k = h + \frac{1}{2} U_i U_i$$

Typical gas transport properties at 300K and one atmosphere.

Fluid	$\mu \times 10^5$, kg/(m)(s)	μ_v/μ	$\kappa \times 10^2$, J/(m)(s)(K)	$\frac{\mu}{\rho} \times 10^5$, m^2/s	Pr
He	1.98	0	15.0	12.2	0.67
Ar	2.27	0	1.77	1.40	0.67
H_2	0.887	32	17.3	10.8	0.71
N_2	1.66	0.8	2.52	1.46	0.71
O_2	2.07	0.4	2.58	1.59	0.72
CO_2	1.50	1,000	1.66	0.837	0.75
Air	1.85	0.6	2.58	1.57	0.71
H_2O (*liquid*)	85.7	3.1	61	0.0857	6.0
Ethyl alcohol	110	4.5	18.3	0.14	15
Glycerine	134,000	0.4	29	109	11,000

$$Pr = \frac{\mu C_p}{\kappa}$$

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5.5 Summary - differential equations of motion

$$
\left.
\begin{aligned}
&\frac{\partial \rho}{\partial t} + \nabla \bullet (\rho \bar{U}) = 0 \\
&\frac{\partial \rho \bar{U}}{\partial t} + \nabla \bullet (\rho \bar{U}\bar{U} + P\bar{\bar{I}} - \bar{\bar{\tau}}) - \rho \bar{G} = 0 \\
&\frac{\partial \rho (e+k)}{\partial t} + \nabla \bullet \left(\rho \bar{U}\left(e + \frac{P}{\rho} + k\right) - \bar{\bar{\tau}} \bullet \bar{U} + \bar{Q}\right) - \rho \bar{G} \bullet \bar{U} = 0
\end{aligned}
\right\}
$$

$$
\left.
\begin{aligned}
&\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho U_i) = 0 \\
&\frac{\partial \rho U_i}{\partial t} + \frac{\partial}{\partial x_j}(\rho U_i U_j + P\delta_{ij} - \tau_{ij}) - \rho G_i = 0 \\
&\frac{\partial \rho (e+k)}{\partial t} + \frac{\partial}{\partial x_i}\left(\rho U_i\left(e + \frac{P}{\rho} + k\right) - \tau_{ij} U_j + Q_i\right) - \rho G_i U_i = 0
\end{aligned}
\right\}.
$$

5.6 Integral form of the equations of motion

Recall the Leibniz rule

$$\frac{D}{Dt}\int_{V(t)} F\,dV = \int_{V(t)} \frac{\partial F}{\partial t}\,dV + \int_{A(t)} F\,\bar{U}_A \bullet \bar{n}\,dA\,.$$

5.6.1 Integral equations on an Eulerian control volume

If the surface of the control volume is fixed in space, ie, the velocity of the surface is zero then

$$\frac{d}{dt}\int_{V} F\,dV = \int_{V} \frac{\partial F}{\partial t}\,dV$$

This is called an Eulerian control volume.

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The integral form of the continuity equation on an Eulerian control volume is derived as follows. Let F= δενσιτψ

$$\frac{d}{dt}\int_V \rho dV = \int_V \frac{\partial \rho}{\partial t} dV$$

Use the differential equation for continuity to replace the partial derivative inside the integral on the right-hand-side

$$\frac{d}{dt}\int_V \rho dV = -\int_V \nabla \bullet (\rho \bar{U}) dV$$

Use the Gauss theorem to convert the volume integral to a surface integral. The integral form of the continuity equation is:

$$\frac{d}{dt}\int_V \rho dV + \int_A \rho \bar{U} \bullet \bar{n} dA = 0$$

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The integral form of the conservation equations on an Eulerian control volume is

$$\frac{d}{dt}\int_V \rho dV + \int_A (\rho\bar{U})\bullet\bar{n}dA = 0$$

$$\frac{d}{dt}\int_V \rho\bar{U}dV + \int_A (\rho\bar{U}\bar{U} + P\bar{\bar{I}} - \bar{\bar{\tau}})\bullet\bar{n}dA - \int_V \rho\bar{G}dV = 0$$

$$\frac{d}{dt}\int_V \rho(e+k)dV + \int_A \left(\rho\bar{U}\left(e + \frac{P}{\rho} + k\right) - \bar{\bar{\tau}}\bullet\bar{U} + \bar{Q}\right)\bullet\bar{n}dA - \int_V (\rho\bar{G}\bullet\bar{U})dV = 0$$

5.6.2 Mixed Eulerian-Lagrangian control volumes

The integral form of the continuity equation on a Mixed Eulerian-Lagrangian control volume is derived as follows. Let F in Liebniz rule be the fluid density.

$$\frac{D}{Dt}\int_{V(t)} \rho dV = \int_{V(t)} \frac{\partial \rho}{\partial t} dV + \int_{A(t)} \rho\, \bar{U}_A \bullet \bar{n} dA$$

Use the differential equation for continuity to replace the partial derivative inside the first integral on the right-hand-side and use the Gauss theorem to convert the volume integral to a surface integral. The integral form of the continuity equation on a Mixed Eulerian-Lagrangian control volume is

$$\frac{D}{Dt}\int_{V(t)} \rho dV = -\int_{A(t)} \rho \bar{U} \bullet \bar{n} dA + \int_{A(t)} \rho\, \bar{U}_A \bullet \bar{n} dA$$

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The integral equations of motion on a general control volume where the surface velocity is not the same as the fluid velocity are derived in a similar way.

