STANFORD
AERONAUTICS &
ASTRONAUTICS

AA210A

Fundamentals of Compressible Flow

Chapter 6 - Several forms of the equations of motion

6.1 The Navier-Stokes equations

Assume a Newtonian stress rate-of-strain relation and a linear thermally conductive medium. The conservation equations become

$$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho U_i) = 0$$

$$\frac{\partial\rho U_i}{\partial t} + \frac{\partial}{\partial x_j}(\rho U_i U_j + P\delta_{ij}) - \rho G_i -$$

$$\frac{\partial}{\partial x_j}(2\mu S_{ij} - ((2/3)\mu - \mu_v)\delta_{ij}S_{kk}) = 0$$

$$\frac{\partial\rho(e+k)}{\partial t} + \frac{\partial}{\partial x_i}(\rho U_i h_t - \kappa(\partial T/\partial x_i)) - \rho G_i U_i -$$

$$\frac{\partial}{\partial x_i}(2\mu U_j S_{ij} - ((2/3)\mu - \mu_v)\delta_{ij}U_j S_{kk}) = 0$$

STANFORD AERONAUTICS & ASTRONAUTICS

6.1.1 The incompressible Navier-Stokes equations

If the density is constant the previous system of equations reduces to

$$\frac{\partial U_i}{\partial x_i} = 0$$

$$\frac{\partial U_i}{\partial t} + \frac{\partial}{\partial x_j}\left(U_i U_j + \frac{P}{\rho}\delta_{ij} - 2\left(\frac{\mu}{\rho}\right)S_{ij}\right) = 0 \qquad (6.2)$$

$$\frac{\partial T}{\partial t} + \frac{\partial}{\partial x_i}\left(U_i T - \left(\frac{\kappa}{\rho C}\right)\frac{\partial T}{\partial x_i}\right) - 2\left(\frac{\mu}{\rho C}\right)S_{ij}S_{ij} = 0$$

in the absence of body forces.

6.2 The momentum equation in terms of vorticity

Assume the two viscosities are constant - this is reasonable if the Mach number is not too large. The momentum equation can be written.

$$\frac{\partial \rho \bar{U}}{\partial t} + \nabla \cdot (\rho \bar{U}\bar{U}) + \nabla P - \mu \nabla^2 \bar{U} - \left(\frac{1}{3}\mu + \mu_v\right)\nabla(\nabla \cdot \bar{U}) - \rho \bar{G} = 0 \qquad \bar{G} = -\nabla \Psi$$

The vorticity is

$$\bar{\Omega} = \nabla \times \bar{U}.$$

Vector identities

$$\bar{U} \bullet \nabla \bar{U} = (\nabla \times \bar{U}) \times \bar{U} + \nabla\left(\frac{\bar{U} \bullet \bar{U}}{2}\right)$$

$$\nabla \times (\nabla \times \bar{U}) = \nabla(\nabla \bullet \bar{U}) - \nabla^2 \bar{U}$$

Now the momentum equation can be written

$$\rho \frac{\partial \bar{U}}{\partial t} + \rho(\bar{\Omega} \times \bar{U}) + \rho \nabla\left(\frac{\bar{U} \bullet \bar{U}}{2}\right) + \nabla P + \rho \nabla \Psi -$$

$$\left(\frac{4}{3}\mu + \mu_v\right)\nabla(\nabla \bullet \bar{U}) + \mu \nabla \times \bar{\Omega} = 0$$

6.3 The momentum equation in terms of entropy

Use the Gibbs equation to replace the gradient of the pressure.

$$\nabla P = \rho \nabla h - \rho T \nabla s$$

$$\frac{\partial \bar{U}}{\partial t} + (\bar{\Omega} \times \bar{U}) + \nabla\left(h + \frac{U^2}{2} + \Psi\right) - T\nabla s -$$

$$\left(\frac{1}{\rho}\right)\left(\frac{4}{3}\mu + \mu_v\right)\nabla(\nabla \bullet \bar{U}) + \left(\frac{\mu}{\rho}\right)\nabla \times \bar{\Omega} = 0$$

The inviscid form of the equation is (Crocco's equation)

$$\frac{\partial \bar{U}}{\partial t} + (\bar{\Omega} \times \bar{U}) + \nabla\left(h + \frac{U^2}{2} + \Psi\right) - T\nabla s = 0,$$

