AA210A

Fundamentals of Compressible Flow

Chapter 8 - Viscous flow along a wall

8.1 The no-slip condition

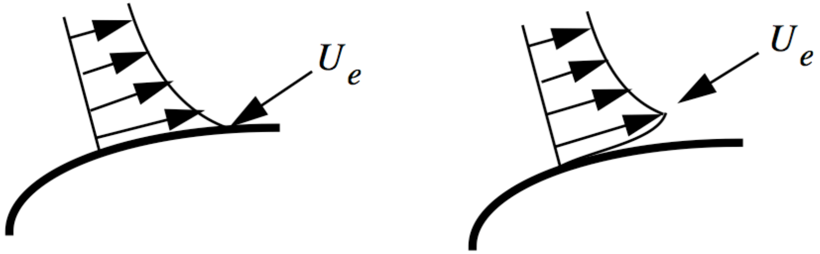


Figure 8.1 Slip versus no-slip flow near a solid surface.

Mean free path in a gas.

$$\lambda = \frac{1}{\sqrt{2}\pi n \sigma^2}$$

Slip velocity.

$$v_{slip} = C\lambda\frac{\partial U}{\partial y}$$

At ordinary temperatures and pressures the mean free path is very small.

8.2 The equations of motion

Steady 2-D flow.

$$\frac{\partial \rho U}{\partial x} + \frac{\partial \rho V}{\partial y} = 0$$

$$\frac{\partial(\rho U U + P - \tau_{xx})}{\partial x} + \frac{\partial(\rho U V - \tau_{xy})}{\partial y} = 0$$

$$\frac{\partial(\rho V U - \tau_{xy})}{\partial x} + \frac{\partial(\rho V V + P - \tau_{yy})}{\partial y} = 0$$

$$\frac{\partial(\rho h U + Q_x)}{\partial x} + \frac{\partial(\rho h V + Q_y)}{\partial y} - \left(U\frac{\partial P}{\partial x} + V\frac{\partial P}{\partial y}\right)$$

$$-\left(\tau_{xx}\frac{\partial U}{\partial x} + \tau_{xy}\frac{\partial U}{\partial y}\right) - \left(\tau_{xy}\frac{\partial V}{\partial x} + \tau_{yy}\frac{\partial V}{\partial y}\right) = 0$$

8.3 Plane, Compressible Couette Flow

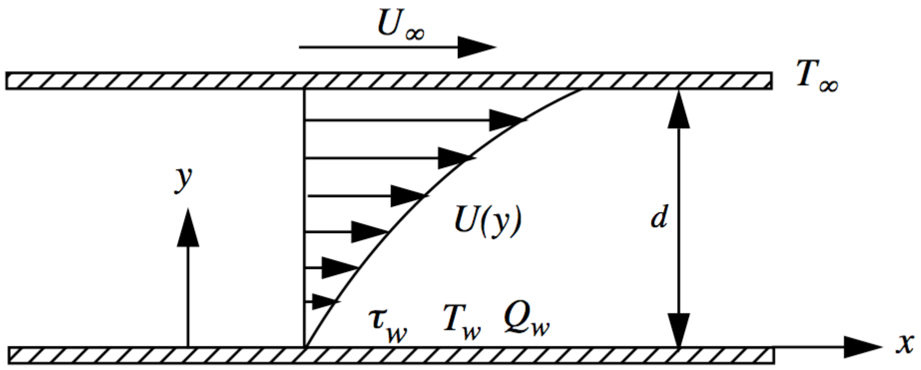


Figure 8.2 Flow produced between two parallel plates in relative motion

The upper wall moves at a velocity U_∞ while the lower wall is at rest. The temperature of the upper wall is T_∞ .

The flow is assumed to be steady and extends to plus and minus infinity in the x-direction. Therefore the velocity and temperature only depend on y.

$$\cancel{\frac{\partial \rho U}{\partial x}} + \cancel{\frac{\partial \rho V}{\partial y}} = 0$$

$$\cancel{\frac{\partial(\rho UU + P - \tau_{xx})}{\partial x}} + \frac{\partial(\cancel{\rho UV} - \underline{\tau_{xy}})}{\partial y} = 0$$

$$\cancel{\frac{\partial(\rho VU - \tau_{xy})}{\partial x}} + \frac{\partial(\cancel{\rho VV} + \underline{P} - \cancel{\tau_{yy}})}{\partial y} = 0$$

$$\cancel{\frac{\partial(\rho hU + Q_x)}{\partial x}} + \frac{\partial(\cancel{\rho hV} + \underline{Q_y})}{\partial y} - \cancel{\left(U\frac{\partial P}{\partial x} + V\frac{\partial P}{\partial y}\right)}$$

$$-\left(\cancel{\tau_{xx}\frac{\partial U}{\partial x}} + \underline{\tau_{xy}\frac{\partial U}{\partial y}}\right) - \left(\cancel{\tau_{xy}\frac{\partial V}{\partial x}} + \cancel{\tau_{yy}\frac{\partial V}{\partial y}}\right) = 0$$

The equations of motion reduce to.

$$\frac{\partial \tau_{xy}}{\partial y} = 0$$

$$\frac{\partial P}{\partial y} = 0$$

$$\frac{\partial (Q_y - \tau_{xy} U)}{\partial y} = 0$$

Both the pressure and shearing stress are uniform throughout the flow. The shearing stress is related to the velocity through the Newtonian constitutive relation.

$$\tau_{xy} = \mu \frac{dU}{dy} = \tau_w = constant$$

Where τ_w is the shear stress at the lower wall.

For gases the viscosity depends only on temperature.

$$\mu = \mu(T)$$

Since the pressure is uniform the density depends only on temperature according to the perfect gas law.

$$\rho(y) = \frac{P}{RT(y)}$$

The solution for the velocity profile can be written as an integral.

$$U(y) = \tau_w \int_0^y \frac{dy}{\mu(T)}$$

To determine the velocity profile we need to know how the viscosity depends on temperature.

The temperature distribution across the channel can be determined from the energy equation. The heat flux is given by

$$Q_y = -\kappa \frac{dT}{dy}$$

The coefficient of heat conductivity, like the viscosity is also only a function of temperature.

$$\kappa = \kappa(T)$$

The Prandtl number is very nearly constant for gases. In many cases the Prandtl number can be taken to be one.

$$P_r = \frac{C_p \mu}{\kappa}$$

The energy equation is

$$\frac{d}{dy}(-Q_y + \tau_w U) = 0$$

Integrating

$$-Q_y + \tau_w U = -Q_w$$

where the integral has been evaluated on the lower wall.

Now insert the expressions for the shear stress and heat flux

$$\kappa\frac{dT}{dy}+\mu U\frac{dU}{dy}=\mu\frac{d}{dy}\left(\frac{1}{P_r}C_pT+\frac{1}{2}U^2\right)=-Q_w$$

where the heat capacity has been assumed constant.
Integrate from the lower wall.

$$C_p(T-T_w)+\frac{1}{2}P_rU^2 = -Q_wP_r\int_0^y\frac{dy}{\mu(T)}$$

Where T_w is the temperature of the lower wall. Note that the integral on the right can be replaced by the velocity.

The result is the so-called energy integral.

$$C_p(T - T_w) + \frac{1}{2}P_r U^2 = -\frac{Q_w}{\tau_w} P_r U$$

At the upper wall the temperature is T_∞ and this can be used to evaluate the lower wall temperature.

$$C_p T_w = C_p T_\infty + P_r\left(\frac{U_\infty^2}{2} + \frac{Q_w}{\tau_w} U_\infty\right)$$

8.9.1 The recovery temperature

Suppose the lower wall is insulated so that $q_w = 0$. What temperature does the lower wall reach? This is called the adiabatic wall recovery temperature.

$$T_{wa} = T_{\infty} + \frac{P_r}{2C_p} U_{\infty}^{2}$$

Introduce the Mach number $M_{\infty}=U_{\infty}/a_{\infty}$

$$\frac{T_{wa}}{T_{\infty}} = 1 + P_r\left(\frac{\gamma - 1}{2}\right) M_{\infty}^{2}$$

Note that the recovery temperature equals the stagnation temperature only for a Prandtl number of one.

For Air $Pr = 0.73$.

Recall

$$\frac{T_{t_{\infty}}}{T_{\infty}} = 1 + \left(\frac{\gamma - 1}{2}\right) M_{\infty}^{2}$$

The recovery factor

$$\frac{T_{wa} - T_{\infty}}{T_{t_{\infty}} - T_{\infty}} = r$$

In the case of Couette flow for a perfect gas with constant C_p

$$\frac{T_{wa} - T_{\infty}}{T_{t_{\infty}} - T_{\infty}} = P_r$$

The heat transfer and shear stress are related by

$$\frac{Q_w}{\tau_w U_{\infty}} = \frac{C_p (T_w - T_{wa})}{P_r U_{\infty}^2}$$

In order to transfer heat into the fluid the lower wall temperature must exceed the recovery temperature.

The last equation can be rearranged to read

$$\frac{\tau_w}{\frac{1}{2}\rho_\infty U_\infty^2} = 2P_r\left(\frac{Q_w}{\rho_\infty U_\infty C_p(T_w - T_{wa})}\right)$$

The friction coefficient is

$$C_f = \frac{\tau_w}{\frac{1}{2}\rho_\infty U_\infty^2}$$

The Stanton number is defined as

$$S_t = \frac{Q_w}{\rho_\infty U_\infty C_p(T_w - T_{wa})}$$

Using these definitions the relation $\frac{q_w}{\tau_w U_\infty} = \frac{C_p(T_w - T_r)}{P_r U_\infty^2}$ is expressed as

$$C_f = 2P_r S_t$$

In general, for compressible flow near a wall, heat transfer and friction are coupled.

8.3.3 The velocity distribution in Couette flow.

Now that the relation between temperature and velocity is known we can integrate the momentum relation for the stress. We use the energy integral written in terms of T_∞ .

$$C_p(T - T_\infty) = P_r \frac{Q_w}{\tau_w}(U_\infty - U) + \frac{1}{2}P_r(U_\infty{}^2 - U^2)$$

with some rearrangement

$$\frac{T}{T_\infty} = 1 + P_r \frac{Q_w}{U_\infty \tau_w}(\gamma - 1)M_\infty^2\left(1 - \frac{U}{U_\infty}\right) + P_r\left(\frac{\gamma - 1}{2}\right)M_\infty^2\left(1 - \frac{U^2}{U_\infty{}^2}\right)$$

The momentum equation is

$$\mu(T)\frac{dU}{dy} = \tau_w$$

or

$$\int_0^U \mu(U)dU = \tau_w y$$

In gases the viscosity dependence on temperature is well approximated by Sutherland's law.

$$\frac{\mu}{\mu_\infty} = \left(\frac{T}{T_\infty}\right)^{3/2}\left(\frac{T_\infty + T_S}{T + T_S}\right)$$

For Air the Sutherland reference temperature is 110.4K.

An approximation that is often used is

$$\frac{\mu}{\mu_\infty} = \left(\frac{T}{T_\infty}\right)^{\omega} \qquad 0.5 < \omega < 1.0$$

For Air the exponent 0.76 is a reasonably accurate approximation to Sutherland's law.

The momentum equation is

$$\int_0^U \left(1 + P_r \frac{Q_w}{U_\infty \tau_w}(\gamma - 1)M_\infty^2\left(1 - \frac{U}{U_\infty}\right) + P_r\left(\frac{\gamma - 1}{2}\right)M_\infty^2\left(1 - \frac{U^2}{U_\infty{}^2}\right)\right)^\omega dU = \frac{\tau_w}{\mu_\infty}y$$

The shear stress is evaluated by integrating over the full height of the channel.

$$\int_0^{U_\infty} \left(1 + P_r \frac{Q_w}{U_\infty \tau_w}(\gamma - 1)M_\infty^2\left(1 - \frac{U}{U_\infty}\right) + P_r\left(\frac{\gamma - 1}{2}\right)M_\infty^2\left(1 - \frac{U^2}{U_\infty{}^2}\right)\right)^\omega dU = \frac{\tau_w}{\mu_\infty}d$$

The simplest case and a reasonable approximation corresponds to $\omega = 1$.
In this case the integral can be carried out.

$$\frac{\tau_w}{\mu_\infty U_\infty}y = \frac{U}{U_\infty} + P_r\left(\frac{\gamma - 1}{2}\right)M_\infty^2\left(\frac{U}{U_\infty} - \frac{1}{3}\left(\frac{U}{U_\infty}\right)^3\right)$$

$Q_w = 0$

The shear stress is found by evaluating at $U/U_\infty = 1$.

$$\frac{\tau_w}{\mu_\infty U_\infty} d = 1 + P_r\left(\frac{\gamma - 1}{3}\right) M_\infty^2$$

The velocity profile is

$$\frac{y}{d} = \frac{\frac{U}{U_\infty} + P_r\left(\frac{\gamma - 1}{2}\right) M_\infty^2 \left(\frac{U}{U_\infty} - \frac{1}{3}\left(\frac{U}{U_\infty}\right)^3\right)}{1 + P_r\left(\frac{\gamma - 1}{3}\right) M_\infty^2}$$

At high Mach number the velocity profile is independent of Mach number and Prandtl number.

$$\lim_{M_\infty \to \infty}\left(\frac{y}{d}\right) = \left(\frac{3}{2}\right)\left(\frac{U}{U_\infty} - \frac{1}{3}\left(\frac{U}{U_\infty}\right)^3\right)$$

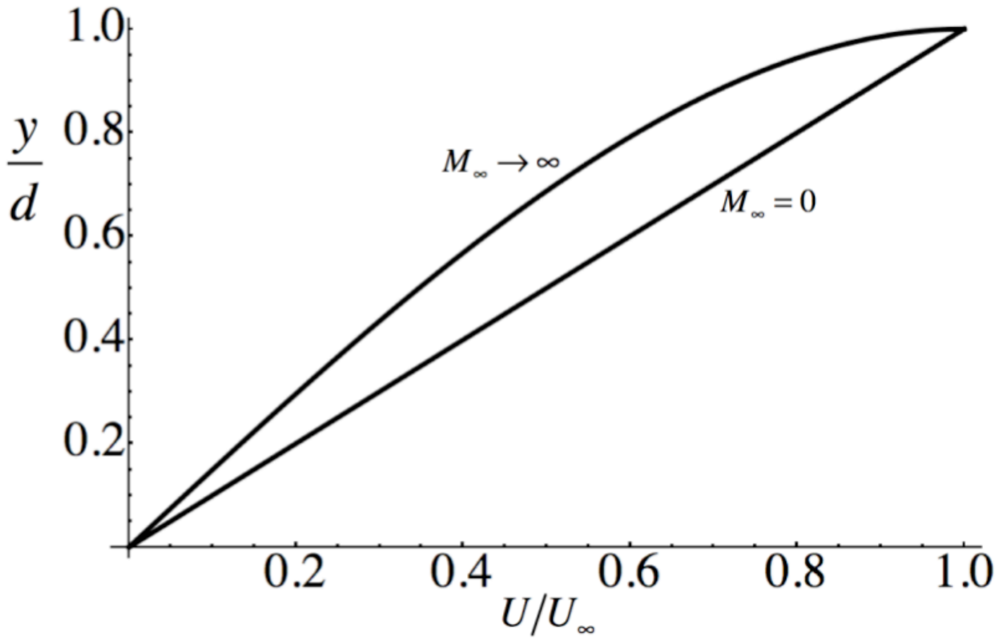


Figure 8.3 Velocity distribution in plane Couette flow for an adiabatic lower wall and $\omega = 1$.

In terms of the friction coefficient and Reynolds number

$$C_f = \frac{\tau_w}{\frac{1}{2}\rho_\infty U_\infty^2} \qquad R_e = \frac{\rho_\infty U_\infty d}{\mu_\infty}$$

the wall friction coefficient for an adiabatic wall is determined in terms of the Prandtl, Reynolds and Mach numbers.

$$C_f = 2\left(\frac{1 + P_r\left(\frac{\gamma - 1}{3}\right)M_\infty^2}{R_e}\right)$$

$Q_w = 0$

The Reynolds number can be expressed as

$$R_e = \frac{\rho_\infty U_\infty d}{\mu_\infty} = \frac{\frac{1}{2}\rho_\infty U_\infty^2}{\frac{1}{2}\mu_\infty \frac{U_\infty}{d}} = \frac{\text{dynamic pressure at the upper plate}}{\text{characteristic shear stress}}$$

For an non-adiabatic wall with $\omega = 1$

$$\int_0^{U/U_\infty} \left(1 + P_r \frac{q_w}{U_\infty \tau_w} (\gamma - 1) M_\infty^2 \left(1 - \frac{U}{U_\infty} \right) + P_r \left(\frac{\gamma - 1}{2} \right) M_\infty^2 \left(1 - \frac{U^2}{U_\infty^2} \right) \right) d\left(\frac{U}{U_\infty} \right) = \frac{\tau_w y}{\mu_\infty U_\infty}$$

Carry out the indicated integration

$$\frac{U}{U_\infty} + P_r \frac{q_w}{U_\infty \tau_w} (\gamma - 1) M_\infty^2 \left(\frac{U}{U_\infty} - \frac{1}{2} \left(\frac{U}{U_\infty} \right)^2 \right) + P_r \left(\frac{\gamma - 1}{2} \right) M_\infty^2 \left(\frac{U}{U_\infty} - \frac{1}{3} \left(\frac{U}{U_\infty} \right)^3 \right) = \frac{\tau_w y}{\mu_\infty U_\infty}$$

Evaluate at the upper wall

$$1 + P_r \frac{q_w}{U_\infty \tau_w} \left(\frac{\gamma - 1}{2} \right) M_\infty^2 + P_r \left(\frac{\gamma - 1}{3} \right) M_\infty^2 = \frac{\tau_w d}{\mu_\infty U_\infty}$$

Evaluate the velocity at the upper wall to determine the shear stress

$$1+P_r\frac{q_w}{U_\infty\tau_w}\left(\frac{\gamma-1}{2}\right)M_\infty^2+P_r\left(\frac{\gamma-1}{3}\right)M_\infty^2=\frac{\tau_w d}{\mu_\infty U_\infty}$$

Using

$$\frac{q_w}{\tau_w U_\infty}=\frac{C_p(T_w-T_r)}{P_r U_\infty^2}$$

The friction coefficient with heat transfer becomes

$$C_f=\frac{2}{R_e}\left(1+\frac{1}{2}\left(1+\left(\frac{\gamma-1}{2}\right)P_rM_\infty^2\right)\left(\frac{T_w}{T_r}-1\right)+\left(\frac{\gamma-1}{3}\right)P_rM_\infty^2\right)$$

$Q_w \neq 0$

8.4 The viscous boundary layer on a wall

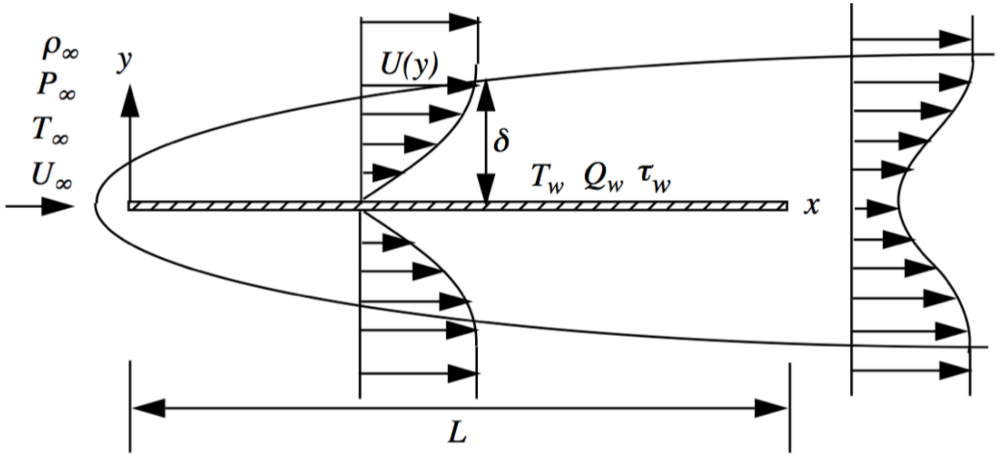


Figure 8.4: *Low Reynolds number flow about a thin flat plate of length L. Re_L is less than a hundred or so. The parabolic envelope which extends upstream of the leading edge roughly delineates the region of rotational flow produced as a consequence of the no slip condition on the plate.*

$$R_{eL} = \frac{\rho_\infty U_\infty L}{\mu_\infty}$$

Reference: Boundary Layer Theory by Schlichting

The figure depicts the flow at low Reynolds number less than 100 or so.

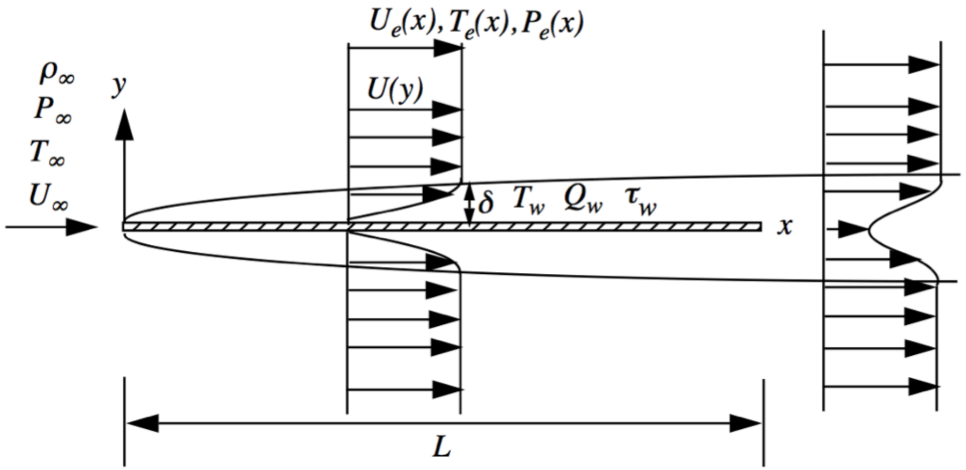


Figure 8.5: *High Reynolds number flow developing from the leading edge of a flat plate of length L.* Re_L *is several hundred or more .*

As the Reynolds number is increased to several hundred or more the velocity profile near the wall becomes quite thin and the guiding effect of the plate leads to a situation where the vertical velocity is small compared to the horizontal velocity.

$$\frac{\delta}{L} \ll 1 \qquad \frac{V}{U} \ll 1 \qquad \frac{\partial(\;)}{\partial x} \ll \frac{\partial(\;)}{\partial y} \qquad U\frac{\partial(\;)}{\partial x} \sim V\frac{\partial(\;)}{\partial y} \qquad \frac{\partial U}{\partial x} \sim -\frac{\partial V}{\partial y}$$

Viscous terms in the equations of motion are comparable to the convective terms.

$$\rho U \frac{\partial U}{\partial x} \approx \mu \frac{\partial^2 U}{\partial y^2} \quad \Rightarrow \quad \frac{\rho_\infty U_\infty^{\;2}}{L} \approx \mu \frac{U_\infty}{\delta^2} \Rightarrow \frac{\delta}{L} \approx \frac{1}{(R_{eL})^{1/2}}$$

First consider the y - momentum equation.

$$\frac{\partial(\rho VU - \tau_{xy})}{\partial x} + \frac{\partial(\rho VV + P - \tau_{yy})}{\partial y} = 0$$

Using the approximations just discussed this equation reduces to.

$$\frac{\partial(P - \tau_{yy})}{\partial y} = 0$$

Integrate from the wall to the edge of the boundary layer.

$$P(x, y) = \tau_{yy}(x, y) + P_e(x)$$

Substitute for the pressure in the x - momentum equation.

$$\rho U\frac{\partial U}{\partial x} + \rho V\frac{\partial U}{\partial y} = -\frac{dP_e}{dx} + \frac{\partial}{\partial x}(\tau_{xx} - \tau_{yy}) + \frac{\partial \tau_{xy}}{\partial y}$$

The energy equation

$$\begin{gathered}\frac{\partial(\rho hU + Q_x)}{\partial x} + \frac{\partial(\rho hV + Q_y)}{\partial y} - \\ \left(U\frac{\partial P}{\partial x} + V\frac{\partial P}{\partial y}\right) - \left(\tau_{xx}\frac{\partial U}{\partial x} + \tau_{xy}\frac{\partial U}{\partial y}\right) - \\ \left(\tau_{xy}\frac{\partial V}{\partial x} + \tau_{yy}\frac{\partial V}{\partial y}\right) = 0\end{gathered}$$

simplifies to

$$\begin{gathered}\rho U\frac{\partial h}{\partial x} + \rho V\frac{\partial h}{\partial y} + \frac{\partial Q_y}{\partial y} - U\frac{dP_e}{dx} + U\frac{\partial}{\partial x}(\tau_{xx} - \tau_{yy}) - \\ \frac{\partial(V\tau_{yy})}{\partial y} - \frac{\partial(U\tau_{xx})}{\partial x} - \tau_{xy}\frac{\partial U}{\partial y} = 0\end{gathered}$$

Neglect the normal stress terms.