The most general integral form of the conservation equations is

$$\frac{D}{Dt}\int_{V(t)} \rho dV + \int_{A(t)} \rho(\bar{U} - \bar{U}_A) \bullet \bar{n} dA = 0$$

$$\frac{D}{Dt}\int_{V(t)} \rho \bar{U} dV + \int_{A(t)} (\rho \bar{U}(\bar{U} - \bar{U}_A) + P\bar{\bar{I}} - \bar{\bar{\tau}}) \bullet \bar{n} dA - \int_{V(t)} \rho \bar{G} dV = 0$$

$$\frac{D}{Dt}\int_{V(t)} \rho(e+k) dV + \int_{A(t)} (\rho(e+k) \bullet (\bar{U} - \bar{U}_A) + P\bar{\bar{I}} \bullet \bar{U} - \bar{\bar{\tau}} \bullet \bar{U} + \bar{Q}) \bullet \bar{n} dA - \int_{V(t)} (\rho \bar{G} \bullet \bar{U}) dV = 0$$

Remember $\bar{U}_A$ is the velocity of the control volume surface.

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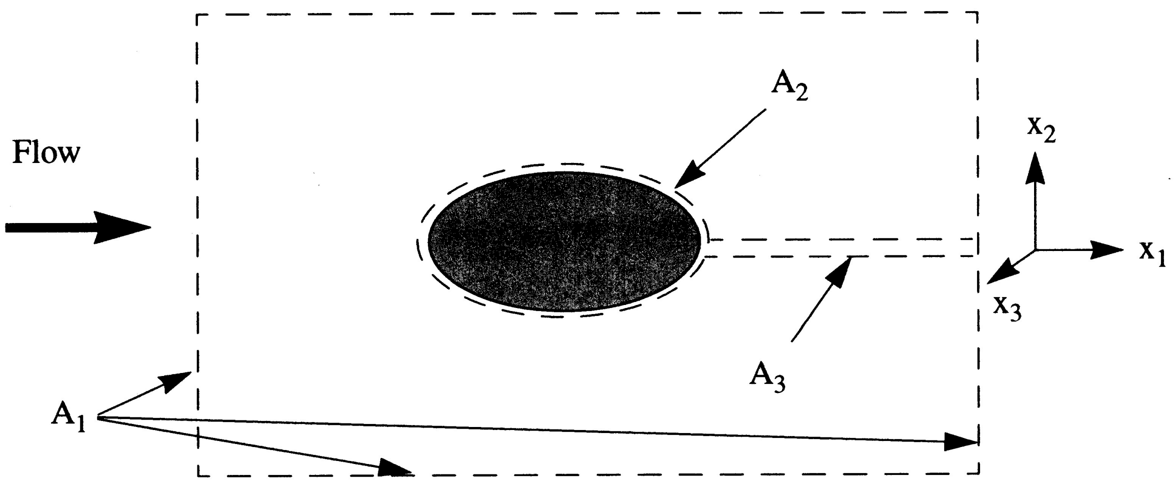
5.7 Applications of control volume analysis

5.7.1 Example 1 - Solid body at rest, steady flow



Integral form of mass conservation

$$\int_{A_1} (\rho \bar{U}) \bullet \bar{n} dA = 0.$$

Integral form of momentum conservation

$$\int_{A_1} (\rho \bar{U}\bar{U} + P\bar{\bar{I}} - \bar{\bar{\tau}}) \bullet \bar{n} dA + \int_{A_2} (P\bar{\bar{I}} - \bar{\bar{\tau}}) \bullet \bar{n} dA = 0$$