If the flow is steady and inviscid

$$(\bar{\Omega} \times \bar{U}) + \nabla\left(h + \frac{U^2}{2} + \Psi\right) - T\nabla s = 0$$

If the stagnation enthalpy and entropy are constant

$$\bar{\Omega} \times \bar{U} = 0$$

STANFORD
AERONAUTICS & ASTRONAUTICS

6.4 Inviscid, Irrotational, homentropic flow

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_k}(\rho U_k) = 0$$

$$\rho \frac{\partial U_i}{\partial t} + \rho \frac{\partial}{\partial x_i}\left(\frac{U_k U_k}{2}\right) + \frac{\partial P}{\partial x_i} = 0$$

$$\frac{P}{P_0} = \left(\frac{\rho}{\rho_0}\right)^{\gamma}$$

These equations are the starting point for a small disturbance analysis that leads to the equations that govern the propagation of sound. We will come back to the topic of acoustics in Chapter 13.

If the flow is steady the equations become

$$\nabla \bullet (\rho \bar{U}) = 0$$

$$\nabla\left(\frac{\bar{U} \bullet \bar{U}}{2}\right) + \frac{\nabla P}{\rho} = 0$$

$$\frac{P}{P_0} = \left(\frac{\rho}{\rho_0}\right)^{\gamma}$$

Use

$$\nabla P = a^2 \nabla \rho$$

and

$$\nabla\left(\frac{P}{\rho}\right) = \frac{\nabla P}{\rho} - \frac{P}{\rho^2}\nabla\rho = \left(\frac{\gamma - 1}{\gamma}\right)\frac{\nabla P}{\rho}$$

To generate

$$\nabla\left(\left(\frac{\gamma}{\gamma - 1}\right)\frac{P}{\rho} + \frac{\bar{U} \bullet \bar{U}}{2}\right) = 0$$

The stagnation enthalpy and entropy are constant everywhere in the flow.

STANFORD AERONAUTICS & ASTRONAUTICS

Continuity

$$\bar{U} \bullet \frac{\nabla P}{\rho} + a^2 \nabla \bullet \bar{U} = 0$$

$$\bar{U} \bullet \nabla a^2 + (\gamma - 1) a^2 \nabla \bullet \bar{U} = 0$$

$$\left(\frac{a^2}{\gamma - 1}\right) = h_t - \frac{\bar{U} \bullet \bar{U}}{2}$$

$$(\gamma - 1)\left(h_t - \frac{\bar{U} \bullet \bar{U}}{2}\right) \nabla \bullet \bar{U} - \bar{U} \bullet \nabla\left(\frac{\bar{U} \bullet \bar{U}}{2}\right) = 0$$

The problem reduces to a single equation for the velocity vector.

STANFORD AERONAUTICS & ASTRONAUTICS

6.5 The Velocity Potential

For irrotational flow the velocity field can be expressed in terms of a scalar potential.

$$\bar{U} = \nabla \Phi .$$

Substitute into the equation derived previously for the velocity

$$(\gamma - 1)\left(h_t - \frac{\bar{U} \bullet \bar{U}}{2}\right) \nabla \bullet \bar{U} - \bar{U} \bullet \nabla\left(\frac{\bar{U} \bullet \bar{U}}{2}\right) = 0$$

Steady, irrotational, homentropic flow is governed by the full potential equation

$$(\gamma - 1)\left(h_t - \frac{\nabla\Phi \bullet \nabla\Phi}{2}\right) \nabla^2 \Phi - \nabla\Phi \bullet \nabla\left(\frac{\nabla\Phi \bullet \nabla\Phi}{2}\right) = 0$$

6.5.1 Unsteady potential flow

The momentum equation for irrotational flow including the gravitational potential is

$$\frac{\partial \bar{U}}{\partial t} + \nabla\left(h + \frac{U^2}{2} + \Psi\right) - T\nabla s - \left(\frac{1}{\rho}\right)\left(\frac{4}{3}\mu + \mu_v\right)\nabla(\nabla \bullet \bar{U}) = 0$$