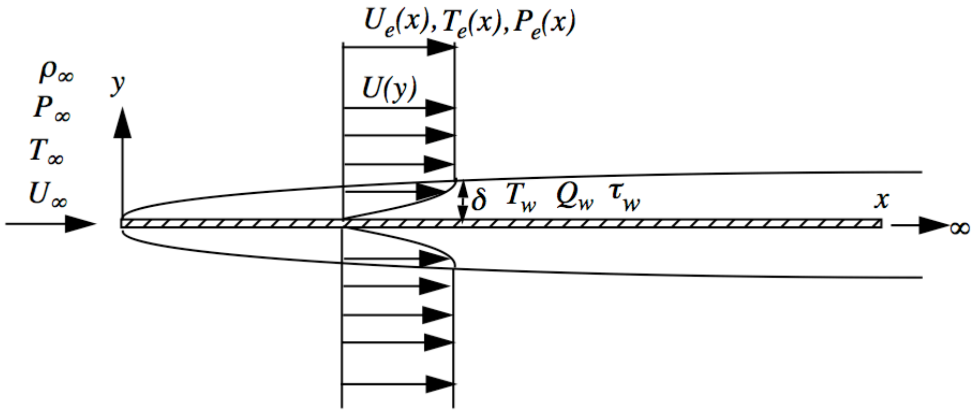
$$\frac{\partial \rho U}{\partial x} + \frac{\partial \rho V}{\partial y} = 0$$

$$\rho U \frac{\partial U}{\partial x} + \rho V \frac{\partial U}{\partial y} = -\frac{dP_e}{dx} + \frac{\partial \tau_{xy}}{\partial y}$$

$$\rho U \frac{\partial h}{\partial x} + \rho V \frac{\partial h}{\partial y} + \frac{\partial Q_y}{\partial y} - U \frac{dP_e}{dx} - \tau_{xy} \frac{\partial U}{\partial y} = 0$$

where

$$\tau_{xy} = \tau_{xy}\Big|_{laminar} + \tau_{xy}\Big|_{turbulent} = \mu \frac{\partial U}{\partial y} + \tau_{xy}\Big|_{turbulent}$$



$$R_{ex} = \frac{\rho_\infty U_\infty x}{\mu_\infty}$$

Figure 8.6 High Reynolds number flow developing from the leading edge of a semi-infinite flat plate.

Newtonian stress.

$$\tau_{ij} = 2\mu S_{ij} - \left(\frac{2}{3}\mu - \mu_v\right)\delta_{ij}S_{kk}$$

$$\tau_{xy} = \mu\left(\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x}\right) \cong \mu\frac{\partial U}{\partial y}$$

Fourier's law.

$$Q_y = -\kappa\frac{\partial T}{\partial y}$$

The laminar compressible boundary layer equations.

$$\boxed{\begin{gathered}
\frac{\partial \rho U}{\partial x} + \frac{\partial \rho V}{\partial y} = 0 \\
\rho U\frac{\partial U}{\partial x} + \rho V\frac{\partial U}{\partial y} = -\frac{dP_e}{dx} + \frac{\partial}{\partial y}\left(\mu\frac{\partial U}{\partial y}\right) \\
\rho U C_p\frac{\partial T}{\partial x} + \rho V C_p\frac{\partial T}{\partial y} = U\frac{dP_e}{dx} + \frac{\partial}{\partial y}\left(\kappa\frac{\partial T}{\partial y}\right) + \mu\left(\frac{\partial U}{\partial y}\right)^2
\end{gathered}}$$

Measures of boundary layer thickness.

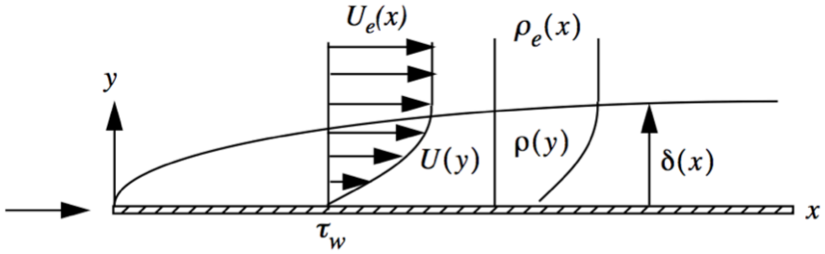
Displacement thickness

$$\delta^* = \int_0^\delta \left(1 - \frac{\rho U}{\rho_e U_e}\right) dy$$

Momentum thickness

$$\theta = \int_0^\delta \frac{\rho U}{\rho_e U_e}\left(1 - \frac{U}{U_e}\right) dy$$

8.5 The compressible von Karman integral equation



$$C_f = \frac{\tau_w}{\frac{1}{2}\rho_e {U_e}^2}$$

Figure 8.7: *Boundary layer velocity and density profiles.*

$$\frac{\partial \rho U}{\partial x} + \frac{\partial \rho V}{\partial y} = 0$$

$$\frac{\partial \rho U^2}{\partial x} + \frac{\partial \rho UV}{\partial y} + \frac{dP_e}{dx} - \frac{\partial \tau_{xy}}{\partial y} = 0$$

Integrate the boundary layer equations with respect to y

$$\int_0^{\delta(x)} \left(\frac{\partial \rho U}{\partial x}\right) dy + \int_0^{\delta(x)} \left(\frac{\partial \rho V}{\partial y}\right) dy = 0$$

$$\int_0^{\delta(x)} \left(\frac{\partial \rho U^2}{\partial x}\right) dy + \int_0^{\delta(x)} \left(\frac{\partial \rho UV}{\partial y}\right) dy + \int_0^{\delta(x)} \left(\frac{dP_e}{dx}\right) dy - \int_0^{\delta(x)} \left(\frac{\partial \tau_{xy}}{\partial y}\right) dy = 0$$

$$\frac{d\theta}{dx} + (2\theta + \delta^*) \frac{1}{U_e}\frac{dU_e}{dx} + \frac{\theta}{\rho_e}\frac{d\rho_e}{dx} = \frac{C_f}{2}$$

$$\frac{d\theta}{dx} + \left(\left(2 - {M_e}^2\right)\theta + \delta^*\right) \frac{1}{U_e}\frac{dU_e}{dx} = \frac{C_f}{2}$$

Shape factor $H = \frac{\delta^*}{\theta}$

$$\boxed{\frac{d\theta}{dx} + \left(2 - {M_e}^2 + H\right) \frac{\theta}{U_e}\frac{dU_e}{dx} = \frac{C_f}{2}}$$

8.6 The laminar incompressible boundary layer

The equations of motion reduce to

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0$$

$$U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} = -\frac{1}{\rho}\frac{dP_e}{dx} + \nu\left(\frac{\partial^2 U}{\partial y^2}\right)$$

$$\nu = \frac{\mu}{\rho}$$

Boundary conditions

$$U(0) = V(0) = 0 \qquad U(\delta) = U_e$$

The pressure

$$P_t = P_e(x) + \frac{1}{2}\rho U_e(x)^2 \Rightarrow \frac{1}{\rho}\frac{dP_e}{dx} = -U_e\frac{dU_e}{dx}$$

Introduce the stream function

$$U = \frac{\partial \psi}{\partial y} \qquad V = -\frac{\partial \psi}{\partial x}$$

The continuity equation is identically satisfied and the momentum equation becomes:

$$\psi_y \psi_{xy} - \psi_x \psi_{yy} = U_e \frac{dU_e}{dx} + \nu \psi_{yyy}$$

Boundary conditions

$$\psi(x,0) = 0 \qquad \psi_y(x,0) = 0 \qquad \psi_y(x,\infty) = U_e$$

The zero pressure gradient, incompressible boundary layer.

$$\psi_y \psi_{xy} - \psi_x \psi_{yy} = \nu \psi_{yyy}$$

Similarity variables

$$\psi = \left(2\nu U_\infty x\right)^{1/2} F(\alpha) \qquad \alpha = y\left(\frac{U_\infty}{2\nu x}\right)^{1/2}$$

Velocity components

$$\frac{U}{U_\infty} = F_\alpha \qquad \frac{V}{U_\infty} = \left(\frac{\nu}{2U_\infty x}\right)^{1/2}(\alpha F_\alpha - F)$$

Reynolds number is based on distance from the leading edge

$$R_{ex} = \frac{U_\infty x}{\nu}$$

Vorticity

$$\omega = \frac{\partial V}{\partial x} - \frac{\partial U}{\partial y} \cong -U_\infty\left(\frac{U_\infty}{2\nu x}\right)^{1/2} F_{\alpha\alpha}$$

Derivatives

$$\psi_{xy} = -\frac{U_\infty}{2x}\alpha F_{\alpha\alpha}$$

$$\psi_{yy} = U_\infty\left(\frac{U_\infty}{2\nu x}\right)^{1/2} F_{\alpha\alpha}$$

$$\psi_{yyy} = \frac{U_\infty^2}{2\nu x} F_{\alpha\alpha\alpha}$$

Substitute into the stream function equation and simplify

$$U_\infty F_\alpha\left(-\frac{U_\infty}{2x}\alpha F_{\alpha\alpha}\right) - U_\infty\left(\left(\frac{\nu}{2U_\infty x}\right)^{1/2}(\alpha F_\alpha - F)\right) U_\infty\left(\frac{U_\infty}{2\nu x}\right)^{1/2} F_{\alpha\alpha} = \nu\frac{U_\infty^2}{2\nu x}F_{\alpha\alpha\alpha}$$

$$-F_\alpha(\alpha F_{\alpha\alpha}) + (\alpha F_\alpha - F)F_{\alpha\alpha} = F_{\alpha\alpha\alpha}$$

$$-\alpha F_\alpha F_{\alpha\alpha} - FF_{\alpha\alpha} + \alpha F_\alpha F_{\alpha\alpha} - F_{\alpha\alpha\alpha} = 0$$

The Blasius equation

$$F_{\alpha\alpha\alpha} + F F_{\alpha\alpha} = 0$$

Boundary conditions

$$F(0) = 0 \qquad F_{\alpha}(0) = 0 \qquad F_{\alpha}(\infty) = 1$$

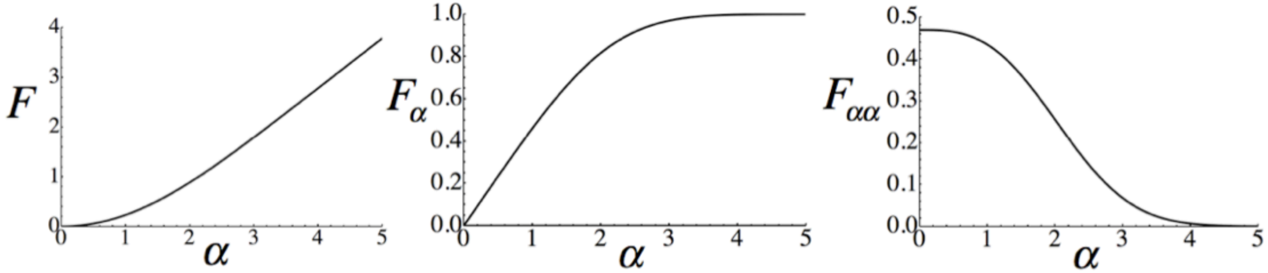


Figure 8.8: *Solution of the Blasius equation* (8.102) *for the stream function velocity and stress (or vorticity) profile in a zero pressure gradient laminar boundary layer.*

Friction coefficient

$$C_f = \frac{\tau_w}{(1/2)\rho U_\infty^2} = \frac{0.664}{\sqrt{R_{ex}}} \qquad C_f = \frac{\sqrt{2}}{\sqrt{R_{ex}}} F_{\alpha\alpha}(0)$$

Normal velocity at the edge of the layer

$$\frac{V_e}{U_\infty} = \frac{0.8604}{\sqrt{R_{ex}}}$$

Boundary layer thickness $\alpha_e = 4.906/\sqrt{2} = 3.469$

$$\frac{\delta_{0.99}}{x} = \frac{4.906}{\sqrt{R_{ex}}} \qquad \frac{\delta^*}{x} = \frac{1.7208}{\sqrt{R_{ex}}} \qquad \frac{\theta}{x} = \frac{0.664}{\sqrt{R_{ex}}}$$

Boundary layer shape factor

$$H = \frac{\delta^*}{\theta} = 2.5916$$

Let $\tau = F_{\alpha\alpha}$

The Blasius equation can be expressed as

$$\frac{d\tau}{\tau} = -F d\alpha$$

$$\frac{\tau}{\tau_w} = e^{-\int_0^{\alpha} F d\alpha}$$

Let $F(\alpha) = \alpha - G(\alpha)$ Then $\lim_{\alpha \to \infty} G(\alpha) = C_1$

$$\left.\frac{\tau}{\tau_w}\right|_{\alpha > \alpha_e} = e^{-\int_0^{\alpha} (\alpha - G(\alpha)) d\alpha} = C_2 e^{C_1 \alpha - \frac{\alpha^2}{2}}$$

Vorticity at the edge of the layer decays exponentially with distance from the wall. This supports the approach where we divide the flow into separate regions of rotational and irrotational flow.

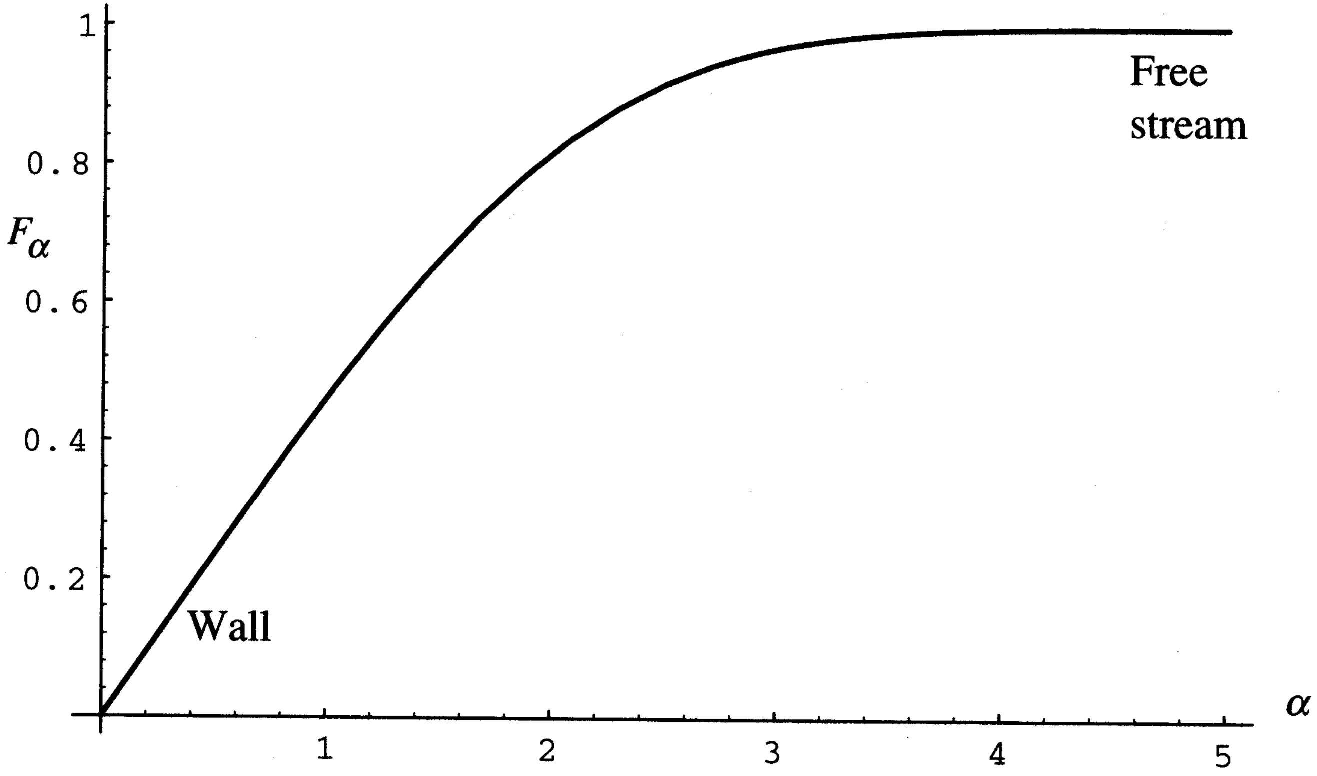


Fig. 10.4. The Blasius velocity profile.

$$C_{f0} = \frac{\tau_0}{\frac{1}{2}\rho U_e^2} = \frac{0.664}{\sqrt{Re}}.$$

Numerical solution

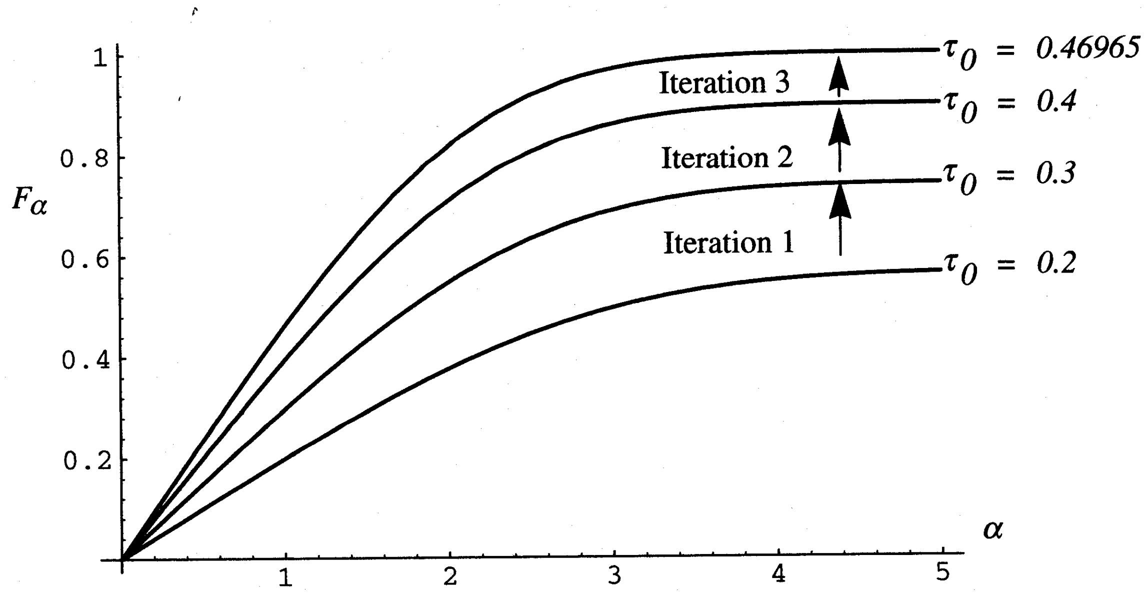


Fig. 10.5. Iteration process leading to the correct match with the free-stream boundary condition $\lim_{\alpha\to\infty} F_\alpha = 1$.

Another perspective: Use the dilational symmetry of the problem

$$\psi_y \psi_{xy} - \psi_x \psi_{yy} = \nu \psi_{yyy}$$

$$\psi(x,0) = 0 \qquad \psi_y(x,0) = 0 \qquad \psi_y(x,\infty) = U_e$$

Transform the governing equation

$$\tilde{x} = e^a x \qquad \tilde{y} = e^b y \qquad \tilde{\psi} = e^c \psi$$

$$\tilde{\psi}_{\tilde{y}} \tilde{\psi}_{\tilde{x}\tilde{y}} - \tilde{\psi}_{\tilde{x}} \tilde{\psi}_{\tilde{y}\tilde{y}} - \nu \tilde{\psi}_{\tilde{y}\tilde{y}\tilde{y}} = e^{2c-a-2b} \psi_y \psi_{xy} - e^{2c-a-2b} \psi_x \psi_{yy} - \nu e^{c-3b} \psi_{yyy} = 0$$

The equation is invariant if and only if

$$2c - a - 2b = c - 3b.$$

$$\tilde{x} = e^a x \qquad \tilde{y} = e^b y \qquad \tilde{\psi} = e^{a-b} \psi$$

Transform boundary curves and boundary functions

$$\tilde{y} = 0 \quad \Rightarrow \quad e^{b} y = 0 \quad \Rightarrow \quad y = 0.$$

At the wall

$$\tilde{\psi}\left(\tilde{x},0\right) = 0\Big|_{all\ \tilde{x}} \Rightarrow e^{a-b}\psi\left(e^{a}x,0\right) = 0 \Rightarrow \psi\left(x,0\right) = 0\Big|_{all\ x}$$

$$\tilde{\psi}_{\tilde{y}}\left(\tilde{x},0\right) = 0\Big|_{all\ \tilde{x}} \Rightarrow e^{a-2b}\psi_{y}\left(e^{a}x,0\right) = 0 \Rightarrow \psi_{y}\left(x,0\right) = 0\Big|_{all\ x}$$

At $y \rightarrow \infty$

$$\tilde{\psi}_{\tilde{y}}\left(\tilde{x},\infty\right) = U_{\infty}\Big|_{all\ \tilde{x}} \Rightarrow e^{a-2b}\psi_{y}\left(e^{a}x,\infty\right) = U_{\infty}\Big|_{all\ x}$$

The freestream boundary condition is invariant if and only if $\underline{a = 2b}$

The governing equations and boundary conditions are invariant under the group:

$$\tilde{x} = e^{2b}x \qquad \tilde{y} = e^{b}y \qquad \tilde{\psi} = e^{b}\psi$$

The infinitesimal transformation. Expand near $b = 0$

$$\xi = 2x \qquad \zeta = y \qquad \eta = \psi$$

Characteristic equations

$$\frac{dx}{2x} = \frac{dy}{y} = \frac{d\psi}{\psi}$$

Invariants $\alpha = \frac{y}{\sqrt{x}}$ $F = \frac{\psi}{\sqrt{x}}$

Since the governing equations and boundary conditions are invariant under the group we can expect that the solution will also be invariant under the group.