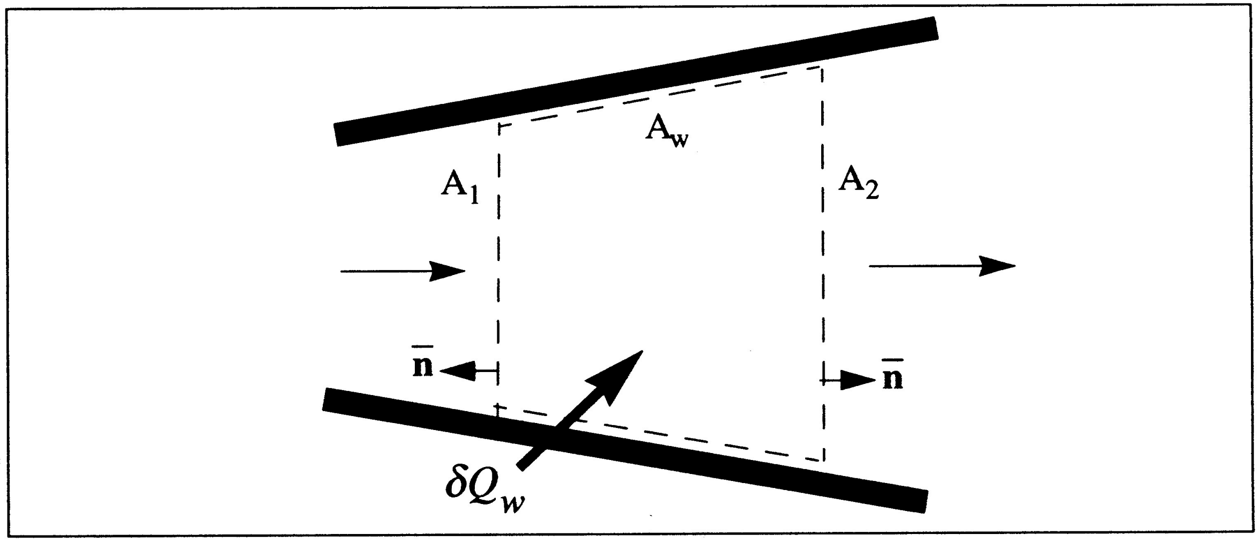
Momentum fluxes in the streamwise and normal directions are equal to the lift and drag forces exerted by the flow on the body.

$$Drag = \int_{A_2} (P\bar{\bar{I}} - \bar{\bar{\tau}}) \bullet \bar{n} dA \Big|_{x_1} \quad ; \qquad Lift = \int_{A_2} (P\bar{\bar{I}} - \bar{\bar{\tau}}) \bullet \bar{n} dA \Big|_{x_2} .$$

$$\int_{A_1} (\rho \bar{U}\bar{U} + P\bar{\bar{I}} - \bar{\bar{\tau}}) \bullet \bar{n} dA \Big|_{x_1} + Drag = 0 .$$

$$\int_{A_1} (\rho \bar{U}\bar{U} + P\bar{\bar{I}} - \bar{\bar{\tau}}) \bullet \bar{n} dA \Big|_{x_2} + Lift = 0 .$$

5.7.2 Example 2 -Channel flow with heat addition



Mass conservation

$$\int_{A_1} (\rho \bar{U}) \bullet \bar{n} dA + \int_{A_2} (\rho \bar{U}) \bullet \bar{n} dA = 0.$$

Energy conservation

$$\int_{A} (\rho \bar{U}(e + k) + P\bar{U} - \bar{\bar{\tau}} \bullet \bar{U} + \bar{Q}) \bullet \bar{n} dA = 0.$$

To a good approximation the energy balance becomes

$$\int_A \rho \bar{U}\left(e + \frac{P}{\rho} + k\right) \bullet \bar{n} dA = -\int_A \bar{Q} \bullet n dA .$$

Most of the conductive heat transfer is through the wall.

$$-\int_A \bar{Q} \bullet n dA \cong -\int_{A_w} \bar{Q} \bullet n dA = \delta Q .$$

The energy balance reduces to

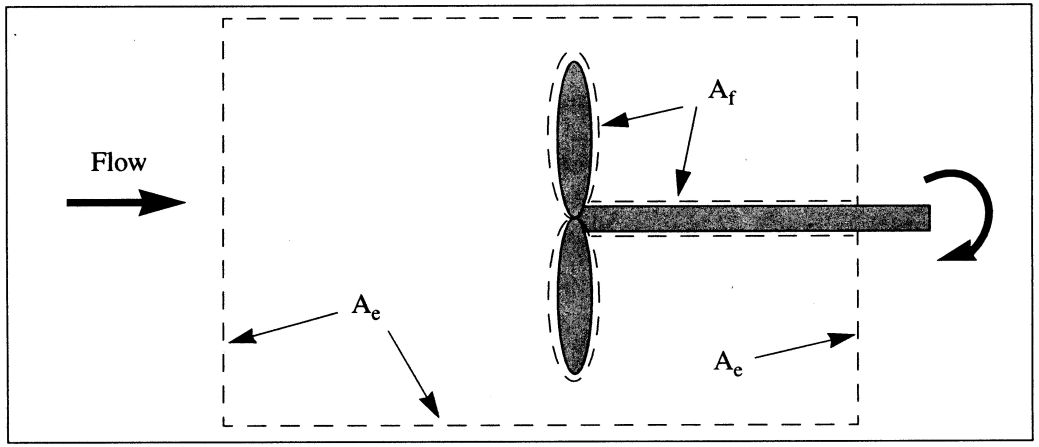
$$\int_{A_2} \rho h_t \bar{U} \bullet \bar{n} dA + \int_{A_1} \rho h_t \bar{U} \bullet \bar{n} dA = \delta Q .$$

When the vector multiplication is carried out the energy balance becomes

$$\int_{A_2} \rho_2 U_2 h_{t2} dA - \int_{A_1} \rho_1 U_1 h_{t1} dA = \delta Q .$$

The heat addition (or removal) per unit mass flow is equal to the change in stagnation enthalpy of the flow.