Substitute the velocity potential

$$\nabla\left(\frac{\partial \Phi}{\partial t} + h + \frac{U^2}{2} + \Psi\right) - T\nabla s - \frac{1}{\rho}\left(\frac{4}{3}\mu + \mu_v\right)\nabla(\nabla^2 \Phi) = 0$$

If the flow is inviscid and homentropic the momentum equation reduces to

$$\nabla\left(\frac{\partial \Phi}{\partial t} + h + \frac{U^2}{2} + \Psi\right) = 0$$

STANFORD
AERONAUTICS & ASTRONAUTICS

The unsteady Bernoulli integral

$$\frac{\partial \Phi}{\partial t} + h + \frac{U^2}{2} + \Psi = F(t)$$

If the flow is calorically perfect

$$\frac{\partial \Phi}{\partial t} + \frac{\gamma}{\gamma - 1}\frac{P}{\rho} + \frac{U^2}{2} + \Psi = F(t)$$

The equations for inviscid, homentropic, unsteady flow with gravity are

$$\frac{1}{\rho}\frac{\partial \rho}{\partial t} + \frac{\nabla \Phi \bullet \nabla \rho}{\rho} + \nabla^2 \Phi = 0$$

$$\nabla\left(\frac{\partial \Phi}{\partial t} + \frac{\gamma}{\gamma - 1}\frac{P}{\rho} + \frac{\nabla \Phi \bullet \nabla \Phi}{2} + \Psi\right) = 0$$

$$\frac{P}{P_{ref}} = \left(\frac{\rho}{\rho_{ref}}\right)^{\gamma}$$

These equations can also be reduced to a single equation for the velocity potential.

$$\left(\frac{dF(t)}{dt} - \Phi_{tt} - \nabla\Phi_t \bullet \nabla\Phi\right) + (\gamma - 1)\left(F(t) - \frac{\partial\Phi}{\partial t} - \frac{\nabla\Phi \bullet \nabla\Phi}{2} - \Psi\right)\nabla^2\Phi -$$

$$\nabla\Phi \bullet \left(\nabla\Phi_t + \nabla\left(\frac{\nabla\Phi \bullet \nabla\Phi}{2}\right)\right) = 0$$

STANFORD AERONAUTICS & ASTRONAUTICS

6.5.2 Incompressible irrotational flow

The momentum equation in the absence of gravity reduces to

$$\frac{\partial \bar{U}}{\partial t} + \nabla\left(\frac{\bar{U} \bullet \bar{U}}{2}\right) + \nabla\left(\frac{P}{\rho}\right) = 0 .$$

Introduce the scalar potential again

$$\nabla\left(\frac{\partial \Phi}{\partial t} + \frac{U^2}{2} + \frac{P}{\rho}\right) = 0 .$$

The incompressible Bernoulli integral

$$\frac{\partial \Phi}{\partial t} + \frac{U^2}{2} + \frac{P}{\rho} = F(t) .$$

The velocity field satisfies Laplace's equation.

$$\nabla \bullet \bar{U} = \nabla^2 \Phi = 0$$

6.6 The vorticity equation

Take the curl of the momentum equation.

$$\nabla \times \left(\frac{\partial \bar{U}}{\partial t} + (\bar{\Omega} \times \bar{U}) + \nabla \left(h + \frac{U^2}{2} + \Psi \right) - T \nabla s - \left(\frac{1}{\rho} \right) \left(\frac{4}{3}\mu + \mu_v \right) \nabla (\nabla \bullet \bar{U}) + \left(\frac{\mu}{\rho} \right) \nabla \times \bar{\Omega} \right) = 0$$

This becomes

$$\frac{\partial \bar{\Omega}}{\partial t} + \nabla \times (\bar{\Omega} \times \bar{U}) - \nabla \times (T \nabla s) - \left(\frac{4}{3}\mu + \mu_v \right) \nabla \times \left(\left(\frac{1}{\rho} \right) \nabla (\nabla \bullet \bar{U}) \right) + \mu \nabla \times \left(\left(\frac{1}{\rho} \right) \nabla \times \bar{\Omega} \right) = 0$$