We can expect the solution to be of the form

$$\psi = \sqrt{x}F(\alpha)$$

The Blasius equation is invariant under a dilation group and this group can be used to generate the solution in one step!

$$\tilde{\alpha} = e^{b}\alpha,$$
$$\tilde{F} = e^{-b}F,$$
$$\tilde{F}_{\tilde{\alpha}} = e^{-2b}F_{\alpha},$$
$$\tilde{F}_{\tilde{\alpha}\tilde{\alpha}} = e^{-3b}F_{\alpha\alpha}.$$

$$1 = e^{-2b}(0.566067) \quad \Rightarrow \quad b = -0.28455'$$

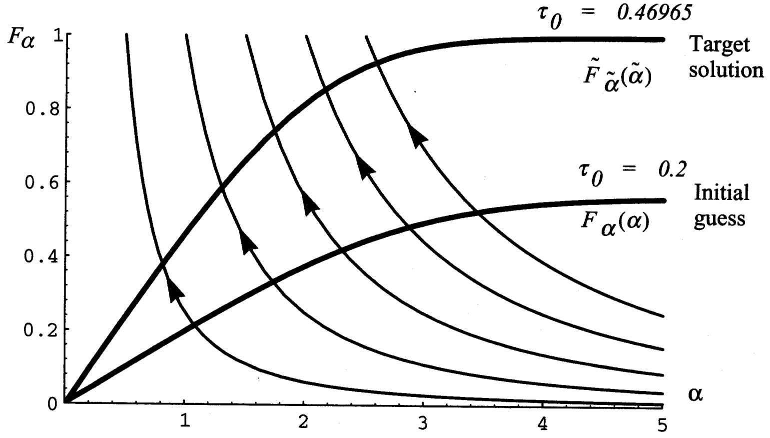


Fig. 10.6. Mapping of an initial guess to the correct solution along the pathlines of the dilation group of the Blasius equation.

$$\tilde{F}_{\tilde{\alpha}\tilde{\alpha}}[0] = e^{-3b}(0.2) \quad \Rightarrow \quad \tilde{F}_{\tilde{\alpha}\tilde{\alpha}}[0] = 0.46965$$

8.7 Falkner-Skan laminar boundary layers

$$\psi_y \psi_{xy} - \psi_x \psi_{yy} - U_e \frac{dU_e}{dx} - \nu \psi_{yyy} = 0 .$$

Free stream velocity .

$$U_e = M x^{\beta}$$

$$\hat{M} = L^{1-\beta} / T .$$

Substitute.

$$\psi_y \psi_{xy} - \psi_x \psi_{yy} - \beta M^2 x^{(2\beta - 1)} - \nu \psi_{yyy} = 0$$

Apply a three-parameter dilation group to the equation.

$$\tilde{x} = e^{a}x \qquad \tilde{y} = e^{b}y \qquad \tilde{\psi} = e^{c}\psi$$

$$\tilde{\psi}_{\tilde{y}}\tilde{\psi}_{\tilde{x}\tilde{y}} - \tilde{\psi}_{\tilde{x}}\tilde{\psi}_{\tilde{y}\tilde{y}} - \beta M^{2}\tilde{x}^{2\beta-1} - \nu\tilde{\psi}_{\tilde{y}\tilde{y}\tilde{y}} =$$

$$e^{2c-a-2b}(\psi_{y}\psi_{xy} - \psi_{x}\psi_{yy}) - e^{(2\beta-1)a}(\beta M^{2}x^{2\beta-1}) - e^{c-3b}(\nu\tilde{\psi}_{\tilde{y}\tilde{y}\tilde{y}}) = 0 .$$

For invariance we require the parameters to be related as follows

$$2c - a - 2b = c - 3b = (2\beta - 1)a .$$

Boundary functions and boundary curves must also be invariant.

$$\tilde{y} = e^{b}y = 0 \Rightarrow y = 0$$

$$\tilde{\psi}(\tilde{x}, 0) = e^{c}\psi(e^{a}x, 0) = 0 \Rightarrow \psi(x, 0) = 0$$

$$\tilde{\psi}_{\tilde{y}}(\tilde{x}, 0) = e^{c-b}\psi_{y}(e^{a}x, 0) = 0 \Rightarrow \psi_{y}(x, 0) = 0$$

Free stream boundary condition.

$$\tilde{\psi}_{\tilde{y}}(\tilde{x}, \infty) = e^{c-b}\psi_y(e^a x, \infty) = e^{\beta a} M x^{\beta}$$

For invariance

$$c - b = \beta a$$

The group that leaves the problem as a whole invariant is

$$\tilde{x} = e^{\frac{2}{1-\beta}b} x \qquad \tilde{y} = e^{b} y \qquad \tilde{\psi} = e^{\frac{1+\beta}{1-\beta}b} \psi$$

The solution should be invariant under the same group

$$\frac{\psi}{x^{\left(\frac{1+\beta}{2}\right)}} = F\left(\frac{y}{x^{\left(\frac{1-\beta}{2}\right)}}\right)$$

In summary

$$\psi_y\psi_{xy} - \psi_x\psi_{yy} - U_e\frac{dU_e}{dx} - \nu\psi_{yyy} = 0.$$

Allow for a virtual origin in x

$$U_e = M(x+x_0)^{\beta}$$

$$\hat{M} = L^{1-\beta}/T .$$

Dimensionless similarity variables

$$\left.\begin{aligned} \alpha &= \left(\frac{M}{2\nu}\right)^{\frac{1}{2}}\frac{y}{(x+x_0)^{(1-\beta)/2}} \\ F &= \frac{\psi}{(x+x_0)^{(1+\beta)/2}(2\nu M)^{1/2}} \end{aligned}\right\}.$$

$$(x+x_0)^{2\beta-1}(F_{\alpha}((1+\beta)F-(1-\beta)\alpha F_{\alpha})_{\alpha} -$$
$$F_{\alpha\alpha}((1+\beta)F-(1-\beta)\alpha F_{\alpha})\; -2\beta - F_{\alpha\alpha\alpha}) \;=\; 0$$

The Falkner-Skan equation.

$$F_{\alpha\alpha\alpha} + (1+\beta)FF_{\alpha\alpha} - 2\beta(F_{\alpha})^2 + 2\beta \;=\; 0$$

$$F[0] = 0 \quad ; \qquad F_{\alpha}[0] = 0 \quad ; \qquad F_{\alpha}[\infty] = 1$$

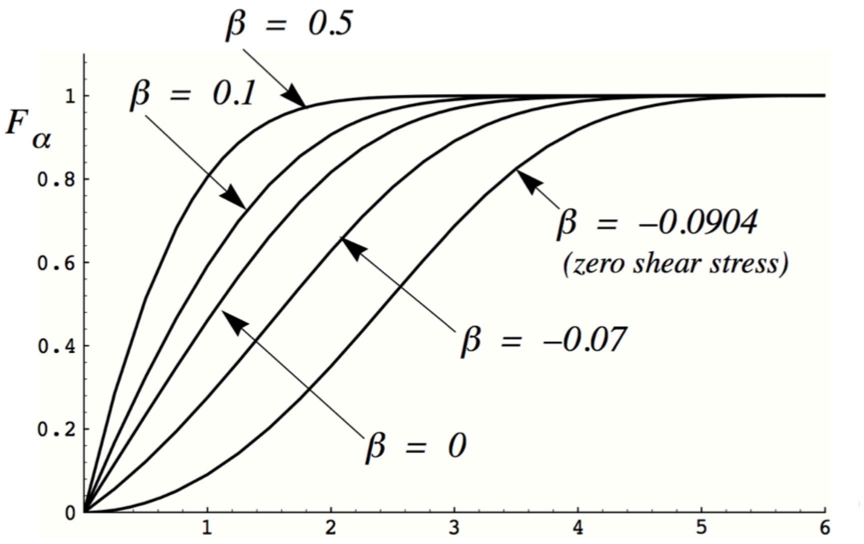


Figure 8.9: *Falkner-Skan velocity profiles.*

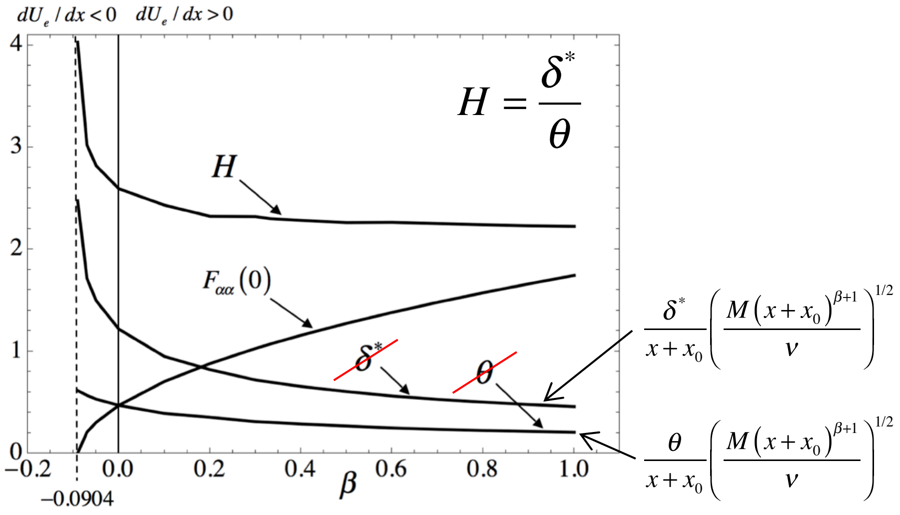
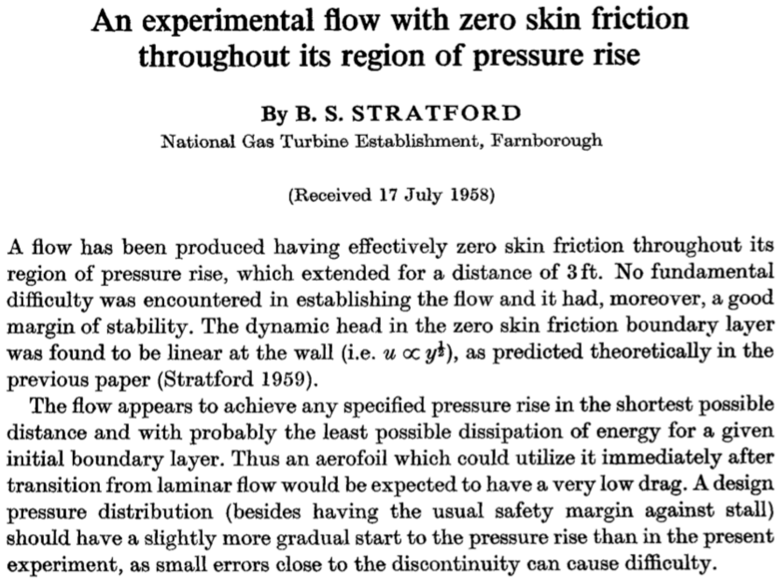


Figure 8.10: *Falkner-Skan boundary layer parameters versus β.*



An experimental flow with zero skin friction throughout its region of pressure rise

By B. S. STRATFORD
National Gas Turbine Establishment, Farnborough

(Received 17 July 1958)

A flow has been produced having effectively zero skin friction throughout its region of pressure rise, which extended for a distance of 3 ft. No fundamental difficulty was encountered in establishing the flow and it had, moreover, a good margin of stability. The dynamic head in the zero skin friction boundary layer was found to be linear at the wall (i.e. $u \propto y^{\frac{1}{2}}$), as predicted theoretically in the previous paper (Stratford 1959).

The flow appears to achieve any specified pressure rise in the shortest possible distance and with probably the least possible dissipation of energy for a given initial boundary layer. Thus an aerofoil which could utilize it immediately after transition from laminar flow would be expected to have a very low drag. A design pressure distribution (besides having the usual safety margin against stall) should have a slightly more gradual start to the pressure rise than in the present experiment, as small errors close to the discontinuity can cause difficulty.

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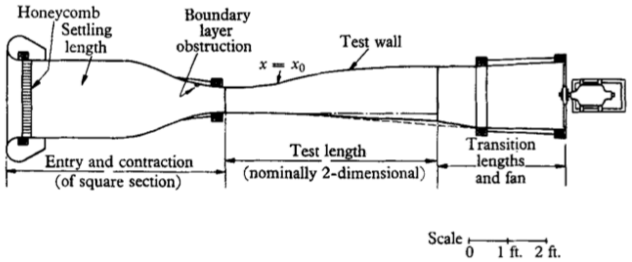


FIGURE 1. Plan-section sketch of the wind tunnel.

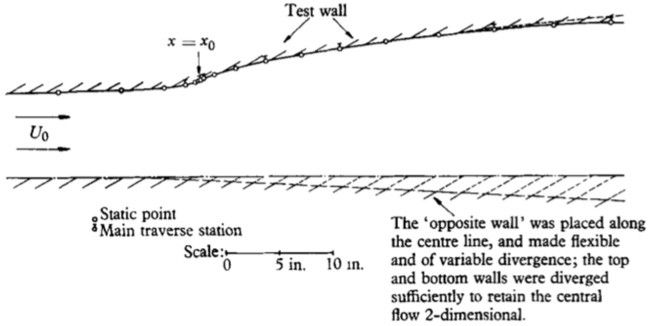


FIGURE 2. Design of the test section.

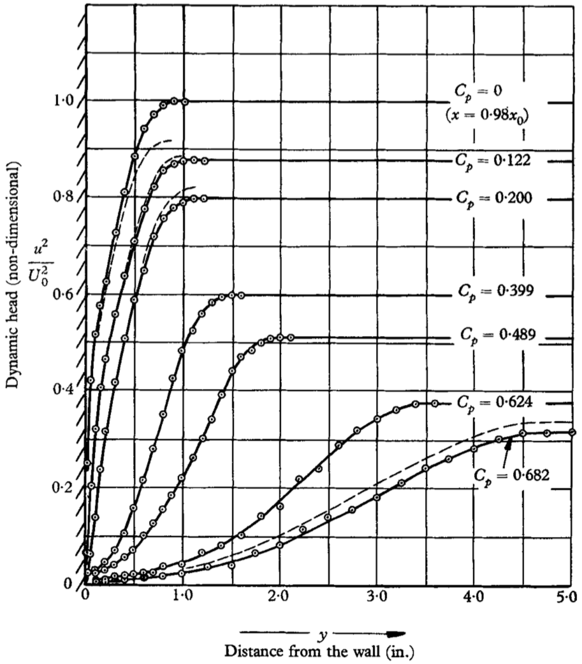


FIGURE 8. The dynamic head profiles. The full line profiles represent total pressure minus static pressure at the wall. Where the static pressure varies across the boundary layer the true dynamic head is represented by the broken lines. The C_p values refer to the wall.

8.8 Thwaites' method for approximate calculation of boundary layer parameters.

From the momentum equation

$$\left.\frac{\partial^2 U}{\partial y^2}\right|_{y=0} = -\frac{U_e}{\nu}\frac{dU_e}{dx}$$

From the von Karman equation

$$\left.\frac{\partial U}{\partial y}\right|_{y=0} = (2+H)\theta\frac{U_e}{\nu}\frac{dU_e}{dx} + \frac{U_e^2}{\nu}\frac{d\theta}{dx}$$

Nondimensionalize using θ and U_e

$$\left(\frac{\theta^2}{U_e}\right)\left.\frac{\partial^2 U}{\partial y^2}\right|_{y=0} = -\frac{\theta^2}{\nu}\frac{dU_e}{dx}$$

$$\left(\frac{\theta}{U_e}\right)\left.\frac{\partial U}{\partial y}\right|_{y=0} = (2+H)\frac{\theta^2}{\nu}\frac{dU_e}{dx} + \frac{U_e}{2\nu}\frac{d\theta^2}{dx}$$

Define

$$m = \left(\frac{\theta^2}{U_e}\right)\left.\frac{\partial^2 U}{\partial y^2}\right|_{y=0} \qquad l(m) = \left(\frac{\theta}{U_e}\right)\left.\frac{\partial U}{\partial y}\right|_{y=0} .$$

Thwaites argued that there should exist a universal function relating m and $l(m)$.

$$\frac{U_e}{\nu}\frac{d\theta^2}{dx} = 2((2+H)m + l(m)) = L(m)$$

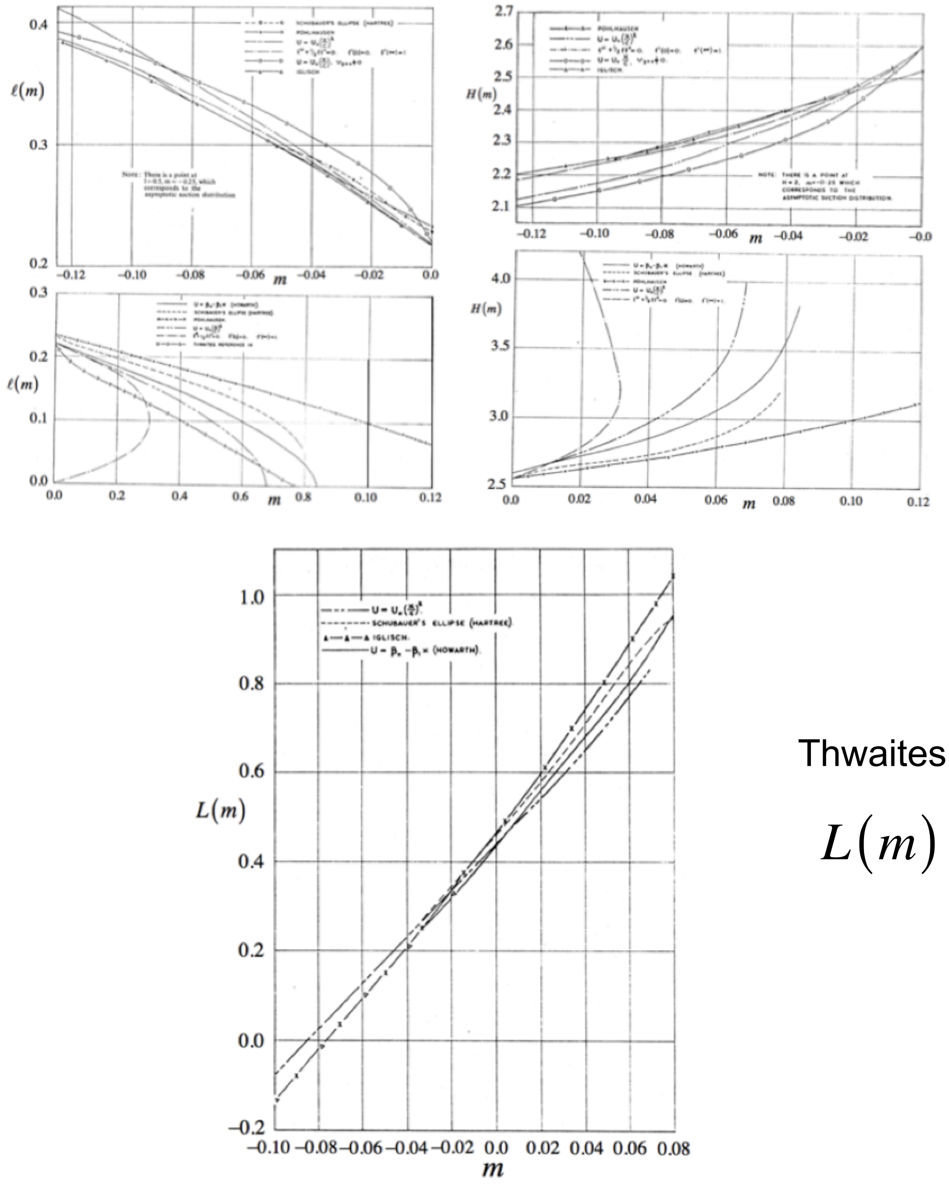


Figure 8.14: *Data on skin friction collected by Thwaites:* $l(m)$ *, shape factor* $H(m)$ *and* $L(m)$ *for a variety of boundary layer solutions.*

Thwaites suggests using

$$L(m) = 0.45 + 6m$$

Thwaites functions can be calculated explicitly for the Falkner-Skan boundary layers

$$m = F_{\alpha\alpha\alpha}(0)\left(\int_0^\alpha F_\alpha(1-F_\alpha)d\alpha\right)^2 = -2\beta\left(\int_0^\alpha F_\alpha(1-F_\alpha)d\alpha\right)^2$$

$$l(m) = F_{\alpha\alpha}(0)\int_0^\alpha F_\alpha(1-F_\alpha)d\alpha$$

$$H(m) = \frac{\int_0^\alpha (1-F_\alpha)d\alpha}{\int_0^\alpha F_\alpha(1-F_\alpha)d\alpha}$$

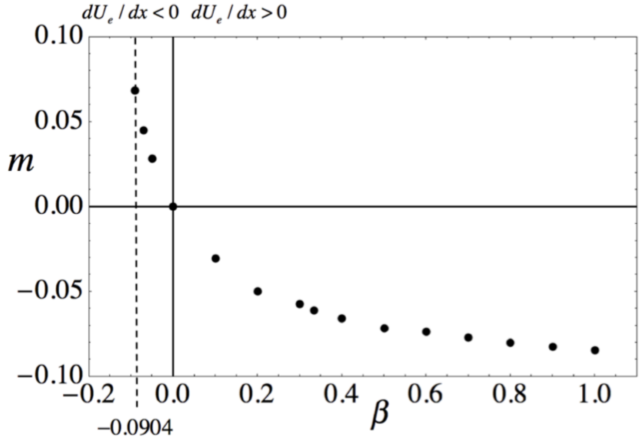


Figure 8.15: *The variable m defined in* (8.154) *versus the free stream velocity exponent*
for Falkner-Skan boundary layers.

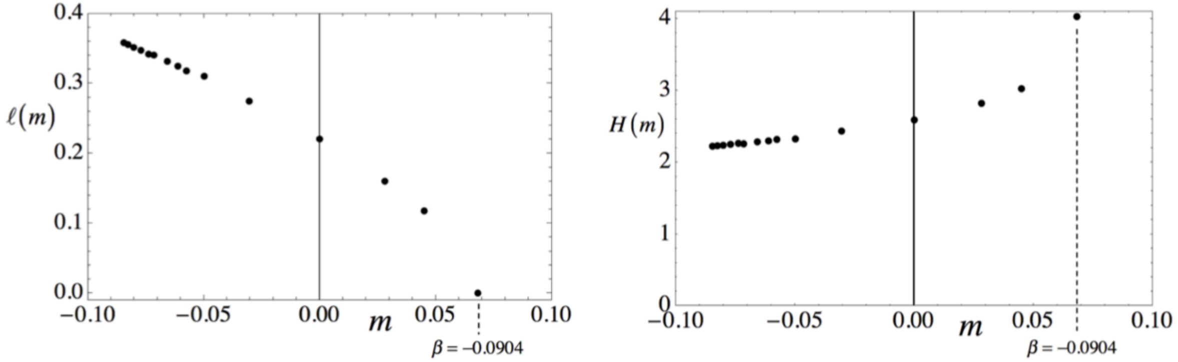
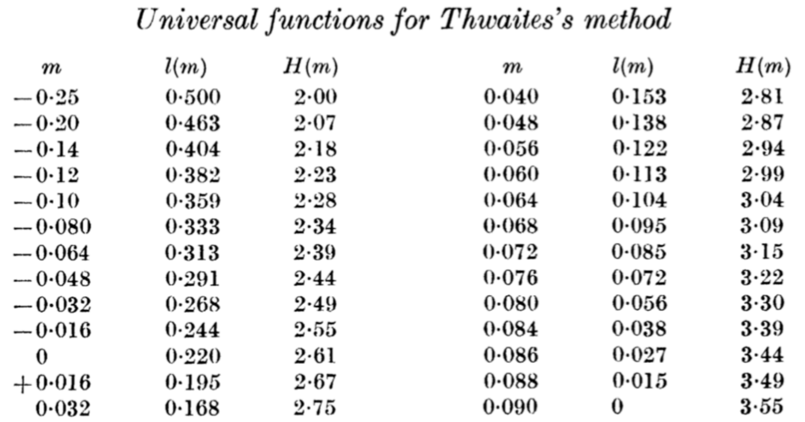


Figure 8.16: *Thwaites functions for the Falkner-Skan solutions* (8.157).