5.7.3 Example 3 - A Rotating fan in a stationary flow



The control volume surface is attached to and moves with the fan surface.

The integrated mass fluxes are zero.

$$\int_{A_e} (\rho \bar{U}) \bullet \bar{n}\, dA = 0$$

Momentum fluxes are equal to the surface forces on the fan

$$\int_{A_e} (\rho \bar{U}\bar{U} + P\bar{\bar{I}} - \bar{\bar{\tau}}) \bullet \bar{n}\, dA + \int_{A_f} (\rho \bar{U}(\bar{U} - \bar{U}_A) + P\bar{\bar{I}} - \bar{\bar{\tau}}) \bullet \bar{n}\, dA = 0$$

$$\int_{A_e} (\rho \bar{U}\bar{U} + P\bar{\bar{I}} - \bar{\bar{\tau}}) \bullet \bar{n}\, dA + \bar{F} = 0$$

The vector force by the flow on the fan is

$$\bar{F} = \int_{A_f} (P\bar{\bar{I}} - \bar{\bar{\tau}}) \bullet \bar{n} dA$$

The flow and fan velocity on the fan surface are the same due to the no-slip condition.

The integrated energy fluxes are equal to the work done by the flow on the fan.

$$\int_{A_e} (\rho\bar{U}(e+k) + P\bar{U} - \bar{\bar{\tau}} \bullet \bar{U}) \bullet \bar{n} dA + \int_{A_f} (P\bar{U} - \bar{\bar{\tau}} \bullet \bar{U}) \bullet \bar{n} dA = 0.$$

$$Work = \int_{A_f} (P\bar{U} - \bar{\bar{\tau}} \bullet \bar{U}) \bullet \bar{n} dA = \delta W$$

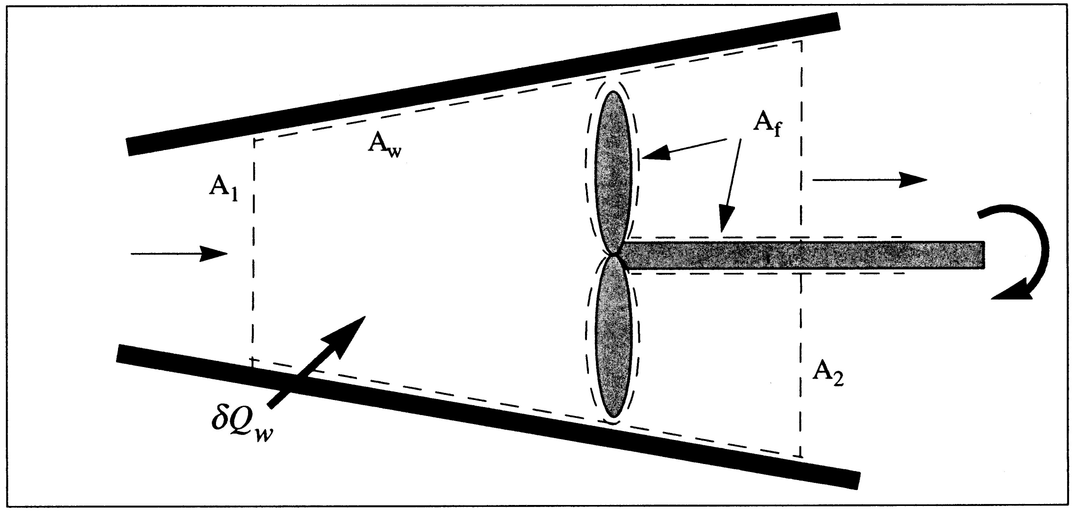
If the flow is adiabatic and work by viscous normal stresses is neglected the energy equation becomes.

$$\int_{A_e} \rho\left(e + \frac{P}{\rho} + k\right)\bar{U} \bullet \bar{n} dA + \delta W = 0$$

The work per unit mass flow is equal to the change in stagnation enthalpy of the flow.