Use vector identities to rearrange

$$\frac{\partial \bar{\Omega}}{\partial t} + \bar{U} \bullet \nabla \bar{\Omega} = (\bar{\Omega} \bullet \nabla) \bar{U} - \bar{\Omega} \nabla \bullet \bar{U} + \nabla T \times \nabla s + \left(\frac{4}{3}\mu + \mu_v \right) \nabla \times \left(\left(\frac{1}{\rho} \right) \nabla (\nabla \bullet \bar{U}) \right) - \mu \nabla \times \left(\left(\frac{1}{\rho} \right) \nabla \times \bar{\Omega} \right) = 0$$

STANFORD
AERONAUTICS & ASTRONAUTICS

For inviscid, homentropic flow

$$\frac{D\bar{\Omega}}{Dt} = \frac{\partial\bar{\Omega}}{\partial t} + \bar{U} \bullet \nabla\bar{\Omega} = (\bar{\Omega} \bullet \nabla)\bar{U} - \bar{\Omega}\nabla \bullet \bar{U}.$$

Vortex stretching term

For viscous, incompressible flow

$$\frac{D\bar{\Omega}}{Dt} = \frac{\partial\bar{\Omega}}{\partial t} + \bar{U} \bullet \nabla\bar{\Omega} = (\bar{\Omega} \bullet \nabla)\bar{U} + \frac{\mu}{\rho}\nabla^2\bar{\Omega}$$

In two dimensions the flow satisfies the convective diffusion equation.

$$\frac{D\Omega}{Dt} = \frac{\mu}{\rho}\nabla^2\Omega.$$

This is the same equation satisfied by the temperature.

$$\frac{DT}{Dt} = \frac{\kappa}{\rho}\nabla^2 T$$

6.7 Fluid flow in three dimensions, the dual stream function

The particle path equations

$$\frac{dx}{dt} = U(\bar{x}) \; ; \qquad \frac{dy}{dt} = V(\bar{x}) \; ; \qquad \frac{dz}{dt} = W(\bar{x}) .$$

$$x = f(\tilde{\bar{x}}, t) \; ; \quad y = g(\tilde{\bar{x}}, t) \; ; \quad z = h(\tilde{\bar{x}}, t)$$

If we eliminate time between these three equations the result is two families of stream-function surfaces.

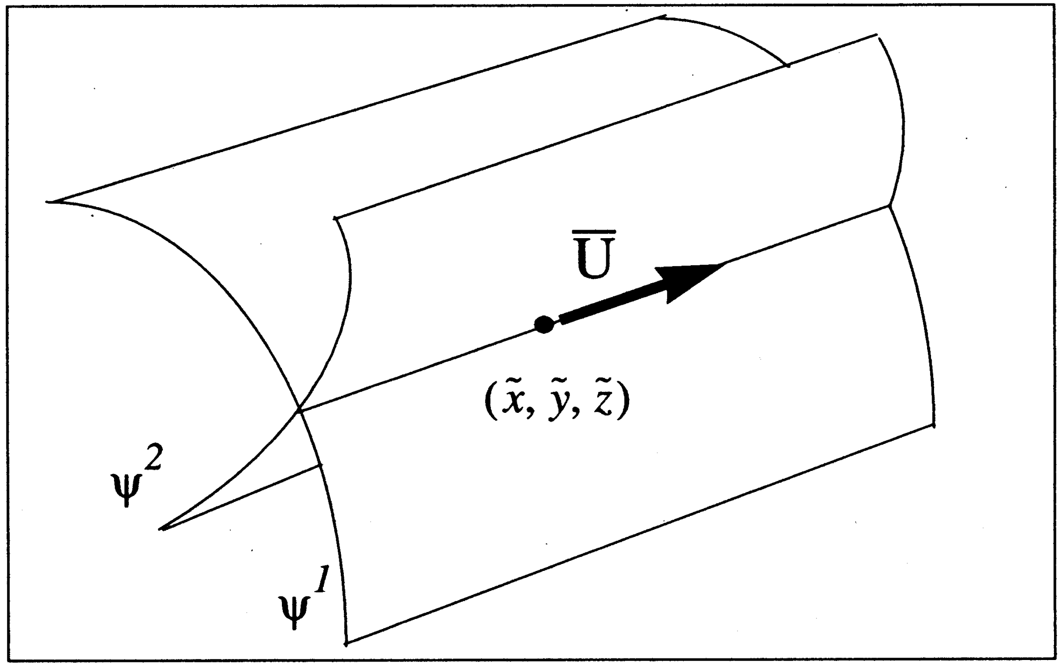
$$\psi^1 = \Psi^1(\bar{x}) \; ; \qquad \psi^2 = \Psi^2(\bar{x}) .$$