N. Curle adjusted Thwaites' functions slightly especially near separation.



Universal functions for Thwaites's method

m	$l(m)$	$H(m)$	m	$l(m)$	$H(m)$
−0·25	0·500	2·00	0·040	0·153	2·81
−0·20	0·463	2·07	0·048	0·138	2·87
−0·14	0·404	2·18	0·056	0·122	2·94
−0·12	0·382	2·23	0·060	0·113	2·99
−0·10	0·359	2·28	0·064	0·104	3·04
−0·080	0·333	2·34	0·068	0·095	3·09
−0·064	0·313	2·39	0·072	0·085	3·15
−0·048	0·291	2·44	0·076	0·072	3·22
−0·032	0·268	2·49	0·080	0·056	3·30
−0·016	0·244	2·55	0·084	0·038	3·39
0	0·220	2·61	0·086	0·027	3·44
+0·016	0·195	2·67	0·088	0·015	3·49
0·032	0·168	2·75	0·090	0	3·55

Figure 8.18: *Curle's functions for Thwaites' method.*

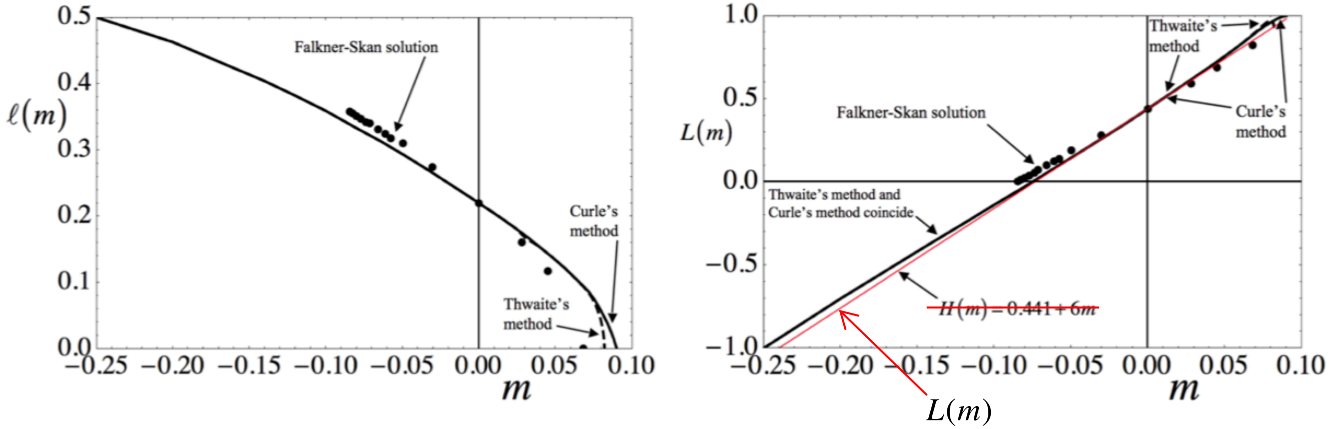


Figure 8.17: *Comparison between Curle's functions and Thwaite's functions.*

Several researchers of the era suggest using

$$L(m) = 0.441 + 6m\,.$$

which is consistent with the friction coefficient for the Blasius case

The von Karman equation becomes

$$U_e \frac{d}{dx}\left(\frac{\theta^2}{\nu}\right) = 0.441 - 6\left(\frac{\theta^2}{\nu}\right)\frac{dU_e}{dx}$$

which integrates to

$$\theta^2 = \frac{0.441\nu}{U_e^6}\int_0^x U_e(x')^5 dx'$$

The procedure for using Thwaite's method is as follows.

1) Use $U_e(x)$ to determine $\theta^2(x)$.

At a given x

2) The parameter m is determined from

$$m = -\frac{\theta^2}{\nu}\frac{dU_e}{dx}.$$

3) The functions $l(m)$ and $H(m)$ are determined from Curl's data.

4) The friction coefficient is determined from

$$C_f = \frac{2\nu}{U_e\theta}l(m).$$

5) The displacement thickness, $\delta^*(m)$, is determined from $H(m)$.

The process is repeated while progressing along the wall to increasing values of x. Separation of the boundary layer is assumed to have occurred if a point is reached where $l(m)=0$.

The key references used in this section are

1) Thwaites, B. 1948 Approximate calculations of the laminar boundary layer, VII International Congress of Applied Mechanics, London. Also Aeronautical Quarterly Vol. 1, page 245, 1949.

2) Curle, N. 1962 The *Laminar Boundary Layer Equations*, Clarendon Press.

Example - surface velocity from the potential flow about a circular cylinder.

$$\frac{U_e}{U_\infty} = 2Sin\left(\frac{x}{R}\right)$$

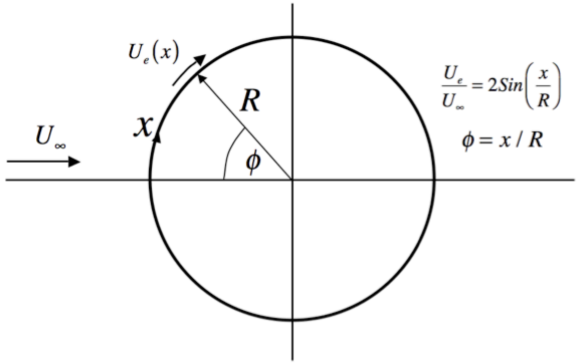


Figure 9.19 Example for Thwaites' method.

$$\left(\frac{\theta}{R}\right)^2 R_e = \frac{0.441}{Sin^6(\phi)}\int_0^{\phi} Sin^5(\phi')d\phi'$$

$$R_e = \frac{U_\infty 2R}{\nu}$$

Thwaites' method gives a finite momentum thickness at the forward stagnation point. This is useful in a wing leading edge calculation.

$$\lim_{\phi \to 0}\left(\frac{\theta}{R}\right)^2 R_e = \frac{0.441}{\phi^6}\int_0^{\phi} \phi'^5 d\phi' = \frac{0.441}{6}$$

The parameter m.

$$m = -\frac{\theta^2}{\nu}\frac{dU_e}{dx} = -\left(\frac{\theta}{R}\right)^2 R_e \frac{d}{d\phi}\left(\frac{U_e}{U_\infty}\right) = \frac{0.882 Cos(\phi)}{Sin^6(\phi)}\int_0^\phi Sin^5(\phi')d\phi'$$

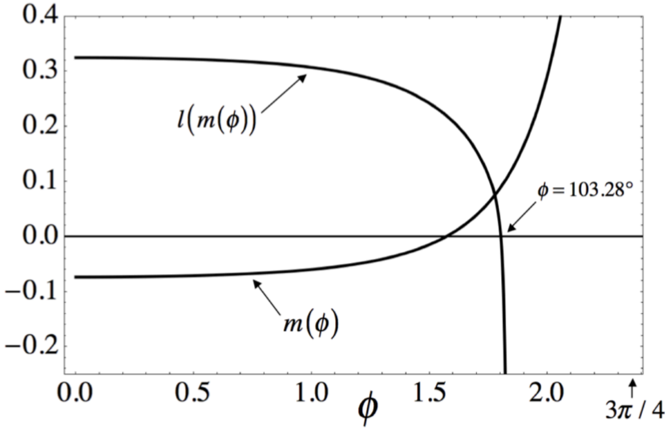


Figure 8.20: *Thwaites' functions for the freestream distribution* (8.163).

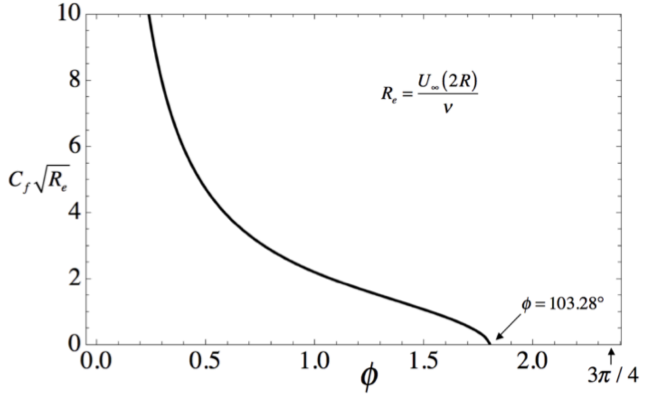


Figure 8.21: *Friction coefficient for the freestream distribution* (8.163).

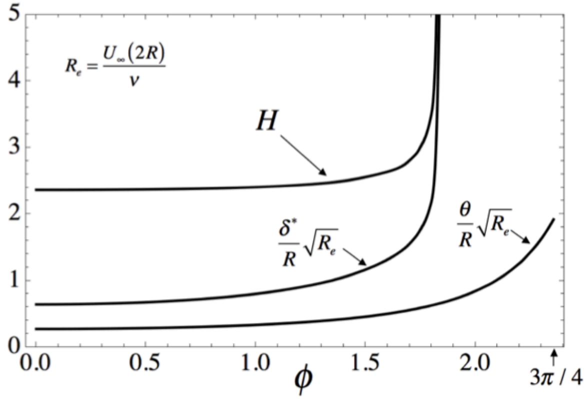


Figure 8.22: *Boundary layer thicknesses and shape factor for the freestream distribution* (8.163).

8.9 Compressible laminar boundary layers

The boundary layer admits an energy integral very similar to the one for Couette flow.

$$\frac{\partial \rho U}{\partial x} + \frac{\partial \rho V}{\partial y} = 0$$

$$\rho U \frac{\partial U}{\partial x} + \rho V \frac{\partial U}{\partial y} + \frac{dP_e}{dx} - \frac{\partial}{\partial y}\left(\mu \frac{\partial U}{\partial y}\right) = 0 \qquad (8.170)$$

$$\rho U C_p \frac{\partial T}{\partial x} + \rho V C_p \frac{\partial T}{\partial y} - U \frac{dP_e}{dx} - \frac{\partial}{\partial y}\left(\kappa \frac{\partial T}{\partial y}\right) - \mu \left(\frac{\partial U}{\partial y}\right)^2 = 0.$$

Let $T = T(U)$. Substitute into the energy equation.

Use the momentum equation to simplify. Introduce the Prandtl number

$$\left(\rho U\frac{\partial U}{\partial x}+\rho V\frac{\partial U}{\partial y}\right)C_p\frac{dT}{dU}-U\frac{dP_e}{dx}-\frac{\partial}{\partial y}\left(\kappa\frac{dT}{dU}\frac{\partial U}{\partial y}\right)-\mu\left(\frac{\partial U}{\partial y}\right)^2=0 \qquad (8.171)$$

Use the momentum equation to replace the factor in parentheses on the left hand side of (8.171)

$$\left(-\frac{dP_e}{dx}+\frac{\partial}{\partial y}\left(\mu\frac{\partial U}{\partial y}\right)\right)C_p\frac{dT}{dU}-U\frac{dP_e}{dx}-\frac{dT}{dU}\frac{\partial}{\partial y}\left(\kappa\frac{\partial U}{\partial y}\right)-\kappa\frac{d^2T}{dU^2}\left(\frac{\partial U}{\partial y}\right)^2-\mu\left(\frac{\partial U}{\partial y}\right)^2=0 \qquad (8.172)$$

which we can write as

$$-\frac{dP_e}{dx}\left(C_p\frac{dT}{dU}+U\right)+C_p\frac{dT}{dU}\frac{\partial}{\partial y}\left(\left(\mu-\frac{\kappa}{C_p}\right)\frac{\partial U}{\partial y}\right)+\left(\kappa\frac{d^2T}{dU^2}+\mu\right)\left(\frac{\partial U}{\partial y}\right)^2=0. \quad (8.173)$$

Introduce the Prandtl number (8.13) which can be assumed to be constant independent of position in the boundary layer. The energy equation becomes

$$-\frac{dP_e}{dx}\left(C_p\frac{dT}{dU}+U\right)+C_p\frac{dT}{dU}\left(\frac{P_r-1}{P_r}\right)\frac{\partial}{\partial y}\left(\mu\frac{\partial U}{\partial y}\right)+\left(\kappa\frac{d^2T}{dU^2}+\mu\right)\left(\frac{\partial U}{\partial y}\right)^2=0. \quad (8.174)$$

There are several important cases to consider.

Adiabatic wall, Prandtl number equals one.

$$T_{wa} - T = \frac{1}{2C_p}U^2$$

$$T_{wa} = T_e + \frac{1}{2C_p}U_e^{\ 2}$$

$$\frac{T_{wa}}{T_e} = 1 + \left(\frac{\gamma - 1}{2}\right)M_e^{\ 2} = \frac{T_{te}}{T_e}$$

Stagnation temperature is constant through the boundary layer.

Non-adiabatic wall, zero pressure gradient, Prandtl number equals one.

$$T_w = T_\infty + \frac{1}{2C_p}U_\infty^{\ 2} + \frac{Q_w}{\tau_w C_p}U_\infty$$

$$C_f = 2S_t$$

$$S_t = \frac{Q_w}{\rho_\infty U_\infty C_p(T_w - T_{wa})}$$

$$\frac{T - T_w}{T_\infty} = \left(1 - \frac{T_w}{T_\infty}\right)\frac{U}{U_\infty} + \left(\frac{U_\infty^2}{2C_pT_\infty}\right)\frac{U}{U_\infty}\left(1 - \frac{U}{U_\infty}\right)$$

8.10 Mapping a compressible to an incompressible boundary layer

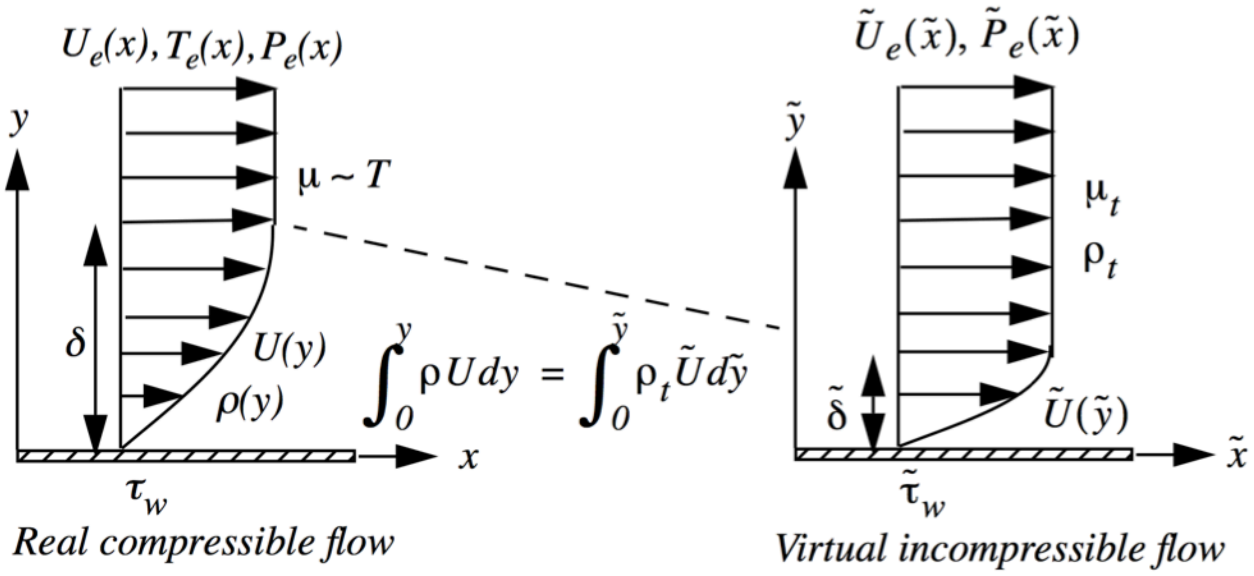


Figure 8.23: *Mapping of a compressible to an incompressible flow.*

Assume flow at the edge of the compressible boundary layer is isentropic.

$$\rho_t = \rho_e\left(1 + \left(\frac{\gamma - 1}{2}\right) M_e^2\right)^{1/(\gamma - 1)}$$

$$\frac{P_t}{P_e} = \left(\frac{T_t}{T_e}\right)^{\gamma/(\gamma - 1)} = \left(\frac{a_t}{a_e}\right)^{(2\gamma)/(\gamma - 1)}$$

Sutherland, W. (1893), "The viscosity of gases and molecular force", Philosophical Magazine, S. 5, 36, pp. 507-531 (1893).

Assume viscosity is linearly proportional to temperature

$$\frac{\mu}{\mu_t} = \sigma\frac{T}{T_t}$$

Viscosity of the virtual flow is the viscosity of the gas evaluated at the stagnation temperature of the gas.

Continuity and momentum equations

$$\frac{\partial \rho U}{\partial x} + \frac{\partial \rho V}{\partial y} = 0$$

$$U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} = -\frac{1}{\rho}\frac{dP_e}{dx} + \frac{1}{\rho}\frac{\partial}{\partial y}\left(\mu\frac{\partial U}{\partial y}\right) + \frac{1}{\rho}\frac{\partial \tau_{xy}}{\partial y}$$

Transformation of coordinates between the real and virtual flow

$$\tilde{x} = \sigma\int_0^x\left(\frac{P_e}{P_t}\left(\frac{a_e}{a_t}\right)\right)dx' = f(x)$$

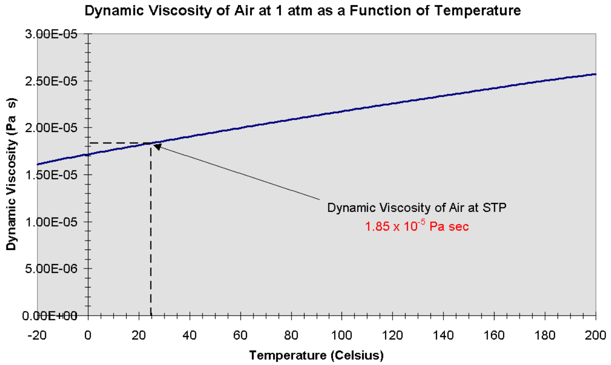
$$\tilde{y} = \left(\frac{a_e}{a_t}\right)\int_0^y\left(\frac{\rho(x, y')}{\rho_t}\right)dy' = g(x, y)$$

$$\frac{\mu}{\mu_t} = \left(\frac{T}{T_t}\right)^{3/2}\left(\frac{T_t + T_S}{T + T_S}\right)$$

$$\sigma = \left(\frac{T_w}{T_t}\right)^{1/2}\left(\frac{T_t + T_S}{T_w + T_S}\right)$$

If $P_r = 1$ then $\sigma = 1$

T_S - Sutherland reference temperature, 110K for Air.



Partial derivatives

$$\frac{\partial\tilde{x}}{\partial x} = f_x = \sigma\left(\frac{P_e}{P_t}\left(\frac{a_e}{a_t}\right)\right)$$

$$\frac{\partial\tilde{x}}{\partial y} = f_y = 0$$

$$\frac{\partial\tilde{y}}{\partial x} = g_x$$

= ??

$$\frac{\partial\tilde{y}}{\partial y} = g_y = \left(\frac{a_e}{a_t}\right)\left(\frac{\rho}{\rho_t}\right)$$

Introduce the stream function for steady compressible flow. Let

$$\rho U = \rho_t\frac{\partial\psi}{\partial y} \qquad \rho V = -\rho_t\frac{\partial\psi}{\partial x}$$

Real and virtual stream functions have the same value.

$$\psi(x, y) = \tilde{\psi}(\tilde{x}(x), \tilde{y}(x, y))$$

Partial derivatives of the stream function from the chain rule.

$$\frac{\partial \psi}{\partial x} = \frac{\partial \tilde{\psi}}{\partial \tilde{x}}\frac{\partial \tilde{x}}{\partial x} + \frac{\partial \tilde{\psi}}{\partial \tilde{y}}\frac{\partial \tilde{y}}{\partial x} = \frac{\partial \tilde{\psi}}{\partial \tilde{x}}\frac{d\tilde{x}}{dx} + \frac{\partial \tilde{\psi}}{\partial \tilde{y}}\frac{\partial \tilde{y}}{\partial x}$$

$$\frac{\partial \psi}{\partial y} = \frac{\partial \tilde{\psi}}{\partial \tilde{x}}\cancel{\frac{\partial \tilde{x}}{\partial y}} + \frac{\partial \tilde{\psi}}{\partial \tilde{y}}\frac{\partial \tilde{y}}{\partial y} = \frac{\partial \tilde{\psi}}{\partial \tilde{y}}\frac{\partial \tilde{y}}{\partial y}$$

Velocities

$$U = \frac{\rho_t}{\rho}\frac{\partial \psi}{\partial y} = \left(\frac{a_e}{a_t}\right)\frac{\partial \tilde{\psi}}{\partial \tilde{y}} = \left(\frac{a_e}{a_t}\right)\tilde{U}$$

$$V = -\frac{\rho_t}{\rho}\frac{\partial \psi}{\partial x} = -\sigma\frac{\rho_t}{\rho}\left(\frac{P_e}{P_t}\frac{a_e}{a_t}\right)\frac{\partial \tilde{\psi}}{\partial \tilde{x}} - \frac{\rho_t}{\rho}\frac{\partial \tilde{\psi}}{\partial \tilde{y}}\frac{\partial \tilde{y}}{\partial x}$$

Partial derivatives of U from the chain rule.

$$\frac{\partial U}{\partial x} = \frac{\partial}{\partial \tilde{x}}\left(\left(\frac{a_e}{a_t}\right)\frac{\partial \tilde{\psi}}{\partial \tilde{y}}\right)\frac{d\tilde{x}}{dx} + \frac{\partial}{\partial \tilde{y}}\left(\left(\frac{a_e}{a_t}\right)\frac{\partial \tilde{\psi}}{\partial \tilde{y}}\right)\frac{\partial \tilde{y}}{\partial x} =$$

$$\sigma\left(\frac{P_e}{P_t}\left(\frac{a_e}{a_t}\right)^2\right)\left(\left(\frac{1}{a_e}\right)\frac{\partial a_e}{\partial \tilde{x}}\frac{\partial \tilde{\psi}}{\partial \tilde{y}} + \frac{\partial^2 \tilde{\psi}}{\partial \tilde{x}\partial \tilde{y}}\right) + \left(\frac{a_e}{a_t}\right)\frac{\partial^2 \tilde{\psi}}{\partial \tilde{y}^2}\frac{\partial \tilde{y}}{\partial x}$$

$$\frac{\partial U}{\partial y} = \frac{\partial}{\partial \tilde{x}}\left(\left(\frac{a_e}{a_t}\right)\frac{\partial \tilde{\psi}}{\partial \tilde{y}}\right)\cancel{\frac{\partial \tilde{x}}{\partial y}} + \frac{\partial}{\partial \tilde{y}}\left(\left(\frac{a_e}{a_t}\right)\frac{\partial \tilde{\psi}}{\partial \tilde{y}}\right)\frac{\partial \tilde{y}}{\partial y} = \left(\frac{a_e}{a_t}\right)^2\left(\frac{\rho}{\rho_t}\right)\left(\frac{\partial^2 \tilde{\psi}}{\partial \tilde{y}^2}\right)$$