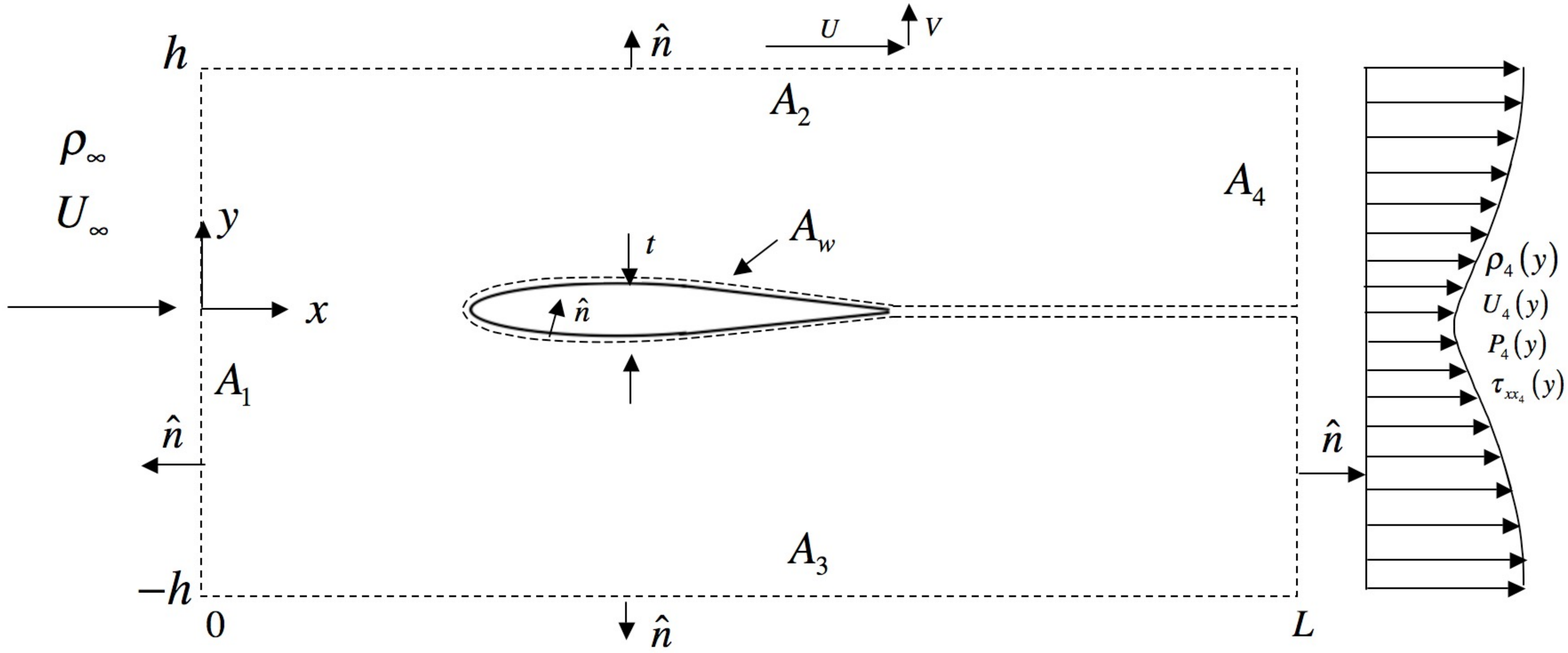
5.7.4 Example 3 - Combined heat transfer and work

In a general situation with heat transfer and work



$$\int_{A_2} \rho_2 U_2 h_{t2} dA - \int_{A_1} \rho_1 U_1 h_{t1} dA = \delta Q - \delta W .$$

Example - Drag of a 2-D airfoil



Continuity

$$\int_{A_1+A_2+A_3+A_4+A_w} \rho\bar{U}\bullet\hat{n}\,dA = 0$$

$$-\rho_\infty U_\infty A_1 + \int_{A_2+A_3} \rho\bar{U}\bullet\hat{n}\,dA + \int_{A_4} \rho(y)U(y)dy = 0$$

Note $U=0$ **on** A_w

$$-\rho_\infty U_\infty 2h + \int_0^L \rho V\,dx\Big|_{A_2} - \int_0^L \rho V\,dx\Big|_{A_3} + \int_{-h}^{h} \rho_4(y)U_4(y)dy = 0$$

Momentum

$$\int_{A_1+A_2+A_3+A_4+A_w} \left(\rho\bar{U}\bar{U} + P\bar{\bar{I}} - \bar{\bar{\tau}}\right)\bullet\hat{n}\,dA = 0$$

$U \doteq U_\infty$ Very small $U \doteq U_\infty$ Very small

$$-\left(\rho_\infty U_\infty^2 + P_\infty - \tau_{xx_\infty}\right)2h + \int_0^L \left(\rho UV - \tau_{xy}\right)dx\Big|_{A_2} - \int_0^L \left(\rho UV - \tau_{xy}\right)dx\Big|_{A_3} + \int_{-h}^{h}\left(\rho_4(y)U_4^2(y) + P_4(y) - \tau_{xx_4}(y)\right)dy + \int_{A_w}\left(P\bar{\bar{I}} - \bar{\bar{\tau}}\right)\bullet\hat{n}\,dA\Big|_x = 0$$

$$-\left(\rho_\infty U_\infty^2 + P_\infty - \tau_{xx_\infty}\right)2h + \int_0^L \left(\rho U_\infty V\right)dx\Big|_{A_2} - \int_0^L \left(\rho U_\infty V\right)dx\Big|_{A_3} + \int_{-h}^{h}\left(\rho_4(y)U_4^2(y) + P_4(y) - \tau_{xx_4}(y)\right)dy + Drag = 0$$