These are integrals of the first order PDE

$$U \cdot \nabla \Psi^j = U\frac{\partial \Psi^j}{\partial x} + V\frac{\partial \Psi^j}{\partial y} + W\frac{\partial \Psi^j}{\partial z} = 0 \; ; \qquad j = 1, 2$$

STANFORD AERONAUTICS & ASTRONAUTICS

The velocity vector lies along the line of intersection of the two surfaces.



$$\bar{U} = \nabla\Psi^1 \times \nabla\Psi^2$$

6.8 The vector potential

The velocity field of an incompressible flow can be represented by the curl of a vector potential.

$$U = \nabla \times \bar{A}.$$

The vorticity and vector potential are related by a vector Poisson equation.

$$\nabla^2 \bar{A} = -\bar{\Omega}.$$

Where we have used the vector identity.

$$\nabla(\nabla \bullet \bar{A}) - \nabla^2 \bar{A} = \nabla \times (\nabla \times \bar{A})$$

The vector potential is related to the dual stream-functions.

$$\bar{A} = \Psi^1 \nabla \Psi^2 = -\Psi^2 \nabla \Psi^1 .$$

STANFORD
AERONAUTICS & ASTRONAUTICS

6.9 Incompressible flow with mass and vorticity sources

A general incompressible flow containing mass sources and distributed vorticity can be constructed from a superposition of the fields generated by a scalar and a vector potential.

$$\bar{U} = \nabla\phi + \nabla \times \bar{A}$$

Where the potentials satisfy the Poisson equation.

$$\left.\begin{aligned} \nabla^2\phi &= Q(x,t) \\ \nabla^2\bar{A} &= -\bar{\Omega}(x,t) \end{aligned}\right\}$$

STANFORD AERONAUTICS & ASTRONAUTICS

6.10 Turbulent flow

Recall the equations of motion.

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho U_i) = 0$$

$$\frac{\partial \rho U_i}{\partial t} + \frac{\partial}{\partial x_j}(\rho U_i U_j + P\delta_{ij}) - \rho G_i -$$

$$\frac{\partial}{\partial x_j}(2\mu S_{ij} - ((2/3)\mu - \mu_v)\delta_{ij} S_{kk}) = 0$$

$$\frac{\partial \rho(e+k)}{\partial t} + \frac{\partial}{\partial x_i}(\rho U_i h_t - \kappa(\partial T/\partial x_i)) - \rho G_i U_i -$$

$$\frac{\partial}{\partial x_i}(2\mu U_j S_{ij} - ((2/3)\mu - \mu_v)\delta_{ij} U_j S_{kk}) = 0$$

Decompose each flow variable into a mean and fluctuating part.

$$Q(x, y, z, t) = \bar{Q}(x, y, z, t) + Q'(x, y, z, t)$$

Consider N realizations of the flow. The mean of any flow variable is defined as

$$\bar{Q} = \frac{1}{N}\sum_{n=1}^{N} Q_n$$

For any term that is linear in the fluctuations the average is

$$\overline{\frac{\partial}{\partial x_j}(P\delta_{ij})} = \overline{\frac{\partial}{\partial x_j}((\bar{P} + P')\delta_{ij})} = \frac{\partial}{\partial x_j}(\bar{P}\delta_{ij}) + \overline{\frac{\partial}{\partial x_j}(P'\delta_{ij})} = \frac{\partial}{\partial x_j}(\bar{P}\delta_{ij})$$

STANFORD AERONAUTICS & ASTRONAUTICS

Nonlinear terms are more complex.