Convective terms of the momentum equation

$$U\frac{\partial U}{\partial x}+V\frac{\partial U}{\partial y}=$$

$$\sigma\left(\frac{P_e}{P_t}\left(\frac{a_e}{a_t}\right)^3\right)\left(\left(\frac{1}{a_e}\right)\frac{\partial a_e}{\partial\tilde{x}}\left(\frac{\partial\tilde{\psi}}{\partial\tilde{y}}\right)^2+\frac{\partial\tilde{\psi}}{\partial\tilde{y}}\frac{\partial^2\tilde{\psi}}{\partial\tilde{x}\partial\tilde{y}}\right)+\underline{\left(\frac{a_e}{a_t}\right)^2\frac{\partial\tilde{\psi}}{\partial\tilde{y}}\left(\frac{\partial^2\tilde{\psi}}{\partial\tilde{y}^2}\frac{\partial\tilde{y}}{\partial x}\right)}-$$

$$\left((\sigma)\left(\frac{P_e a_e}{P_t a_t}\right)\left(\frac{a_e}{a_t}\right)^2\left(\frac{\partial^2\tilde{\psi}}{\partial\tilde{y}^2}\right)\frac{\partial\tilde{\psi}}{\partial\tilde{x}}+\underline{\left(\frac{a_e}{a_t}\right)^2\left(\frac{\partial^2\tilde{\psi}}{\partial\tilde{y}^2}\right)\frac{\partial\tilde{\psi}}{\partial\tilde{y}}\frac{\partial\tilde{y}}{\partial x}}\right)$$

Cancel terms

$$U\frac{\partial U}{\partial x}+V\frac{\partial U}{\partial y}=$$

$$\sigma\left(\frac{P_e}{P_t}\left(\frac{a_e}{a_t}\right)^3\right)\left(\left(\frac{1}{a_e}\right)\frac{\partial a_e}{\partial\tilde{x}}\left(\frac{\partial\tilde{\psi}}{\partial\tilde{y}}\right)^2+\frac{\partial\tilde{\psi}}{\partial\tilde{y}}\frac{\partial^2\tilde{\psi}}{\partial\tilde{x}\partial\tilde{y}}-\left(\frac{\partial^2\tilde{\psi}}{\partial\tilde{y}^2}\right)\frac{\partial\tilde{\psi}}{\partial\tilde{x}}\right)$$

Pressure gradient term

$$\left(\frac{P_e}{P_t}\right) = \left(\frac{a_e}{a_t}\right)^{\frac{2\gamma}{(\gamma-1)}}$$

$$\frac{dP_e}{dx} = \frac{2\gamma P_t}{(\gamma-1)}\left(\frac{a_e}{a_t}\right)^{\frac{\gamma+1}{\gamma-1}}\frac{1}{a_t}\frac{da_e}{d\tilde{x}}\frac{d\tilde{x}}{dx} = \sigma\left(\frac{P_e}{P_t}\left(\frac{a_e}{a_t}\right)\right)\frac{2\gamma P_t}{(\gamma-1)}\left(\frac{a_e}{a_t}\right)^{\frac{\gamma+1}{\gamma-1}}\frac{1}{a_t}\frac{da_e}{d\tilde{x}}$$

$$\frac{dP_e}{dx} = \sigma\left(\frac{P_e}{P_t}\left(\frac{a_e}{a_t}\right)^2\right)\left(\frac{2\gamma P_t}{(\gamma-1)}\left(\frac{a_e}{a_t}\right)^{\frac{\gamma+1}{\gamma-1}}\frac{1}{a_e}\frac{da_e}{d\tilde{x}}\right)$$

Now

$$U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} + \frac{1}{\rho}\frac{dP_e}{dx} =$$

$$\sigma\left(\frac{P_e}{P_t}\left(\frac{a_e}{a_t}\right)^3\right)\left(\frac{\partial\tilde{\psi}}{\partial\tilde{y}}\frac{\partial^2\tilde{\psi}}{\partial\tilde{x}\partial\tilde{y}} - \left(\frac{\partial^2\tilde{\psi}}{\partial\tilde{y}^2}\right)\frac{\partial\tilde{\psi}}{\partial\tilde{x}}\right) +$$

$$\left(\frac{\partial\tilde{\psi}}{\partial\tilde{y}}\right)^2 = U^2\left(\frac{a_t}{a_e}\right)^2$$

$$\sigma\left(\frac{P_e}{P_t}\left(\frac{a_e}{a_t}\right)^3\right)\left(U^2\left(\frac{a_t}{a_e}\right)^2 + \frac{1}{\rho}\frac{2\gamma P_t}{(\gamma-1)}\left(\frac{a_e}{a_t}\right)^{\frac{2}{\gamma-1}}\right)\left(\left(\frac{1}{a_e}\right)\frac{da_e}{d\tilde{x}}\right)$$

At the edge of the boundary layer

$$a_t^2 = a_e^2 + \left(\frac{\gamma - 1}{2}\right) U_e^{\,2} \qquad \frac{1}{a_e}\frac{da_e}{d\tilde{x}} = -\left(\frac{\gamma - 1}{2a_e^2}\right) U_e \frac{dU_e}{d\tilde{x}}$$

Now

$$U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} + \frac{1}{\rho}\frac{dP_e}{dx} =$$

$$\sigma\left(\frac{P_e}{P_t}\left(\frac{a_e}{a_t}\right)^3\right)\left(\frac{\partial\tilde{\psi}}{\partial\tilde{y}}\frac{\partial^2\tilde{\psi}}{\partial\tilde{x}\partial\tilde{y}} - \left(\frac{\partial^2\tilde{\psi}}{\partial\tilde{y}^2}\right)\frac{\partial\tilde{\psi}}{\partial\tilde{x}}\right) -$$

$$\sigma\left(\frac{P_e}{P_t}\left(\frac{a_e}{a_t}\right)^3\right)\left(U^2\frac{(\gamma - 1)}{2a_e^2}\left(\frac{a_t}{a_e}\right)^2 + \frac{1}{\rho}\frac{\gamma P_t}{a_e^2}\left(\frac{a_e}{a_t}\right)^{\frac{2}{\gamma - 1}}\right)\left(U_e\frac{dU_e}{d\tilde{x}}\right)$$

Note that

$$U^2\frac{(\gamma-1)}{2a_e^2}\left(\frac{a_t}{a_e}\right)^2+\frac{1}{\rho}\frac{\gamma P_t}{a_e^2}\left(\frac{a_e}{a_t}\right)^{\frac{2}{\gamma-1}} =$$

$$U^2\frac{(\gamma-1)}{2a_e^2}\left(\frac{a_t}{a_e}\right)^2+\frac{\rho_e}{\rho}\frac{\gamma P_e}{\rho_e}\frac{1}{a_e^2}\left(\frac{a_t}{a_e}\right)^{\frac{2\gamma}{\gamma-1}}\left(\frac{a_e}{a_t}\right)^{\frac{2}{\gamma-1}} =$$

$$\left(\frac{a_t}{a_e}\right)^4\left(\frac{a^2+\frac{(\gamma-1)}{2}U^2}{a_t^2}\right)$$

Where we have used

$$P_t/P_e = (a_t/a_e)^{2\gamma/(\gamma-1)} \qquad \gamma P_e/\rho = (\gamma P)/\rho = a^2$$

Viscous term. Note $\rho T = P/R = P_e/R$

$$\tau_{xy}\big|_{laminar} = \mu\frac{\partial U}{\partial y} = \sigma\mu_t\left(\frac{T}{T_t}\right)\frac{\partial}{\partial y}\left(\left(\frac{a_e}{a_t}\right)\frac{\partial\tilde{\psi}}{\partial\tilde{y}}\right) = \sigma\mu_t\left(\frac{T}{T_t}\right)\left(\frac{a_e}{a_t}\right)\frac{\partial}{\partial y}\left(\frac{\partial\tilde{\psi}}{\partial\tilde{y}}\right) =$$

$$\sigma\mu_t\left(\frac{a_e}{a_t}\right)^2\left(\frac{T}{T_t}\right)\left(\frac{\rho}{\rho_t}\right)\frac{\partial}{\partial\tilde{y}}\left(\frac{\partial\tilde{\psi}}{\partial\tilde{y}}\right) = \sigma\left(\frac{a_e}{a_t}\right)^2\left(\frac{\rho T}{\rho_t T_t}\right)\left(\mu_t\frac{\partial\tilde{U}}{\partial\tilde{y}}\right) = \sigma\left(\frac{a_e}{a_t}\right)^2\left(\frac{P_e}{P_t}\right)\tilde{\tau}_{\tilde{x}\tilde{y}}\big|_{laminar}$$

$$\frac{1}{\rho}\frac{\partial}{\partial y}\left(\mu\frac{\partial U}{\partial y}\right) = \sigma\frac{\mu_t}{\rho_t}\frac{P_e}{P_t}\left(\frac{a_e}{a_t}\right)^3\left(\frac{\partial^3\tilde{\psi}}{\partial\tilde{y}^3}\right)$$

$$\frac{1}{\rho}\frac{\partial}{\partial y}\left(\tau_{xy}\big|_{turbulent}\right) = \sigma\frac{1}{\rho_t}\frac{P_e}{P_t}\left(\frac{a_e}{a_t}\right)^3\frac{\partial}{\partial\tilde{y}}\left(\tilde{\tau}_{xy}\big|_{turbulent}\right)$$

The boundary layer momentum equation becomes

$$U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} + \frac{1}{\rho}\frac{dP_e}{dx} - \frac{1}{\rho}\frac{\partial}{\partial y}\left(\mu\frac{\partial U}{\partial y}\right) - \frac{1}{\rho}\frac{\partial \tau_{xy}}{\partial y} =$$

$$\sigma\left(\frac{P_e}{P_t}\left(\frac{a_e}{a_t}\right)^3\right)\left(\frac{\partial\tilde{\psi}}{\partial\tilde{y}}\frac{\partial^2\tilde{\psi}}{\partial\tilde{x}\partial\tilde{y}} - \left(\frac{\partial^2\tilde{\psi}}{\partial\tilde{y}^2}\right)\frac{\partial\tilde{\psi}}{\partial\tilde{x}}\right) -$$

$$\sigma\left(\frac{P_e}{P_t}\left(\frac{a_e}{a_t}\right)^3\right)\left(\left(\frac{a_t}{a_e}\right)^4\left(\frac{a^2 + \frac{(\gamma - 1)}{2}U^2}{a_t^2}\right)\right)\left(U_e\frac{dU_e}{d\tilde{x}}\right) -$$

$$\sigma\frac{\mu_t}{\rho_t}\frac{P_e}{P_t}\left(\frac{a_e}{a_t}\right)^3\left(\frac{\partial^3\tilde{\psi}}{\partial\tilde{y}^3}\right) - \sigma\frac{1}{\rho_t}\frac{P_e}{P_t}\left(\frac{a_e}{a_t}\right)^3\frac{\partial}{\partial\tilde{y}}\left(\tilde{\tau}_{xy}\Big|_{turbulent}\right) = 0$$

Drop the common multiplying factors

$$\left(\frac{\partial\tilde{\psi}}{\partial\tilde{y}}\frac{\partial^2\tilde{\psi}}{\partial\tilde{x}\partial\tilde{y}} - \left(\frac{\partial^2\tilde{\psi}}{\partial\tilde{y}^2}\right)\frac{\partial\tilde{\psi}}{\partial\tilde{x}}\right) - \left(\frac{a_t}{a_e}\right)^4\left(\frac{a^2 + \frac{(\gamma - 1)}{2}U^2}{a_t^2}\right)U_e\frac{dU_e}{d\tilde{x}} -$$

$$\frac{\mu_t}{\rho_t}\left(\frac{\partial^3\tilde{\psi}}{\partial\tilde{y}^3}\right) - \frac{1}{\rho_t}\frac{\partial}{\partial\tilde{y}}(\tilde{\tau}_{xy}|_{turbulent}) = 0$$

$$\tilde{U}_e = \frac{a_t}{a_e}U_e$$

$$\tilde{U}_e\frac{d\tilde{U}_e}{d\tilde{x}} = \left(\frac{a_t}{a_e}\right)^2\left(\frac{1}{a_e^2}\right)\left(a_e^2 + \left(\frac{\gamma - 1}{2}\right)U_e^2\right)U_e\frac{dU_e}{d\tilde{x}} =$$

$$\left(\frac{a_t}{a_e}\right)^4 U_e\frac{dU_e}{d\tilde{x}}$$

Now the momentum equation is expressed entirely in tildaed variables.

$$\left(\frac{\partial\tilde{\psi}}{\partial\tilde{y}}\frac{\partial^2\tilde{\psi}}{\partial\tilde{x}\partial\tilde{y}} - \left(\frac{\partial^2\tilde{\psi}}{\partial\tilde{y}^2}\right)\frac{\partial\tilde{\psi}}{\partial\tilde{x}}\right) - \left(\frac{a^2 + \frac{(\gamma-1)}{2}U^2}{a_t^2}\right)\tilde{U}_e\frac{d\tilde{U}_e}{d\tilde{x}} -$$

$$\frac{\mu_t}{\rho_t}\left(\frac{\partial^3\tilde{\psi}}{\partial\tilde{y}^3}\right) - \frac{1}{\rho_t}\frac{\partial}{\partial\tilde{y}}\left(\tilde{\tau}_{xy}\big|_{turbulent}\right) = 0$$

$$\tilde{U} = \frac{\partial\tilde{\psi}}{\partial\tilde{y}} \qquad \tilde{V} = -\frac{\partial\tilde{\psi}}{\partial x}$$

$$\left(\tilde{U}\frac{\partial\tilde{U}}{\partial\tilde{x}} + \left(\frac{\partial\tilde{U}}{\partial\tilde{y}}\right)\tilde{V}\right) - \left(\frac{a^2 + \frac{(\gamma-1)}{2}U^2}{a_e^2 + \frac{(\gamma-1)}{2}U_e^2}\right)\tilde{U}_e\frac{d\tilde{U}_e}{d\tilde{x}} -$$

$$\nu_t\left(\frac{\partial^2\tilde{U}}{\partial\tilde{y}^2}\right) - \frac{1}{\rho_t}\frac{\partial}{\partial\tilde{y}}\left(\tilde{\tau}_{xy}\big|_{turbulent}\right) = 0$$

For an adiabatic wall, and a Prandtl number of one the factor in brackets is one and the equation maps exactly to the incompressible form.

$$\left(\tilde{U}\frac{\partial \tilde{U}}{\partial \tilde{x}} + \left(\frac{\partial \tilde{U}}{\partial \tilde{y}}\right)\tilde{V}\right) - \tilde{U}_e\frac{d\tilde{U}_e}{d\tilde{x}} - \nu_t\left(\frac{\partial^2 \tilde{U}}{\partial \tilde{y}^2}\right) - \frac{1}{\rho_t}\frac{\partial}{\partial \tilde{y}}\left(\tilde{\tau}_{xy}\big|_{turbulent}\right) = 0$$

with boundary conditions

$$\tilde{U}(0) = 0 \qquad \tilde{V}(0) = 0 \qquad \tilde{U}(\tilde{\delta}) = \tilde{U}_e$$

Skin friction

$$\tilde{C}_f = \frac{\tilde{\tau}_w}{(1/2)\rho_t\tilde{U}_e^2} = \frac{\frac{1}{\sigma}\left(\left(\frac{a_t}{a_e}\right)^2\frac{P_t}{P_e}\right)\tau_w}{(1/2)\left(\frac{\rho_t}{\rho_e}\right)\rho_e\left(\frac{a_t}{a_e}\right)^2 U_e^2} = \frac{1}{\sigma}\frac{\rho_e P_t}{\rho_t P_e}\left(\frac{\tau_w}{(1/2)\rho_e U_e^2}\right) = \frac{1}{\sigma}\frac{T_t}{T_e}C_f$$

$$\frac{C_f}{\tilde{C}_f} = \frac{1}{1 + \left(\frac{\gamma - 1}{2}\right)M_e^2}$$

For an adiabatic wall and $P_r = 1$ the factor in brackets is equal to one. In this case the momentum equation maps exactly to the incompressible form

$$\left(\tilde{U}\frac{\partial \tilde{U}}{\partial \tilde{x}} + \tilde{V}\left(\frac{\partial \tilde{U}}{\partial \tilde{y}}\right)\right) - \tilde{U}_e\frac{d\tilde{U}_e}{d\tilde{x}} - \nu_t\left(\frac{\partial^2 \tilde{U}}{\partial \tilde{y}^2}\right) - \frac{1}{\rho_t}\frac{\partial}{\partial \tilde{y}}\left(\tilde{\tau}_{\tilde{x}\tilde{y}}|_{turbulent}\right) = 0 \tag{8.228}$$

with boundary conditions

$$\begin{aligned} &\tilde{U}\left(0\right) = 0 \\ &\tilde{V}\left(0\right) = 0 \\ &\tilde{U}\left(\tilde{\delta}\right) = \tilde{U}_e. \end{aligned} \tag{8.229}$$

The implication of (8.228) and (8.229) is that the effects of compressibility on the boundary layer can be almost completely accounted for by the scaling of coordinates presented in (8.197) which is driven in the y direction by the decrease in density near the wall due to heating and in the x direction by the isentropic changes in free stream temperature and boundary layer pressure due to flow acceleration or deceleration imposed by the surrounding potential flow.

In order to solve for the physical velocity profiles we need to determine the temperature in the boundary layer. Look at the case

$$dU_e/dx = 0 \qquad \tau_{xy}\Big|_{turbulent} = 0$$

The energy equation was integrated earlier $T = T_t - \frac{1}{2C_p}U^2$

$$\frac{T}{T_e} = 1 + \left(\frac{\gamma - 1}{2}\right)M_e^2\left(1 - \left(\frac{U}{U_e}\right)^2\right)$$

Use $U/U_e = \tilde{U}/\tilde{U}_e$ and $\rho T = \rho_e T_e$

$$\frac{T}{T_e} = 1 + \left(\frac{\gamma - 1}{2}\right)M_e^2\left(1 - \left(\frac{\tilde{U}}{\tilde{U}_e}\right)^2\right) = \frac{\rho_e}{\rho}$$

We need to relate wall normal coordinates in the real and virtual flow

$$dy = \left(\frac{a_t}{a_e}\right)\left(\frac{\rho_t}{\rho_e}\right)\left(\frac{\rho_e}{\rho}\right)d\tilde{y} = \left(\frac{a_t}{a_e}\right)\left(\frac{\rho_t}{\rho_e}\right)\left(1 + \left(\frac{\gamma - 1}{2}\right)M_e^2\left(1 - \left(\frac{\tilde{U}}{\tilde{U}_e}\right)^2\right)\right)d\tilde{y}$$

The spatial similarity variable in the virtual flow is

$$\tilde{\alpha} = \tilde{y}\left(\frac{\tilde{U}_e}{2\nu_t\tilde{x}}\right)^{1/2}$$

$$dy = \left(\frac{a_t}{a_e}\right)\left(\frac{\rho_t}{\rho_e}\right)\left(\frac{2\nu_t\tilde{x}}{\tilde{U}_e}\right)^{1/2}\left(1+\left(\frac{\gamma-1}{2}\right)M_e^2\left(1-\left(\frac{\tilde{U}}{\tilde{U}_e}\right)^2\right)\right)d\left(\tilde{y}\left(\frac{\tilde{U}_e}{2\nu_t\tilde{x}}\right)^{1/2}\right)$$

$$d\left(y\left(\frac{U_e}{2\nu_e x}\right)^{1/2}\right) =$$

$$\left(\frac{a_t}{a_e}\right)\left(\frac{\rho_t}{\rho_e}\right)\left(\frac{2\nu_t\tilde{x}}{\tilde{U}_e}\right)^{1/2}\left(\frac{U_e}{2\nu_e x}\right)^{1/2}\left(1+\left(\frac{\gamma-1}{2}\right)M_e^2\left(1-\left(\frac{\tilde{U}}{\tilde{U}_e}\right)^2\right)\right)d\left(\tilde{y}\left(\frac{\tilde{U}_e}{2\nu_t\tilde{x}}\right)^{1/2}\right)$$

$$\left(\frac{a_t}{a_e}\right)\left(\frac{\rho_t}{\rho_e}\right)\left(\frac{2\nu_t\tilde{x}}{\tilde{U}_e}\frac{U_e}{2\nu_e x}\right)^{1/2} = \left(\frac{a_t}{a_e}\right)\left(\frac{\rho_t}{\rho_e}\right)\left(\frac{\mu_t}{\mu_e}\frac{\rho_e}{\rho_t}\frac{U_e}{\tilde{U}_e}\frac{\tilde{x}}{x}\right)^{1/2} =$$

$$\left(\frac{a_t}{a_e}\right)\left(\frac{\rho_t}{\rho_e}\right)\left(\frac{T_t}{T_e}\frac{\rho_e}{\rho_t}\frac{a_e}{a_t}\frac{P_e}{P_t}\frac{a_e}{a_t}\right)^{1/2} = \left(\frac{\rho_t}{\rho_e}\right)\left(\frac{T_t}{T_e}\frac{\rho_e}{\rho_t}\frac{\rho_e}{\rho_t}\frac{T_e}{T_t}\right)^{1/2} = 1$$

Spatial similarity variables in the two flows are related by

$$d\alpha = \left(1 + \left(\frac{\gamma - 1}{2}\right) M_e^2 \left(1 - \left(\frac{\tilde{U}}{\tilde{U}_e}\right)^2\right)\right) d\tilde{\alpha}$$

$$\alpha(\tilde{\alpha}) = \tilde{\alpha} + \left(\frac{\gamma - 1}{2}\right) M_e^2 \int_0^{\tilde{\alpha}} \left(1 - \left(\frac{\tilde{U}}{\tilde{U}_e}\right)^2\right) d\tilde{\alpha}'$$

$$\tilde{\alpha}_e = 4.906 / \sqrt{2} = 3.469$$

$$\alpha_e = \tilde{\alpha}_e + \left(\frac{\gamma - 1}{2}\right) M_e^2 \int_0^{\tilde{\alpha}_e} \left(1 - \left(\frac{\tilde{U}}{\tilde{U}_e}\right)^2\right) d\tilde{\alpha} =$$

$$\alpha_e = 3.469 + 1.67912 \left(\frac{\gamma - 1}{2}\right) M_e^2$$

The thickness of the compressible layer increases with Mach number.