Subtract the continuity equation multiplied by U_∞

$$-\rho_\infty U_\infty^2 2h + \int_0^L \rho U_\infty V\,dx\Big|_{A_2} - \int_0^L \rho U_\infty V\,dx\Big|_{A_3} + \int_{-h}^{h} \rho_4(y)U_\infty U_4(y)dy = 0$$

$$\int_{-h}^{h}\left(\left(\rho_4(y)U_4^2(y)-\rho_4(y)U_\infty U_4(y)\right)+\left(P_4(y)-P_\infty\right)-\left(\tau_{xx_4}(y)-\tau_{xx_\infty}\right)\right)dy+Drag=0$$

$$Drag=\int_{-h}^{h}\left(\rho_4(y)U_\infty U_4(y)-\rho_4(y)U_4^2(y)\right)+\left(P_\infty-P_4(y)\right)-\left(\tau_{xx_\infty}-\tau_{xx_4}(y)\right)dy$$

$$C_D=2\int_{-h/t}^{h/t}\left(\frac{\rho_4(y/t)U_4(y/t)}{\rho_\infty U_\infty}\left(1-\frac{U_4(y/t)}{U_\infty}\right)\right)+\left(\frac{P_\infty-P_4(y/t)}{\rho_\infty U_\infty{}^2}\right)-\left(\frac{\tau_{xx_\infty}-\tau_{xx_4}(y/t)}{\rho_\infty U_\infty{}^2}\right)d\left(\frac{y}{t}\right)$$

In the far wake the pressure and viscous normal stress terms are vanishingly small and $\frac{\rho_4(y)U_4(y)}{\rho_\infty U_\infty}\doteq 1$

$$C_D\doteq 2\int_{\infty}^{\infty}\left(1-\frac{U_4(y/t)}{U_\infty}\right)d\left(\frac{y}{t}\right) \qquad C_D=\frac{Drag}{\frac{1}{2}\rho_\infty U_\infty^2 t}$$

The Reynolds number

$$R_e=\frac{\rho_\infty U_\infty t}{\mu_\infty}=\frac{\left(\frac{1}{2}\rho_\infty U_\infty^2\right)}{\left(\frac{1}{2}\mu_\infty\frac{U_\infty}{t}\right)}\approx\frac{\text{Dynamic Pressure}}{\text{Characteristic viscous stress}}$$

5.8 Stagnation enthalpy, temperature and pressure

5.8.1 Stagnation enthalpy of a fluid element

Using the energy equation

$$\frac{\partial\rho(e+k)}{\partial t} + \nabla\bullet\left(\rho\bar{U}\left(e+\frac{P}{\rho}+k\right)-\bar{\bar{\tau}}\bullet\bar{U}+\bar{Q}\right)-\rho\bar{G}\bullet\bar{U} = 0$$

Along with the continuity equation

$$\frac{D\rho}{Dt} = -\rho\nabla\bullet\bar{U}$$

and the identity

$$\frac{D}{Dt}\left(\frac{P}{\rho}\right) = \frac{1}{\rho}\frac{DP}{Dt}-\frac{P}{\rho^2}\frac{D\rho}{Dt}$$

One can derive the transport equation for the stagnation enthalpy.

$$\rho\frac{Dh_t}{Dt} = \nabla\bullet(\bar{\bar{\tau}}\bullet\bar{U}-\bar{Q})+\rho\bar{G}\bullet\bar{U}+\frac{\partial P}{\partial t}.$$

5.8.2 Blowdown from a pressure vessel revisited

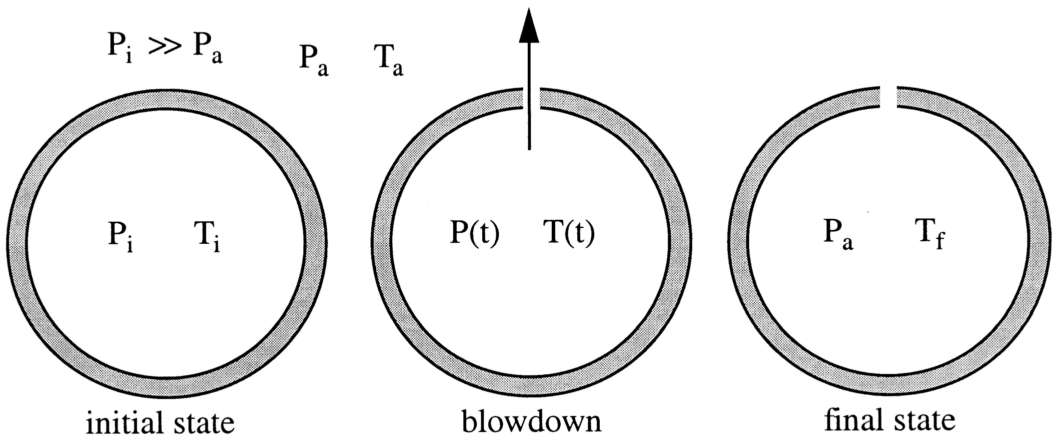


Figure 5.5 A adiabatic pressure vessel exhausting to the surroundings.