$$
\begin{aligned}
\rho &= \bar{\rho} + \rho' \\
U &= \bar{U} + U' \\
V &= \bar{V} + V' \\
W &= \bar{W} + W'
\end{aligned}
$$

$$\overline{\rho U V} = \overline{(\bar{\rho} + \rho')(\bar{U} + U')(\bar{V} + V')} =$$

$$\bar{\rho}\bar{U}\bar{V} + \underset{0}{\overline{\rho'}}\bar{U}\bar{V} + \bar{\rho}\underset{0}{\overline{U'}}\bar{V} + \bar{\rho}\bar{U}\underset{0}{\overline{V'}} + \overline{\rho' U'}\bar{V} + \overline{\rho' V'}\bar{U} + \bar{\rho}\overline{U' V'} + \overline{\rho' U' V'}$$

Nonzero correlations of fluctuations are effective turbulent stresses known as Reynolds stresses.

$$\overline{\rho U V} = \bar{\rho}\bar{U}\bar{V} + (\overline{\rho' U'}\bar{V} + \overline{\rho' V'}\bar{U} + \bar{\rho}\overline{U' V'} + \overline{\rho' U' V'}) = \bar{\rho}\bar{U}\bar{V} + \tau_{xy}\big|_{turbulent}$$

$$\overline{\rho U U} = \bar{\rho}\bar{U}\bar{U} + (2\overline{\rho' U'}\bar{U} + \bar{\rho}\overline{U' U'} + \overline{\rho' U' U'}) = \bar{\rho}\bar{U}\bar{U} + \tau_{xx}\big|_{turbulent}$$

$$\overline{\rho V V} = \bar{\rho}\bar{V}\bar{V} + (2\overline{\rho' V'}\bar{V} + \bar{\rho}\overline{V' V'} + \overline{\rho' V' V'}) = \bar{\rho}\bar{V}\bar{V} + \tau_{yy}\big|_{turbulent}$$

Time dependent terms are ensemble averaged in the same way.

$$\overline{\frac{\partial \rho}{\partial t}} = \frac{1}{N}\sum_{n=1}^{N} \left.\frac{\partial \rho}{\partial t}\right|_{n} = \frac{\partial \bar{\rho}}{\partial t} + \overline{\frac{\partial \rho'}{\partial t}} = \frac{\partial \bar{\rho}}{\partial t}$$

$$\overline{\frac{\partial \rho U_i}{\partial t}} = \frac{1}{N}\sum_{n=1}^{N} \left.\frac{\partial \rho U_i}{\partial t}\right|_{n} = \frac{\partial \bar{\rho}\bar{U}_i}{\partial t} + \overline{\frac{\partial \rho'\bar{U}_i}{\partial t}} + \overline{\frac{\partial \bar{\rho}U'_i}{\partial t}} + \overline{\frac{\partial \rho'U'_i}{\partial t}} = \frac{\partial \bar{\rho}\bar{U}_i}{\partial t} + \frac{\partial \overline{\rho'U'_i}}{\partial t}$$

Incompressible flow - Navier Stokes Equations.

$$\frac{\partial U_j}{\partial x_j} = 0$$

$$\frac{\partial U_i}{\partial t} + \frac{\partial U_i U_j}{\partial x_j} + \frac{1}{\rho}\frac{\partial P}{\partial x_i} - \left(\frac{\mu}{\rho}\right)\frac{\partial^2 U_i}{\partial x_j \partial x_j} = 0$$

Incompressible flow - Reynolds Averaged Navier Stokes Equations (RANS).

$$\frac{\partial \bar{U}_j}{\partial x_j} = 0$$

$$\frac{\partial \bar{U}_i}{\partial t} + \frac{\partial \bar{U}_i \bar{U}_j}{\partial x_j} = -\frac{1}{\rho}\frac{\partial \bar{P}}{\partial x_i} + \left(\frac{\mu}{\rho}\right)\frac{\partial^2 \bar{U}_i}{\partial x_j \partial x_j} - \frac{\partial \overline{U'_i U'_j}}{\partial x_j}$$

$$\tau_{ij}\Big|_{turbulent} = \rho \overline{U'_i U'_j}$$

A model relating the Reynolds stresses to the mean flow is needed to close the equations.

6.11 Problems

Problem 1 - Derive equation (6.6) beginning with the Navier-Stokes equations. Do the same for equation (6.47).

Problem 2 - Show that for homentropic flow of an ideal gas $\nabla P = a^2 \nabla \rho$ where a is the local speed of sound.