Now

$$\frac{T(\alpha(\tilde{\alpha}))}{T_e} = \frac{\rho_e}{\rho(\alpha(\tilde{\alpha}))} = 1 + \left(\frac{\gamma - 1}{2}\right) M_e^2 \left(1 - \left(\frac{\tilde{U}(\tilde{\alpha})}{\tilde{U}_e}\right)^2\right)$$

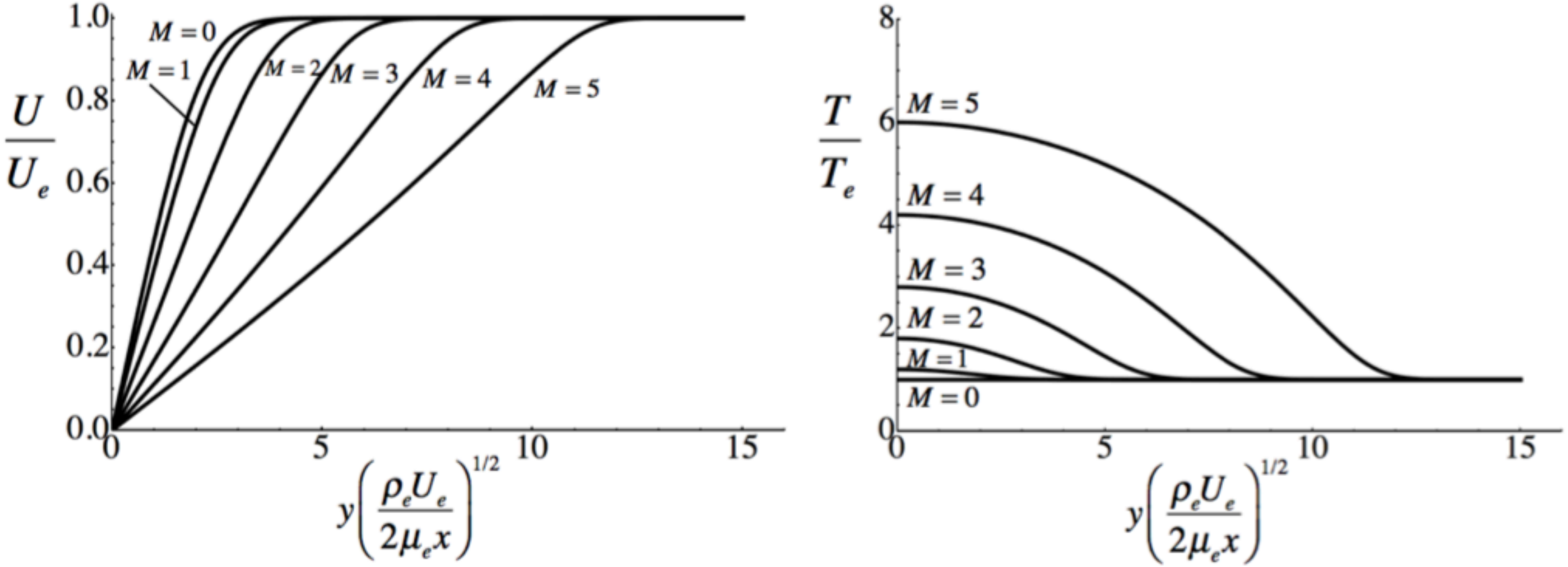


Figure 8.24: *Compressible boundary layer profiles on an adiabatic plate for* $P_r = 1$, *viscosity exponent* $\omega = 1$ *and* $\gamma = 1.4$.

8.11 Turbulent boundary layers

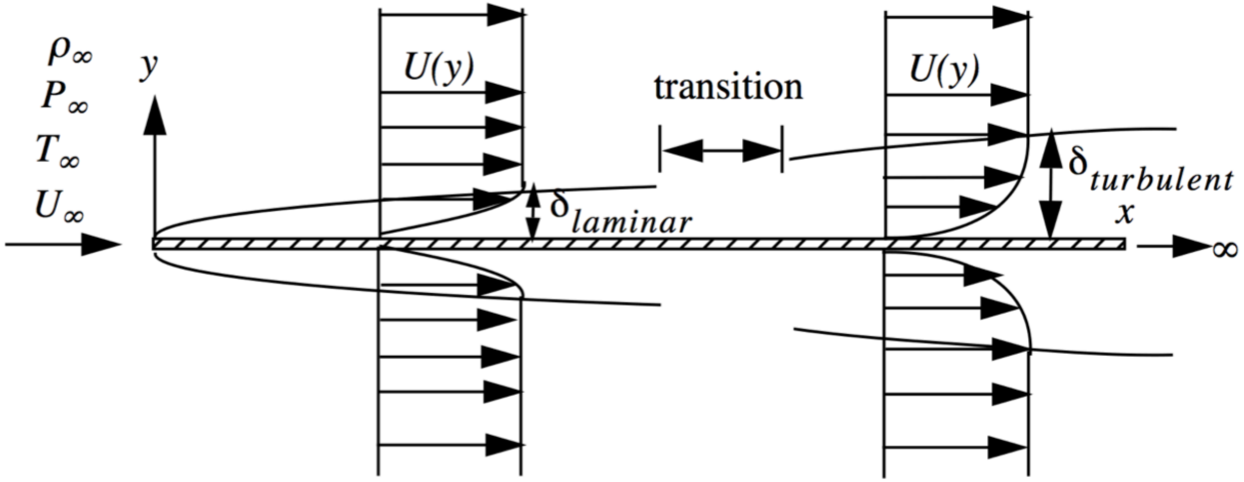


Figure 8.25: *Sketch of boundary layer growth in the laminar and turbulent regions.*

Impirical relations for the thickness of the incompressible case, useful over a limited range of Reynolds number.

$$\frac{\delta}{x} = \frac{0.37}{R_{ex}^{1/5}}$$

Or for a wider range of Reynolds number

$$\frac{\delta}{x} = \frac{0.14}{ln(R_{ex})} G(Ln(R_{ex}))$$

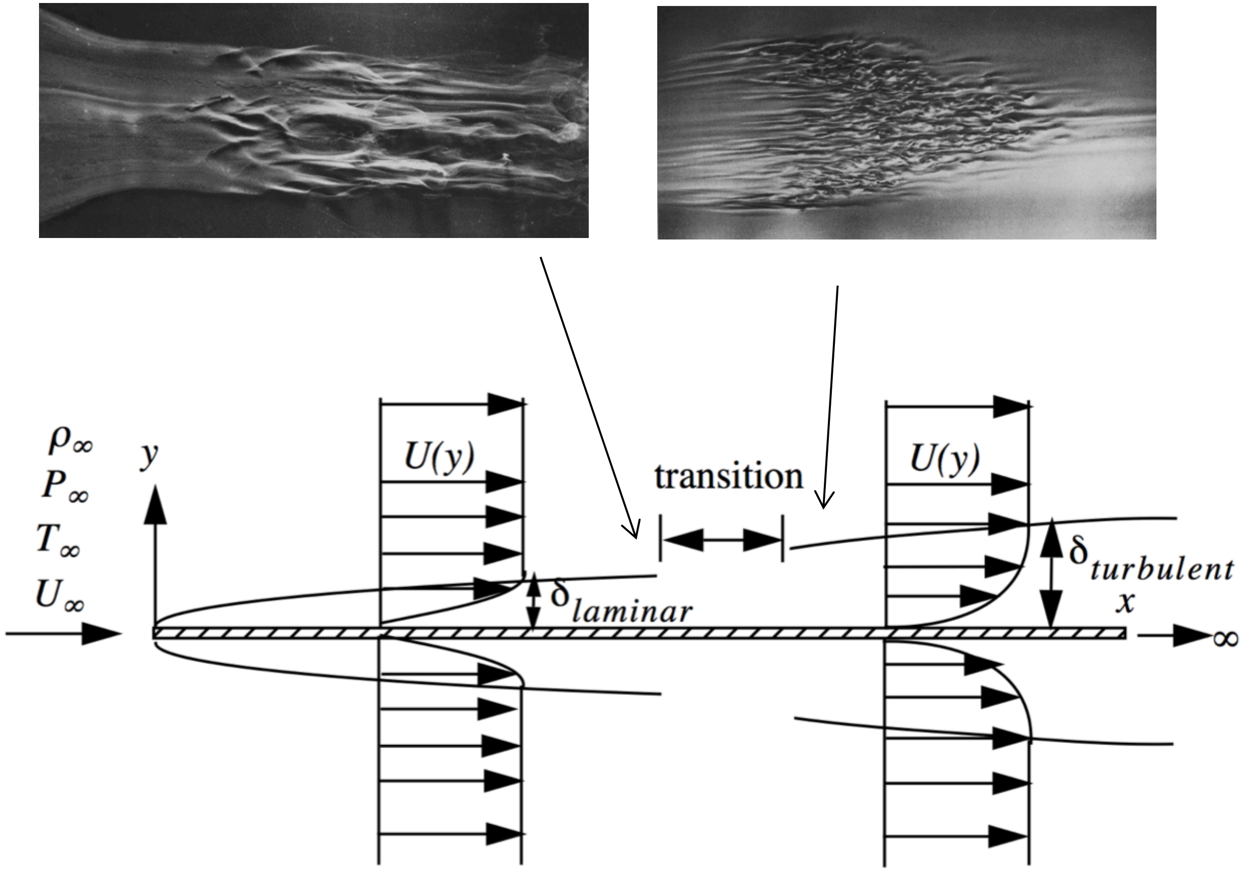
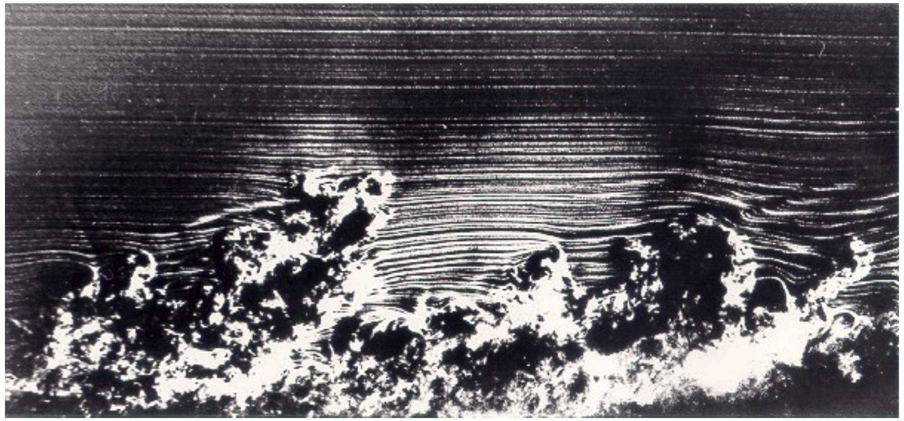
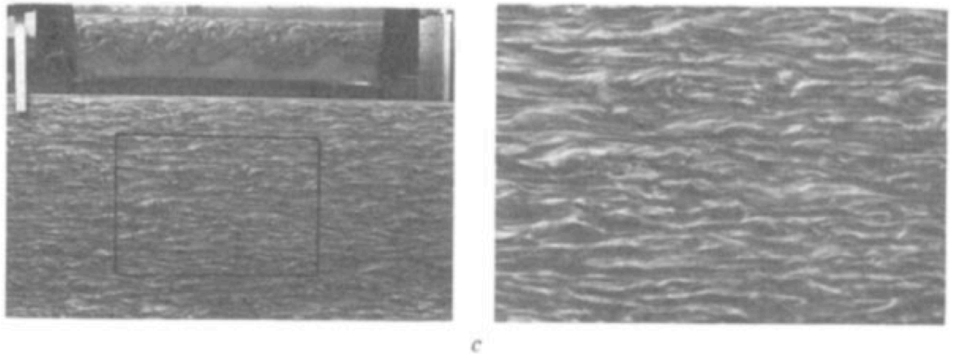


Figure 8.25: *Sketch of boundary layer growth in the laminar and turbulent regions.*

Turbulent boundary layer visualization

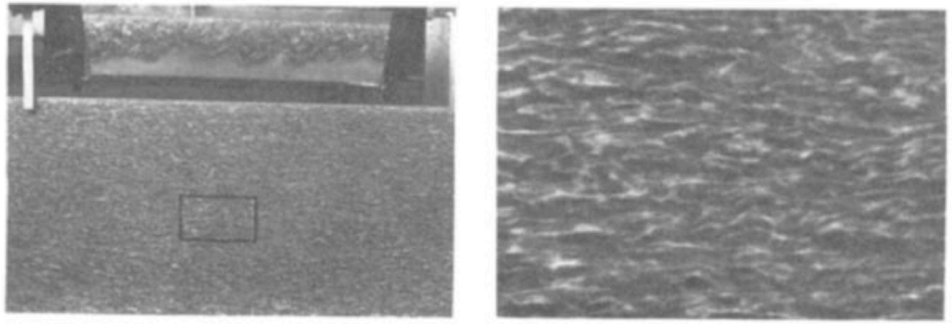


side view – light sheet



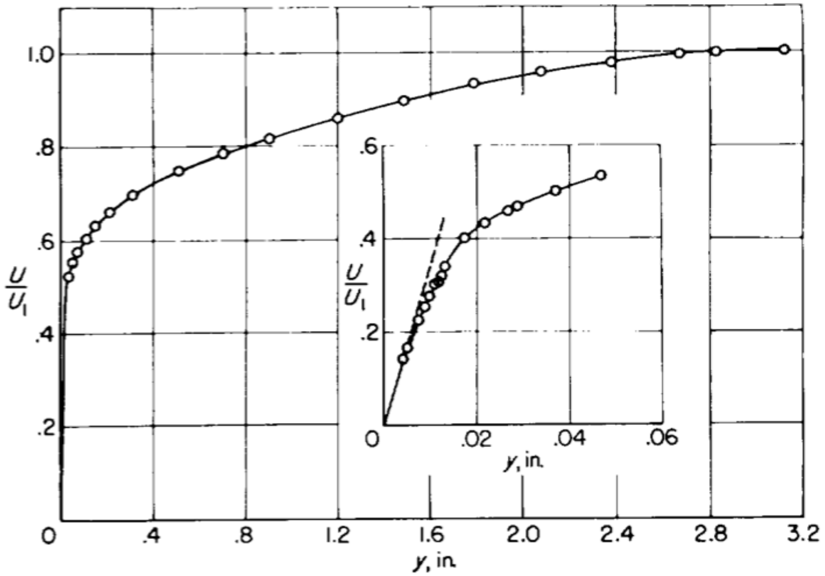
View from below a glass wall.

$$R_\delta = 9000$$

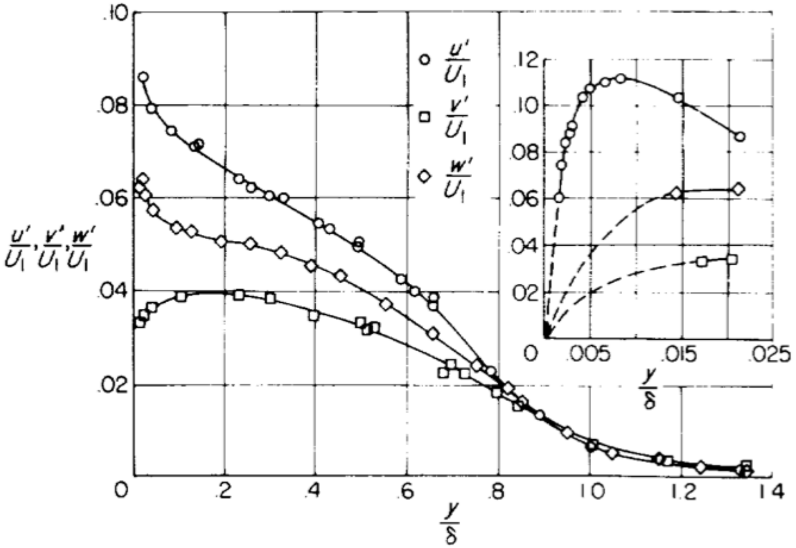


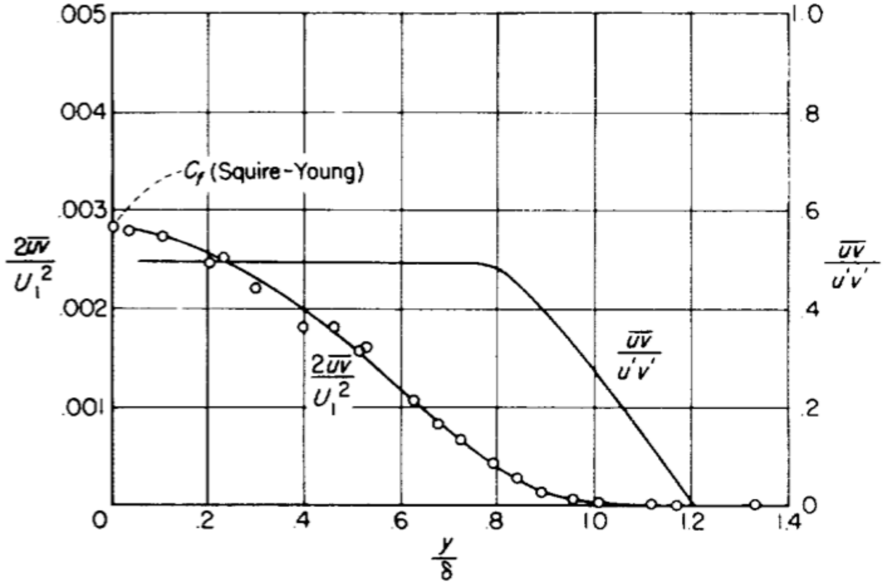
$$R_\delta = 26000$$

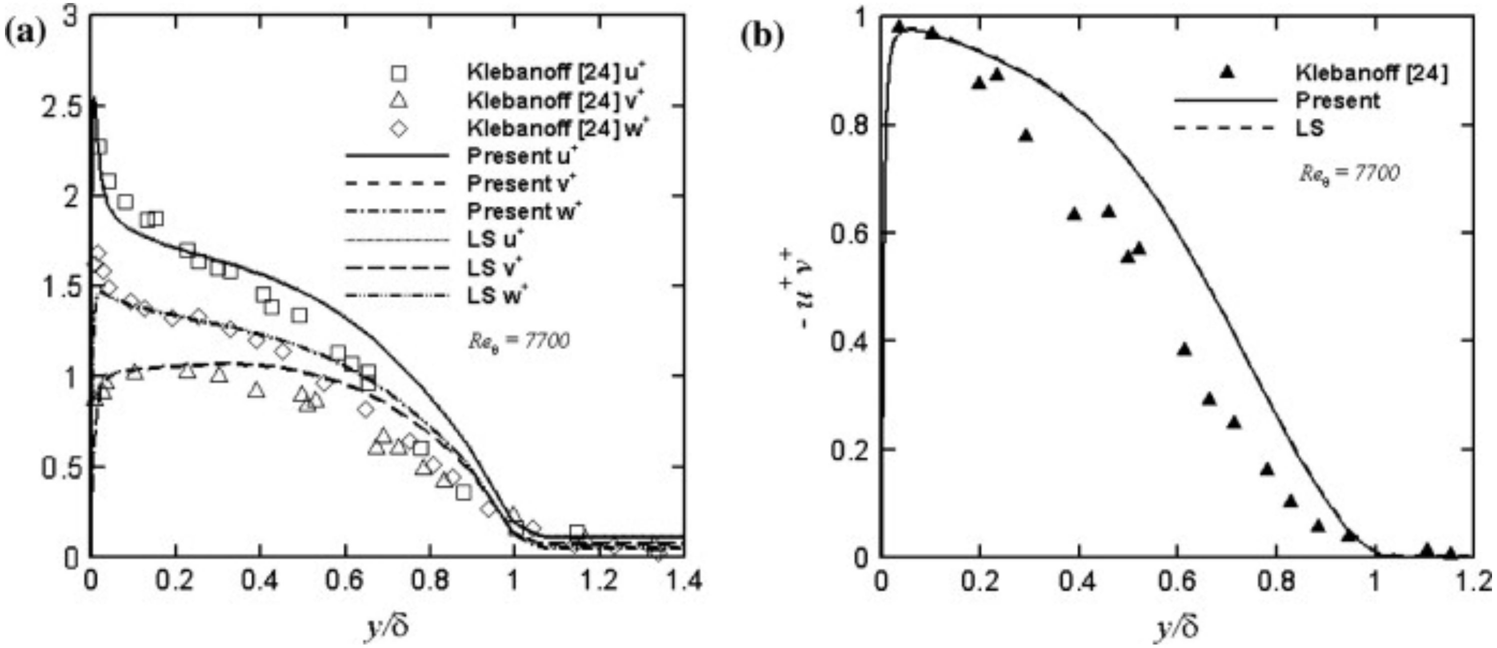
Turbulent boundary layer data – P.S. Klebanoff NACA 1247, 1955



$$R_{ex} = 4.2 \times 10^6$$






(a)
Klebanoff [24] u+
Klebanoff [24] v+
Klebanoff [24] w+
Present u+
Present v+
Present w+
LS u+
LS v+
LS w+
Reθ = 7700
0
0.5
1
1.5
2
2.5
3
0
0.2
0.4
0.6
0.8
1
1.2
1.4
y/δ
(b)
Klebanoff [24]
Present
LS
Reθ = 7700
- u+ v+
0
0.2
0.4
0.6
0.8
1
0
0.2
0.4
0.6
0.8
1
1.2
y/δ

The incompressible wall friction coefficient

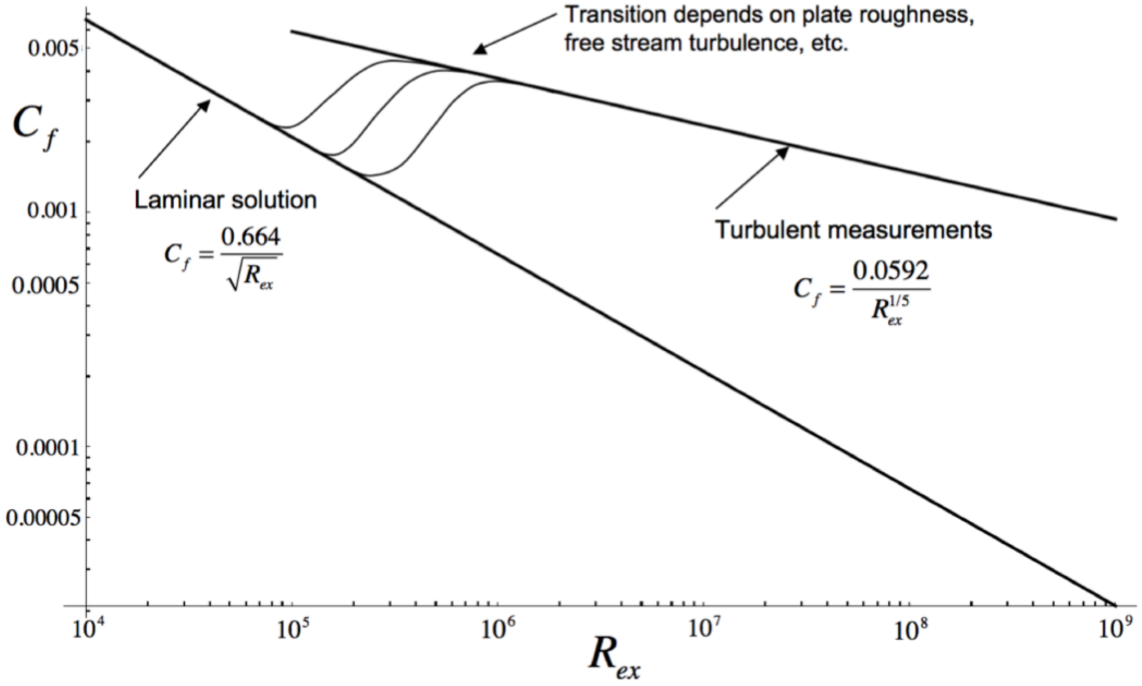


Figure 8.26: *Laminar and turbulent wall friction coefficient in a zero pressure gradient flat plate boundary layer.*

An impirical form of the velocity profile; the so-called 1/7th power law

$$\frac{U}{U_e} = \left(\frac{y}{\delta}\right)^{1/7}$$

The problem with this profile is that it fails to capture one of the most important features of the turbulent boundary layer profile which is that the actual shape of the profile depends on Reynolds number.