Assume the divergence term is small then

$$\rho \frac{Dh_t}{Dt} = \frac{\partial P}{\partial t}.$$

Neglect the kinetic energy and assume the temperature and pressure are constant over the interior of the sphere. Then

$$\rho \frac{dh}{dt} = \frac{dP}{dt}$$

Which is the Gibbs equation for an isentropic process. If the gas is calorically perfect

$$\frac{T_f}{T_i} = \left(\frac{P_a}{P_i}\right)^{\frac{\gamma - 1}{\gamma}}.$$

5.8.3 Stagnation enthalpy and temperature in steady flow

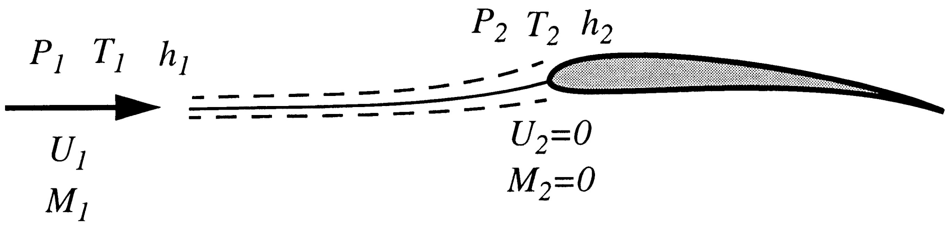


Figure 5.6 Schematic of a stagnation process in steady flow

For flow in an adiabatic streamtube

$$\int_{A_2} \rho_2 U_2 h_{t2} dA = \int_{A_1} \rho_1 U_1 h_{t1} dA$$

Since the mass flow at any point in the tube is the same then the stagnation enthalpy per unit mass is also the same and we would expect

$$h_{t1} = e_1 + \frac{P_1}{\rho_1} + k_1 = h_1 + \frac{1}{2}U_1^2 = h_2 + \frac{1}{2}U_2^2 = h_2$$

Neglecting viscous work, the stagnation enthalpy is conserved along an adiabatic path.

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The stagnation temperature is defined by the enthalpy relation

$$h_t - h = \int_T^{T_t} C_p dT = \frac{1}{2} U_i U_i. \tag{5.75}$$

The stagnation temperature is the temperature reached by an element of gas brought to rest adiabatically. For a calorically perfect ideal gas with constant specific heat in the range of temperatures between T and T_t (5.75) can be written

$$C_p T_t = C_p T + \frac{1}{2} U_i U_i. \tag{5.76}$$

In terms of the Mach number

$$\boxed{\frac{T_t}{T} = 1 + \left(\frac{\gamma - 1}{2}\right) M^2}$$

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5.8.4 Frames of reference

Stagnation temperature in fixed and moving frames

$$C_p T_t = C_p T + \frac{1}{2} U_i U_i$$

$$C_p T_t' = C_p T + \frac{1}{2} U_i' U_i'$$

Transformation of kinetic energy

$$\frac{1}{2} U_i' U_i' = \frac{1}{2} U_i U_i + \frac{1}{2} \dot{X}(\dot{X} - 2U) + \frac{1}{2} \dot{Y}(\dot{Y} - 2V) + \frac{1}{2} \dot{Z}(\dot{Z} - 2W)$$

Stagnation temperature

$$C_p T_t' = C_p T_t + \frac{1}{2} \dot{X}(\dot{X} - 2U) + \frac{1}{2} \dot{Y}(\dot{Y} - 2V) + \frac{1}{2} \dot{Z}(\dot{Z} - 2W)$$

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5.8.5 Stagnation pressure

Gibbs equation

$$ds = C_p \frac{dT}{T} - R\frac{dP}{P}$$

along an isentropic path

$$R\int_{P_1}^{P_2} \frac{dP}{P} = \int_{T_1}^{T_2} C_p(T)\frac{dT}{T}$$

Integrate the pressure term

$$\frac{P_2}{P_1} = Exp\left(\frac{1}{R}\int_{T_1}^{T_2} C_p(T)\frac{dT}{T}\right)$$

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If an element of fluid is brought to rest isentropically

$$\frac{P_t}{P} = Exp\left(\frac{1}{R}\int_T^{T_t} C_p(T)\frac{dT}{T}\right)$$

If the heat capacity is constant

$$\frac{P_t}{P} = \left(\frac{T_t}{T}\right)^{\frac{\gamma}{\gamma-1}} = \left(1 + \left(\frac{\gamma-1}{2}\right)M^2\right)^{\frac{\gamma}{\gamma-1}}.$$