A much better, though still impirical, relation is the law of the wake developed by Don Coles at Caltech coupled with the universal law of the wall. In this approach the velocity profile is normalized by the wall friction velocity.

$$u^* = \sqrt{\frac{\tau_w}{\rho}} \qquad \tau_w = \mu \frac{\partial U}{\partial y}\bigg|_{y=0}$$

Define dimensionless wall variables

$$y^+ = \frac{yu^*}{\nu} \qquad U^+ = \frac{U}{u^*}$$

Reference: D. Coles, The Law of the Wake in the Turbulent Boundary Layer, J. Fluid Mech. Vol 1, 1956

The thickness of the boundary layer in wall units is

$$\delta^+ = \frac{\delta u^*}{\nu}$$

and

$$\frac{u^*}{U_e} = \left(\frac{\tau_w}{\rho U_e^2}\right)^{1/2} = \left(\frac{C_f}{2}\right)^{1/2} = \left(\frac{0.0592}{2R_{ex}^{1/5}}\right)^{1/2} = \frac{0.172}{R_{ex}^{1/10}}$$

$$\delta^+ = \frac{\delta}{x}\frac{u^*}{U_e}\frac{U_e x}{\nu} = \left(\frac{0.37}{R_{ex}^{1/5}}\right)\left(\frac{0.172}{R_{ex}^{1/10}}\right)R_{ex} = 0.0636R_{ex}^{7/10}$$

Once the Reynolds number is known most of the important properties of the boundary layer are known.

Velocity profile

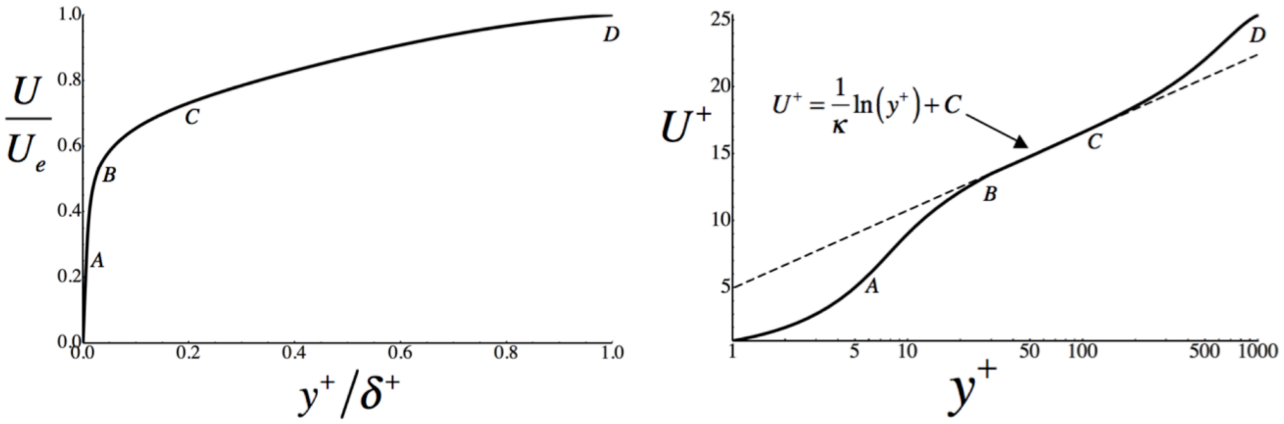


Figure 8.27: *Turbulent boundary layer velocity profile in linear and log-linear coordinates. The Reynolds number is* $R_{ex} = 10^6$.

Viscous sublayer - wall to A $0 \leq y^+ < 7$

$$U^+ = y^+$$

Buffer layer - A to B $7 \leq y^+ < 30$

$$y^+ = U^+ + e^{-\kappa C}\left(e^{\kappa U^+} - 1 - \kappa U^+ - \frac{1}{2}(\kappa U^+)^2 - \frac{1}{6}(\kappa U^+)^3 - \frac{1}{24}(\kappa U^+)^4\right)$$

Logarithmic and outer layer - B to C to D $dP_e/dx = 0, \Pi = 0.62$

$$U^+ = \frac{1}{\kappa}ln(y^+) + C + 2\frac{\Pi(x)}{\kappa}Sin^2\left(\frac{\pi y^+}{2\delta^+}\right)$$

$C = 5.1$ $\kappa = 0.4$

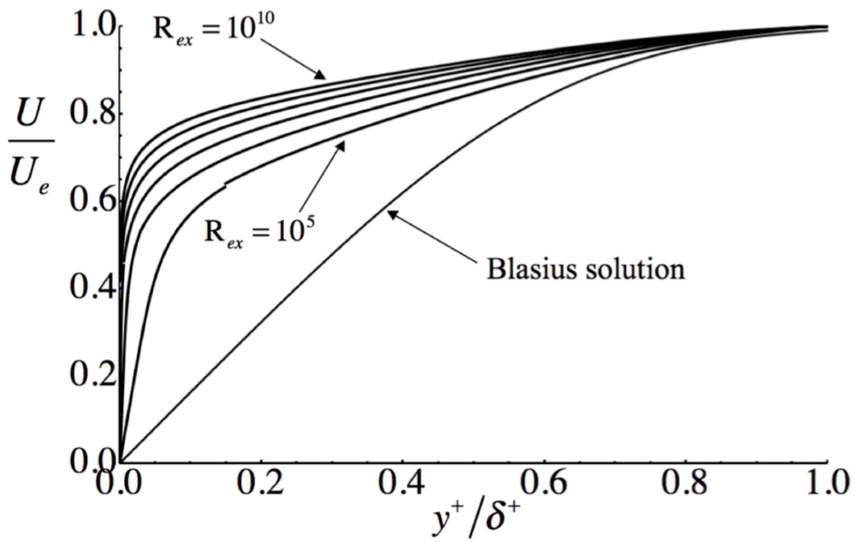


Figure 8.28: *Incompressible turbulent boundary layer profiles at several Reynolds numbers compared to the Blasius solution for a laminar boundary layer.*

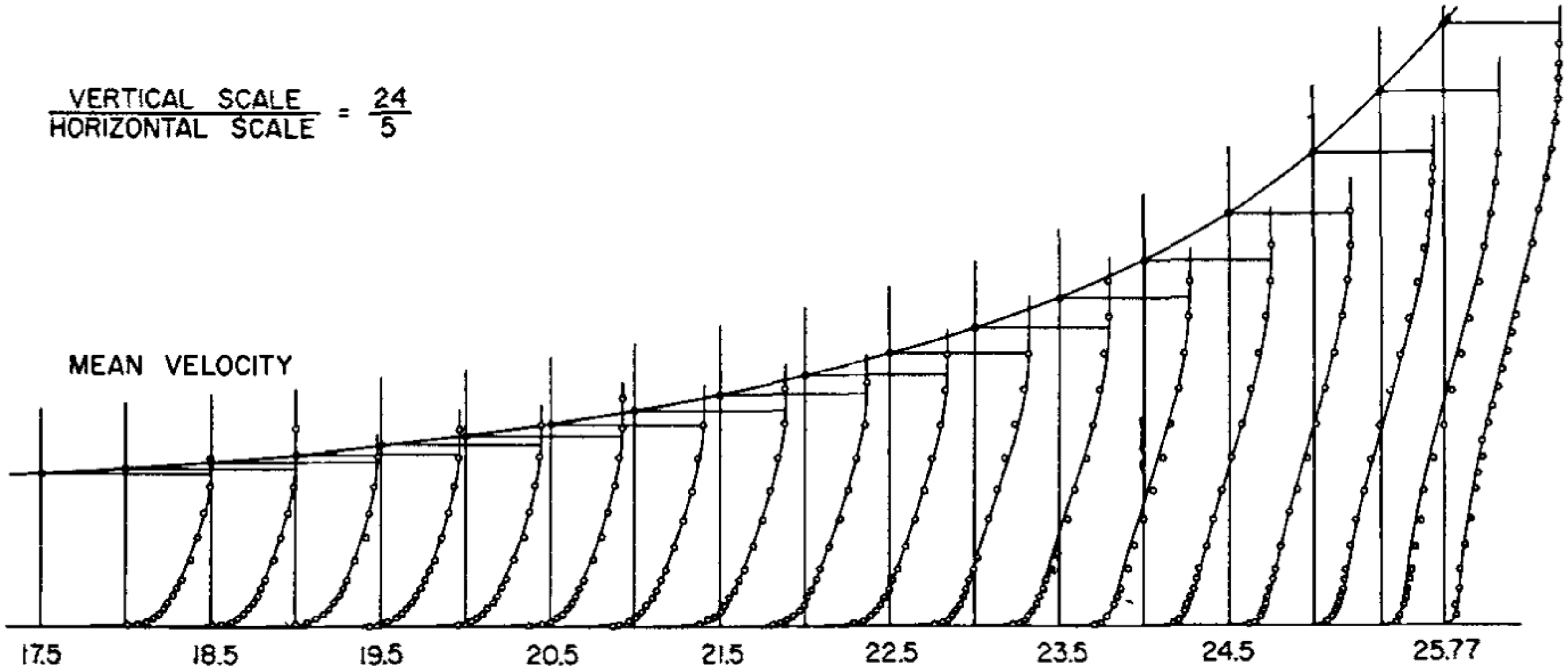
Measurements of velocity in the logarithmic layer can be used to infer the skin friction from the law of the wall.

$$\frac{U}{U^*} = \frac{1}{\kappa} ln\left(\left(\frac{yU^*}{\nu}\right) + C\right)$$

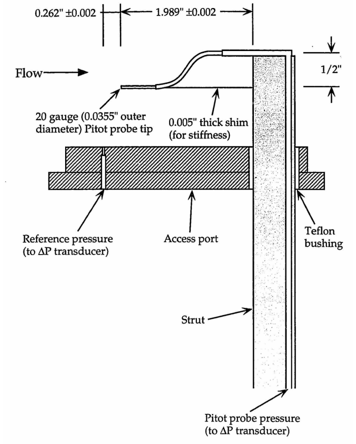
C increase with increasing roughness Reynolds number

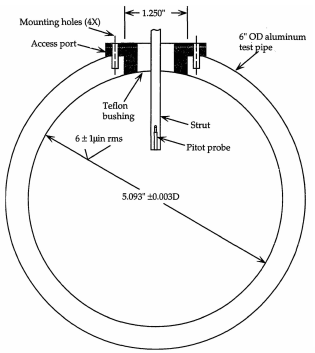
k_s	Roughness height
$R_{es} = \frac{k_s u^*}{\nu} < 3$	Hydraulically smooth
$R_{es} = \frac{k_s u^*}{\nu} > 100$	Fully rough

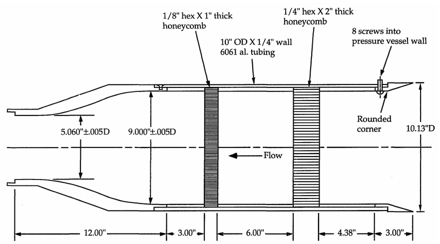
Separating turbulent boundary layer

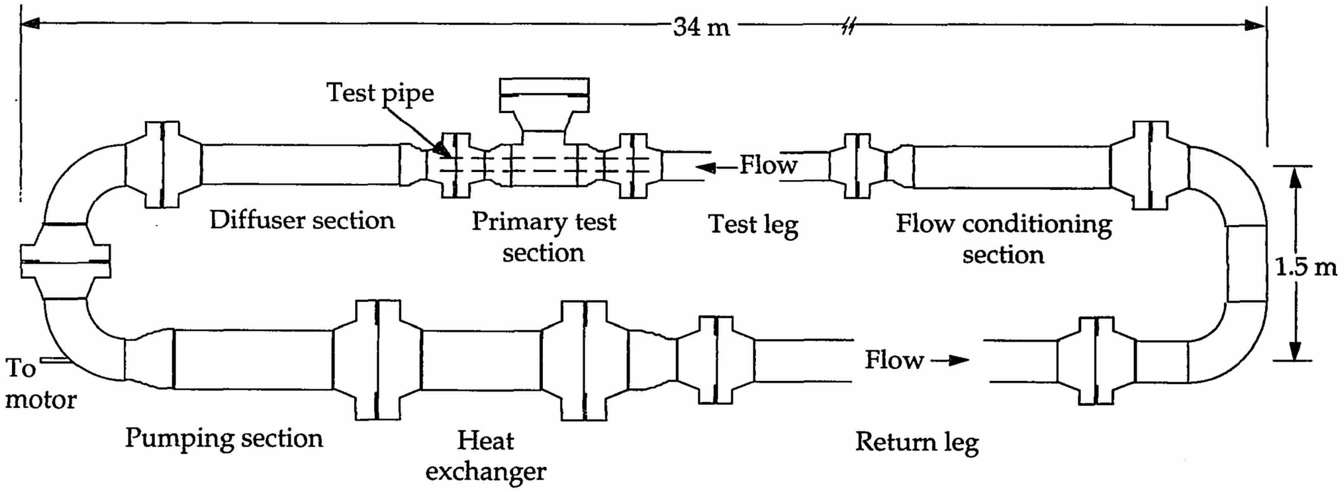


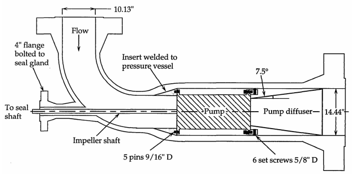
The Princeton Superpipe (PSP) Facility

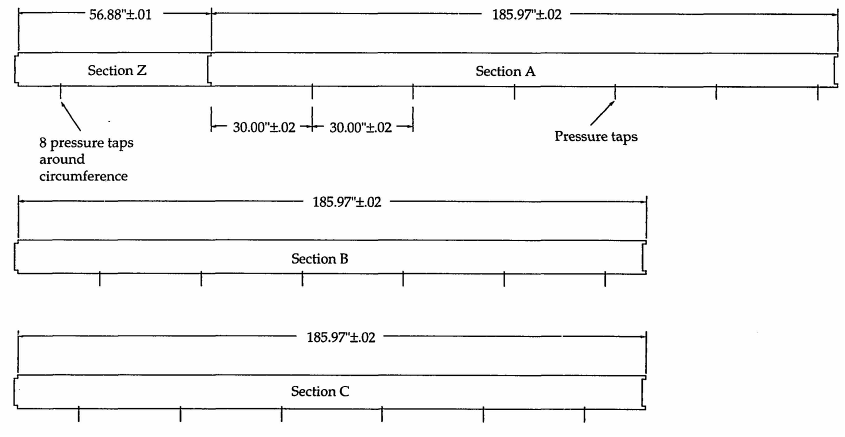


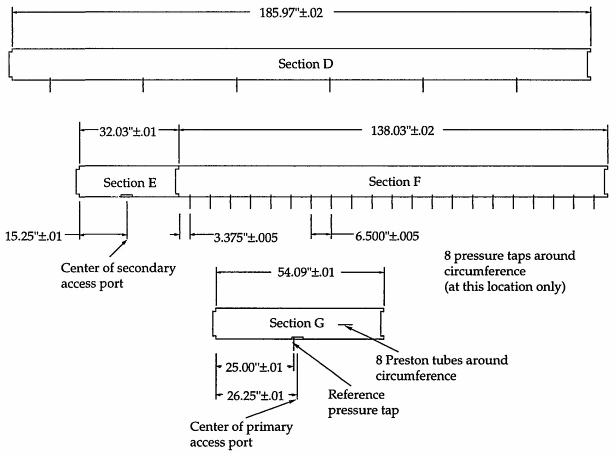




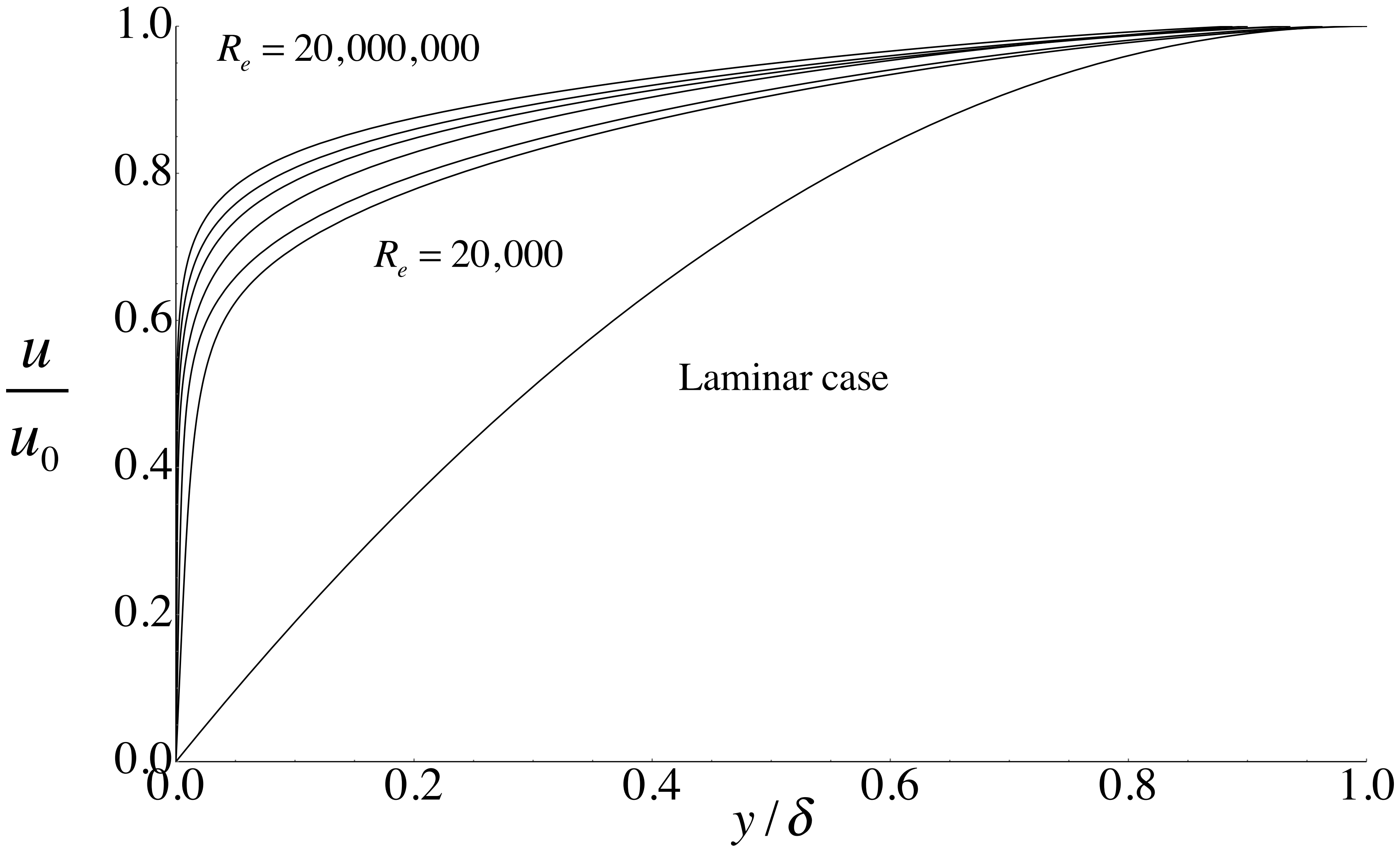




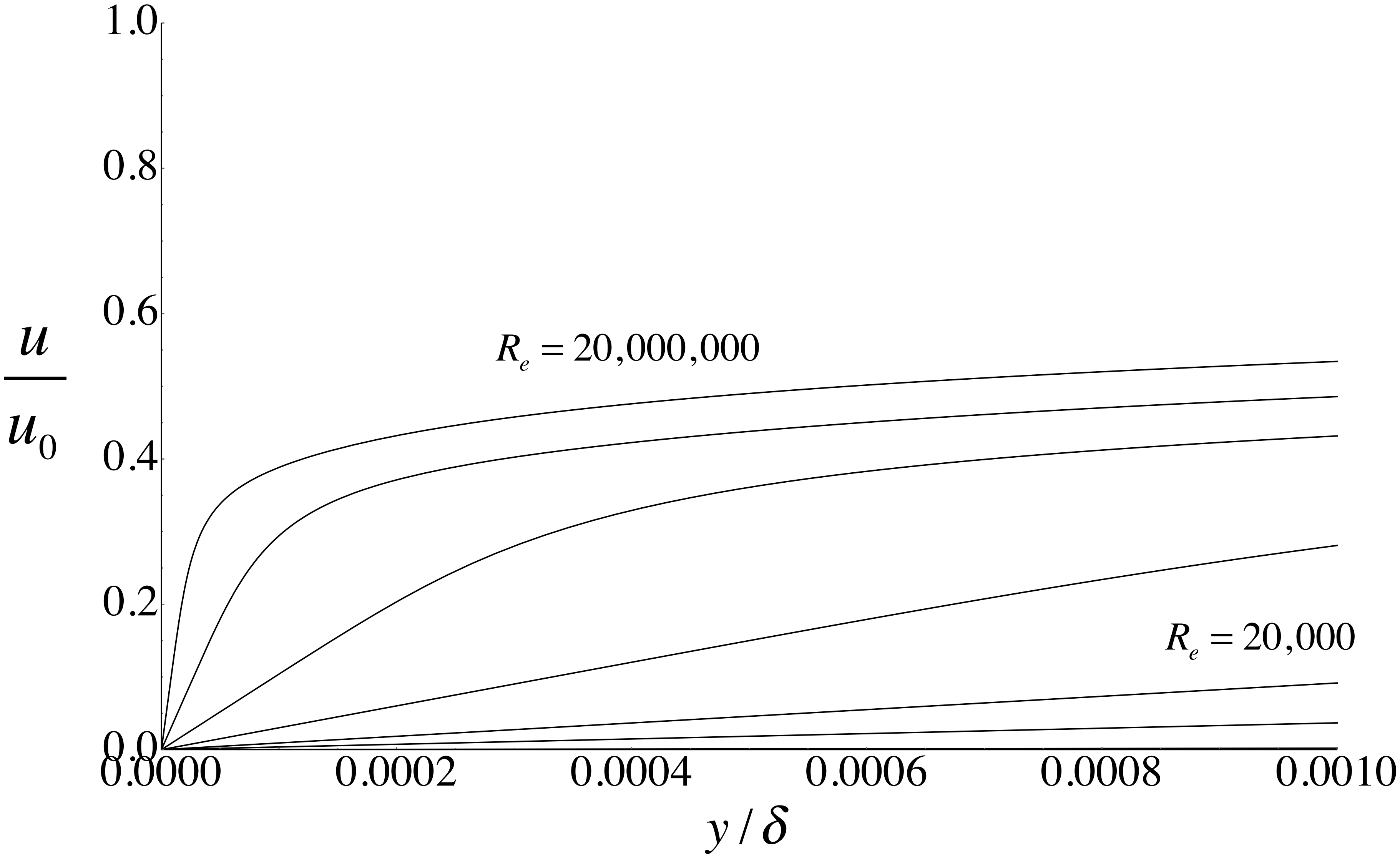




Velocity profiles



Velocity profiles – near wall



8.12 Transformation between flat plate and curved wall boundary layers

Boundary layer equations

$$\frac{\partial \rho U}{\partial x} + \frac{\partial \rho V}{\partial y} = 0$$

$$\rho U \frac{\partial U}{\partial x} + \rho V \frac{\partial U}{\partial y} + \frac{dP_e}{dx} - \frac{\partial \tau_{xy}}{\partial y} = 0$$

$$\rho U C_p \frac{\partial T}{\partial x} + \rho V C_p \frac{\partial T}{\partial y} - U \frac{dP_e}{dx} + \frac{\partial Q_y}{\partial y} - \tau_{xy} \frac{\partial U}{\partial y} = 0$$