Changes in the stagnation state are related by the Gibbs equation.

$$T_t\frac{Ds}{Dt} = \frac{Dh_t}{Dt} - \frac{1}{\rho_t}\frac{DP_t}{Dt}$$

5.8.6 Transforming the stagnation pressure between fixed and moving frames

$$\frac{P_t}{P} = \left(\frac{T_t}{T}\right)^{\frac{\gamma}{\gamma-1}}$$

$$\frac{P_t'}{P} = \left(\frac{T_t'}{T}\right)^{\frac{\gamma}{\gamma-1}}$$

Divide out the static pressure and temperature

$$\frac{P_t'}{P_t} = \left(\frac{T_t'}{T_t}\right)^{\frac{\gamma}{\gamma-1}}$$

$$P_t' = P_t\left(1 + \frac{\dot{X}(\dot{X} - 2U) + \dot{Y}(\dot{Y} - 2V) + \dot{Z}(\dot{Z} - 2W)}{2C_pT_t}\right)^{\frac{\gamma}{\gamma-1}}.$$

5.9 Problems

Problem 1 - Work out the time derivative of the following integral.

$$I(t) = \int_{t^2}^{Sin(t)} e^{xt} dx \tag{5.92}$$

Obtain dI/dt in two ways: (1) by directly integrating, then differentiating the result and (2) by applying Leibniz' rule (5.9) then carrying out the integration.

Problem 2 - In Chapter 2, Problem 2 we worked out a hypothetical incompressible steady flow with the velocity components

$$(U, V) = (Cos(x)\, Cos(y), Sin(x)\, Sin(y)). \tag{5.93}$$

This 2-D flow clearly satisfies the continuity equation (conservation of mass), could it possibly satisfy conservation of momentum for an inviscid fluid? To find out work out the substantial derivatives of the velocity components and equate the results to the partial derivatives of the pressure that appear in the momentum equation. The differential of the pressure is

$$dP = \frac{\partial P}{\partial x} dx + \frac{\partial P}{\partial y} dy. \tag{5.94}$$

Show by the cross derivative test whether a pressure field exists that could enable (5.93) to satisfy momentum conservation. If such a pressure field exists work it out.

Problem 3 - Consider steady flow in one dimension where $\bar{U} = (U(x), 0, 0)$ and all velocity gradients are zero except

$$A_{11} = \frac{\partial U}{\partial x}. \tag{5.95}$$

Work out the components of the Newtonian viscous stress tensor τ_{ij}. Note the role of the bulk viscosity.

Problem 4 - A cold gas thruster on a spacecraft uses Helium (atomic weight 4) at a chamber temperature of $300\,K$ and a chamber pressure of one atmosphere. The gas exhausts adiabatically through a large area ratio nozzle to the vacuum of space. Estimate the maximum speed of the exhaust gas.

Problem 5 - Work out equation (5.67).

Problem 6 - Steady flow through the empty test section of a wind tunnel with parallel walls and a rectangular cross-section is shown below. Use a control volume balance to relate the integrated velocity and pressure profiles at stations 1 and 2 to an integral of the wall shear stress.

Figure 5.11: *Steady flow in an empty wind tunnel*

State any assumptions used.

Problem 7 - Use a control volume balance to show that the drag of a circular cylinder at low Mach number can be related to an integral of the velocity and stress profile in the wake downstream of the cylinder. Be sure to use the continuity equation to help account for the x-momentum convected out of the control volume through the upper and lower surfaces. State any assumptions used.

Problem 8 - Use a control volume balance to evaluate the lift of a three dimensional wing in an infinite steady stream. Assume the Mach number is low enough so that there are no shock waves formed.

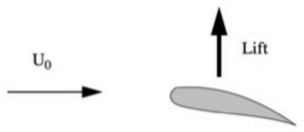

Figure 5.12: *3-D wing in an infinite stream*

1) Select an appropriate control volume.

2) Write down the integral form of the mass conservation equation.

3) Write down the integral form of the momentum conservation equation.

4) Evaluate the various terms on the control volume boundary so as to express the lift of the wing in terms of an integral over the downstream wake.

5) Why did I stipulate that there are no shock waves? Briefly state any other assumptions that went in to your solution.

Problem 9 - Suppose a model 3-D wing is contained in a finite sized wind tunnel test section with horizontal and vertical walls as shown below.

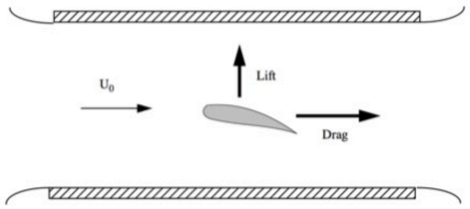

Figure 5.13: *3-D wing in a wind tunnel*

What would a test engineer have to measure to determine lift and drag in the absence of sensors on the model or a mechanical balance for directly measuring forces? Consider a control volume that coincides with the wind tunnel walls.