Transform variables by adding an arbitrary function of x to the y coordinate

$$
\begin{aligned}
\tilde{x} &= x \\
\tilde{y} &= y + g(x) \\
\tilde{U}(\tilde{x}, \tilde{y}) &= U(x, y) \\
\tilde{V}(\tilde{x}, \tilde{y}) &= V(x, y) + U(x, y)\frac{dg(x)}{dx} \\
\tilde{\rho}(\tilde{x}, \tilde{y}) &= \rho(x, y) \\
\tilde{\tau}_{xy}(\tilde{x}, \tilde{y}) &= \tau_{xy}(x, y) \\
\tilde{Q}_y(\tilde{x}, \tilde{y}) &= Q_y(x, y) \\
P_e(\tilde{x}) &= P_e(x)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \tilde{U}}{\partial \tilde{x}} &= \frac{\partial U}{\partial x} - \frac{dg}{dx}\frac{\partial U}{\partial y} \\
\frac{\partial \tilde{U}}{\partial \tilde{y}} &= \frac{\partial U}{\partial y} \\
\frac{\partial^2 \tilde{U}}{\partial \tilde{y}^2} &= \frac{\partial^2 U}{\partial y^2} \\
\frac{\partial \tilde{V}}{\partial \tilde{y}} &= \frac{\partial V}{\partial y} + \frac{dg}{dx}\frac{\partial U}{\partial y} \\
\frac{\partial \tilde{\rho}}{\partial \tilde{x}} &= \frac{\partial \rho}{\partial x} - \frac{dg}{dx}\frac{\partial \rho}{\partial y}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \tilde{\rho}}{\partial \tilde{y}} &= \frac{\partial \rho}{\partial y} \\
\frac{\partial \tilde{T}}{\partial \tilde{x}} &= \frac{\partial T}{\partial x} - \frac{dg}{dx}\frac{\partial T}{\partial y} \\
\frac{\partial \tilde{T}}{\partial \tilde{y}} &= \frac{\partial T}{\partial y} \\
\frac{\partial \tilde{Q}_y}{\partial \tilde{y}} &= \frac{\partial Q_y}{\partial y} \\
\frac{\partial \tilde{\tau}_{xy}(\tilde{x}, \tilde{y})}{\partial \tilde{y}} &= \frac{\partial \tau_{xy}(x, y)}{\partial y}
\end{aligned}
$$

$$
\tilde{\rho}\tilde{U}\frac{\partial \tilde{U}}{\partial \tilde{x}} + \tilde{\rho}\tilde{V}\frac{\partial \tilde{U}}{\partial \tilde{y}} + \frac{\partial \tilde{P}_e}{\partial \tilde{x}} - \frac{\partial \tilde{\tau}_{\tilde{x}\tilde{y}}}{\partial \tilde{y}} = \rho U\left(\frac{\partial U}{\partial x} \cancel{- \frac{dg}{dx}\frac{\partial U}{\partial y}}\right) + \rho\left(V \cancel{+ U\frac{dg}{dx}}\right)\frac{\partial U}{\partial y} + \frac{\partial P_e}{\partial x} - \frac{\partial \tau_{xy}}{\partial y} = \rho U\frac{\partial U}{\partial x} + \rho V\frac{\partial U}{\partial y} + \frac{\partial P_e}{\partial x} - \frac{\partial \tau_{xy}}{\partial y}
$$

Insert the transformations of variables and derivatives into the equations of motion. The result is that the equations are mapped to themselves.

$$\frac{\partial \tilde{\rho}\tilde{U}}{\partial \tilde{x}} + \frac{\partial \tilde{\rho}\tilde{V}}{\partial \tilde{y}} = \frac{\partial \rho U}{\partial x} + \frac{\partial \rho V}{\partial y} = 0$$

$$\tilde{\rho}\tilde{U}\frac{\partial \tilde{U}}{\partial \tilde{x}} + \tilde{\rho}\tilde{V}\frac{\partial \tilde{U}}{\partial \tilde{y}} + \frac{d\tilde{P}_e}{d\tilde{x}} - \frac{\partial \tilde{\tau}_{xy}}{\partial \tilde{y}} = \rho U\frac{\partial U}{\partial x} + \rho V\frac{\partial U}{\partial y} + \frac{dP_e}{dx} - \frac{\partial \tau_{xy}}{\partial y} = 0$$

$$\tilde{\rho}\tilde{U}C_p\frac{\partial \tilde{T}}{\partial \tilde{x}} + \tilde{\rho}\tilde{V}C_p\frac{\partial \tilde{T}}{\partial \tilde{y}} - \tilde{U}\frac{d\tilde{P}_e}{d\tilde{x}} + \frac{\partial \tilde{Q}_y}{\partial \tilde{y}} - \tilde{\tau}_{xy}\frac{\partial \tilde{U}}{\partial \tilde{y}} =$$

$$\rho UC_p\frac{\partial T}{\partial x} + \rho VC_p\frac{\partial T}{\partial y} - U\frac{dP_e}{dx} + \frac{\partial Q_y}{\partial y} - \tau_{xy}\frac{\partial U}{\partial \tilde{y}} = 0$$

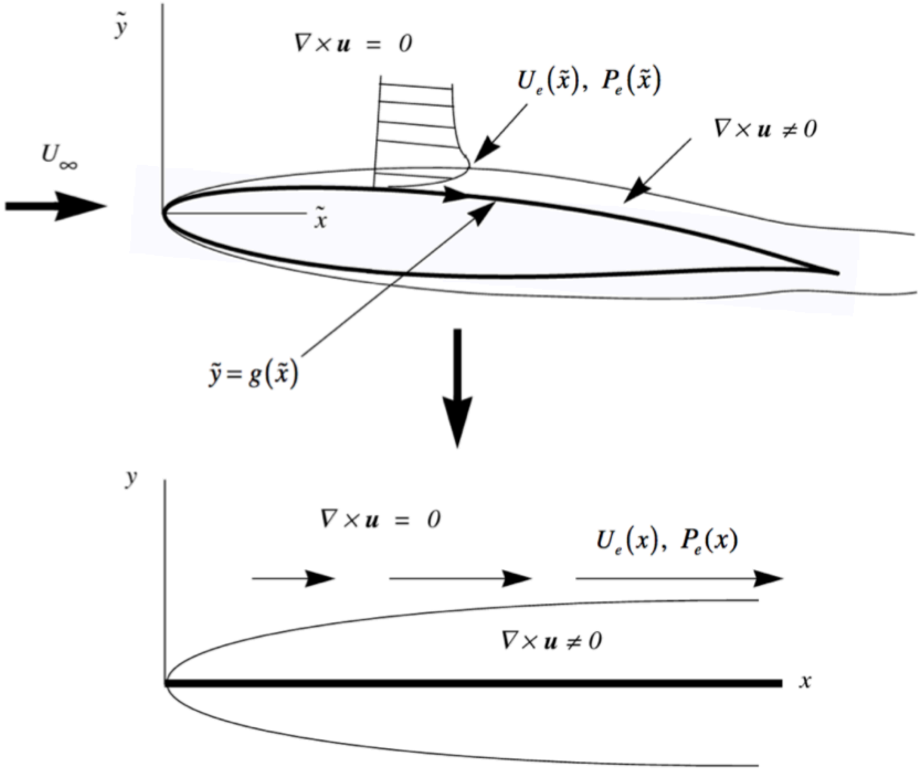
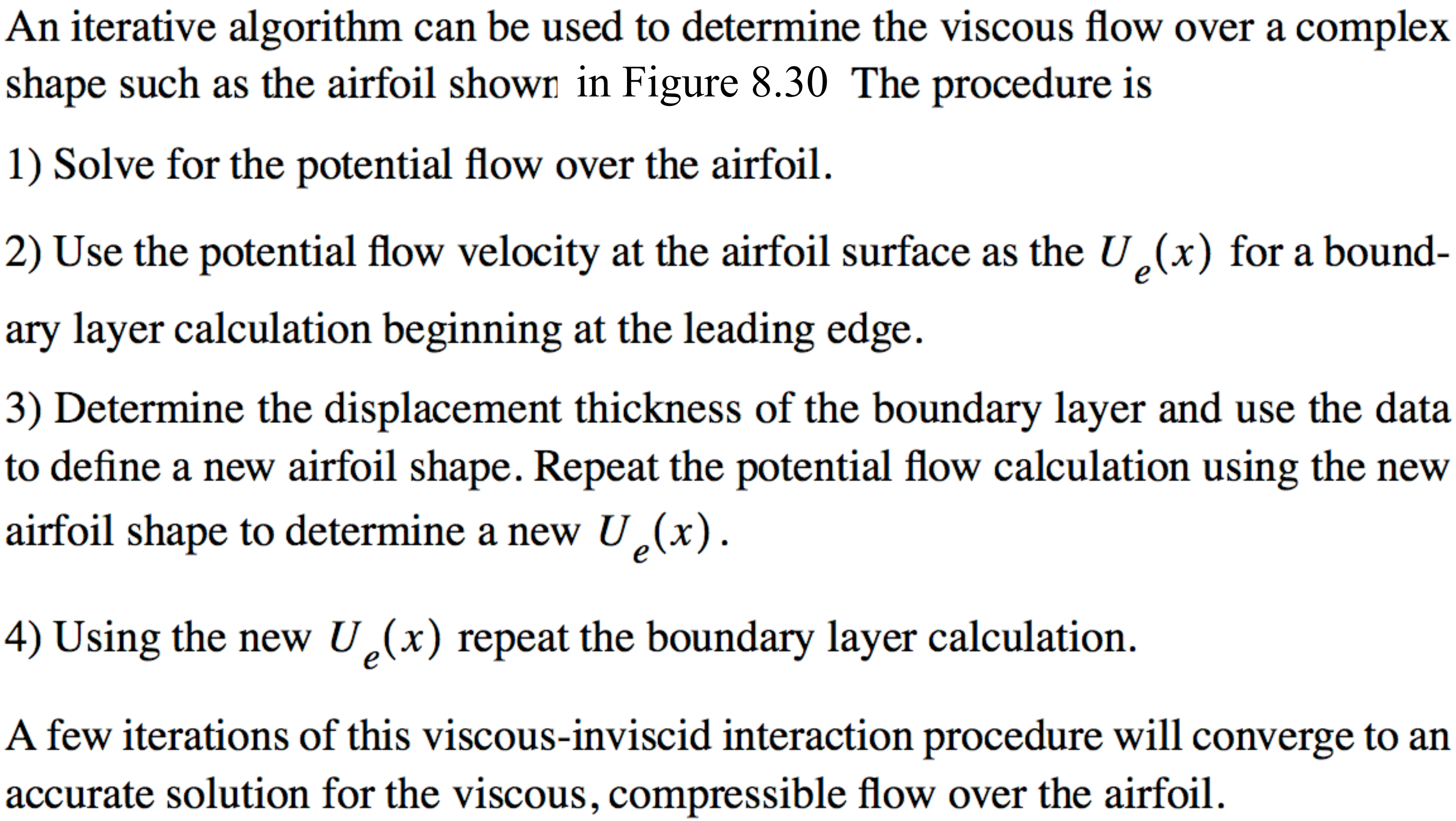


Figure 8.30: *Mapping of the boundary layer developing over an airfoil to the boundary layer on a flat plate with the same pressure gradient.*

Viscous-inciscid interaction algorithm



An iterative algorithm can be used to determine the viscous flow over a complex shape such as the airfoil shown in Figure 8.30 The procedure is

1) Solve for the potential flow over the airfoil.

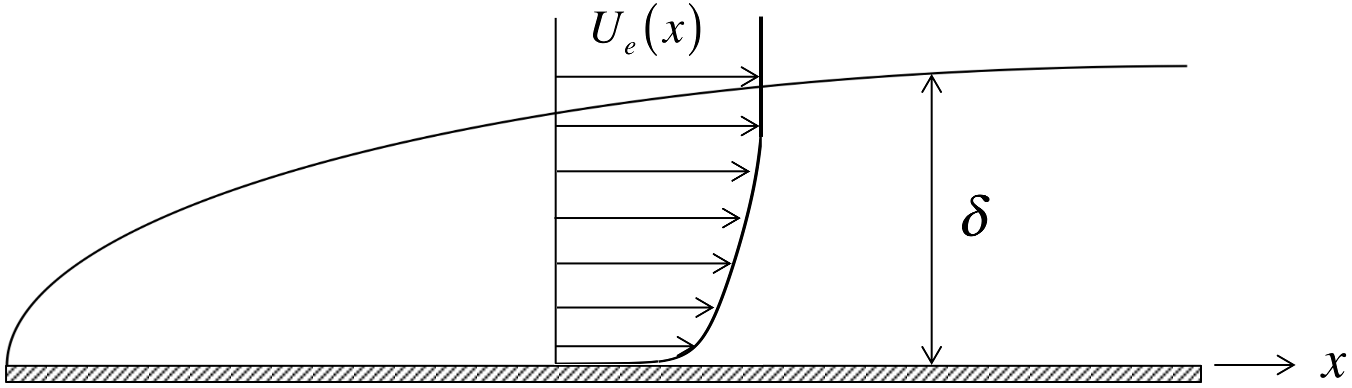
2) Use the potential flow velocity at the airfoil surface as the $U_e(x)$ for a boundary layer calculation beginning at the leading edge.

3) Determine the displacement thickness of the boundary layer and use the data to define a new airfoil shape. Repeat the potential flow calculation using the new airfoil shape to determine a new $U_e(x)$.

4) Using the new $U_e(x)$ repeat the boundary layer calculation.

A few iterations of this viscous-inviscid interaction procedure will converge to an accurate solution for the viscous, compressible flow over the airfoil.

8.13 Head's method for approximate calculation of turbulent boundary layer characteristics



At any position x the area flow in the boundary layer is

$$Q = \int_0^{\delta} U\,dy$$

This can be arranged to read

$$Q = \int_0^{\delta} U\,dy = \int_0^{\delta} U_e\,dy - \int_0^{\delta} U_e\left(1 - \frac{U}{U_e}\right)dy = U_e\left(\delta - \delta^*\right)$$

Entrainment velocity

$$V_e = \frac{d}{dx}\left(U_e\left(\delta - \delta^*\right)\right)$$

Reference: M.R. Head, Entrainment in the Turbulent Boundary Layer, Aero. Res. Council. R&M 3152, 1960

Head defined the boundary layer shape factor

$$H_1 = \frac{\left(\delta - \delta^*\right)}{\theta}$$

His model consists of two assumptions:

1) Assume

$$\frac{V_e}{U_e} = \frac{1}{U_e}\frac{d}{dx}\left(U_e\left(\delta - \delta^*\right)\right) = F\left(H_1\right)$$

2) Assume

$$H_1 = G\left(H\right) \qquad H = \frac{\delta^*}{\theta}$$

In addition he assumed that the skin friction followed the impirical formula due to Ludweig and Tillman

$$C_f = \frac{0.246}{10^{0.678H}\left(\frac{U_e}{U_\infty}R_\theta\right)^{0.268}} \qquad R_\theta = \frac{U_\infty \theta}{\nu}$$

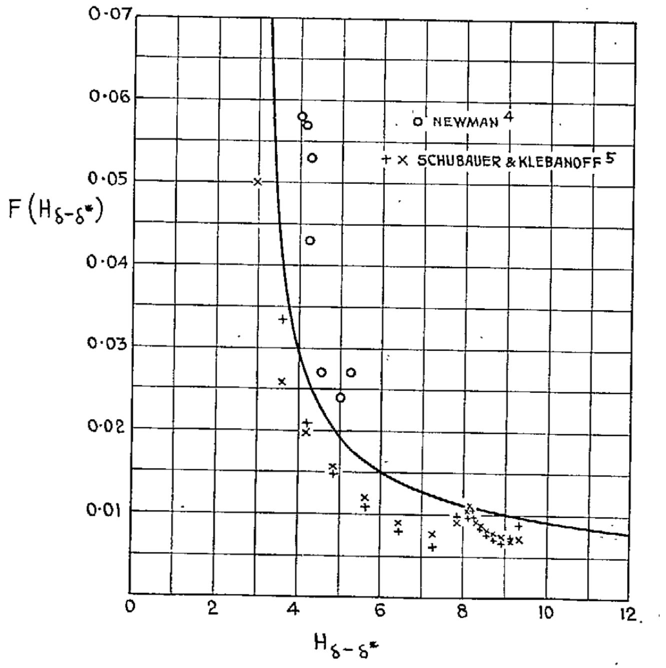


FIG. 1. $\frac{1}{U}\frac{d}{dx}[U(\delta - \delta^*)]$ as a function of $H_{\delta-\delta^*}$.

$$H_1 = H_{\delta-\delta^*}$$

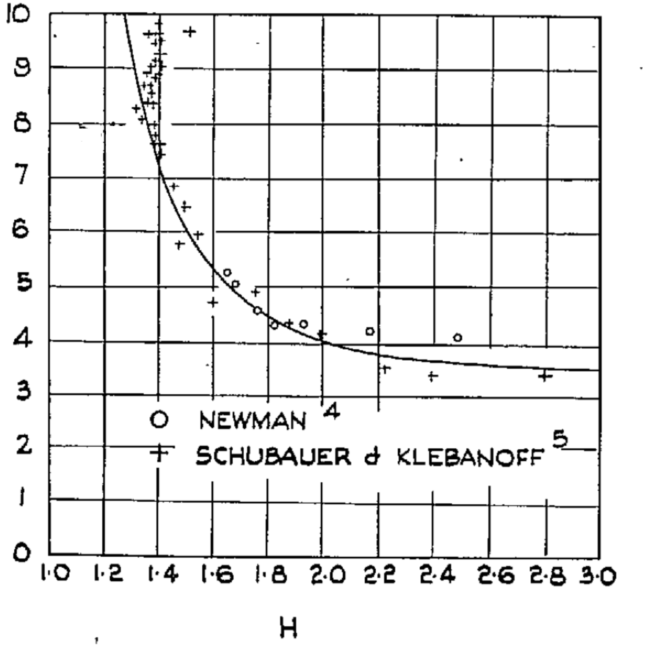


FIG. 2. Variation of $H_{\delta-\delta^*}$ with conventional form parameter H.

We will use $$F(H_1) = \frac{0.0306}{(H_1 - 3.0)^{0.6169}}$$

$$G(H) = 3.0445 + \frac{0.8702}{(H - 1.1)^{1.2721}}$$

Several classical references recommend different functions for F and G

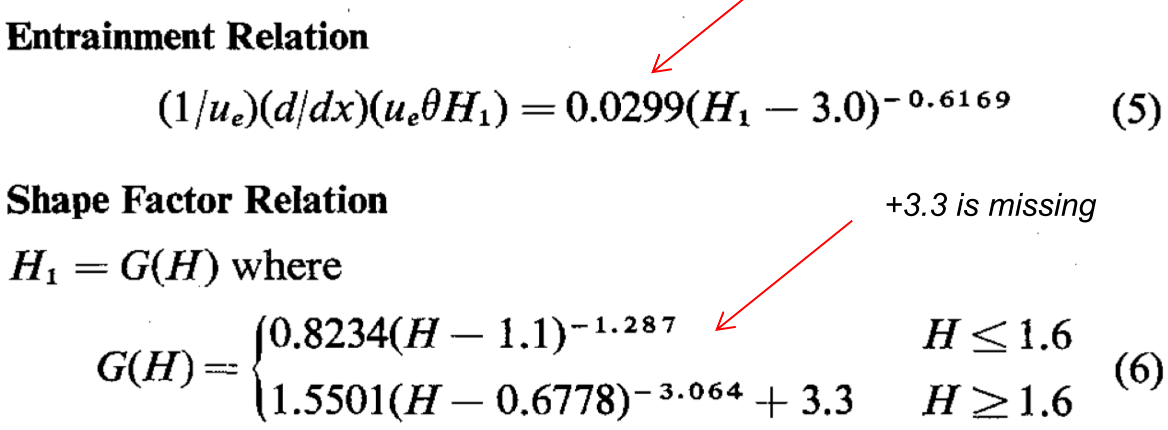
Calculation of Separation Points in Incompressible Turbulent Flows
T. CEBECI, G. J. MOSINSKIS, AND A. M. O. SMITH
Douglas Aircraft Company, Long Beach, Calif.
J. AIRCRAFT VOL. 9, NO. 9

Also

Boundary Layer Theory *H. Schlichting*

Recommend

Schlichting uses 0.0306



Entrainment Relation

$$(1/u_e)(d/dx)(u_e\theta H_1) = 0.0299(H_1 - 3.0)^{-0.6169} \qquad (5)$$

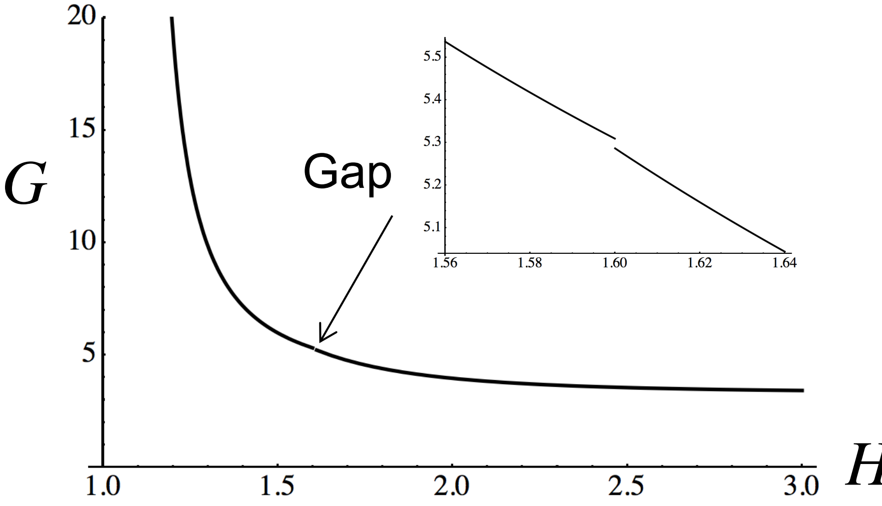
Shape Factor Relation

+3.3 is missing

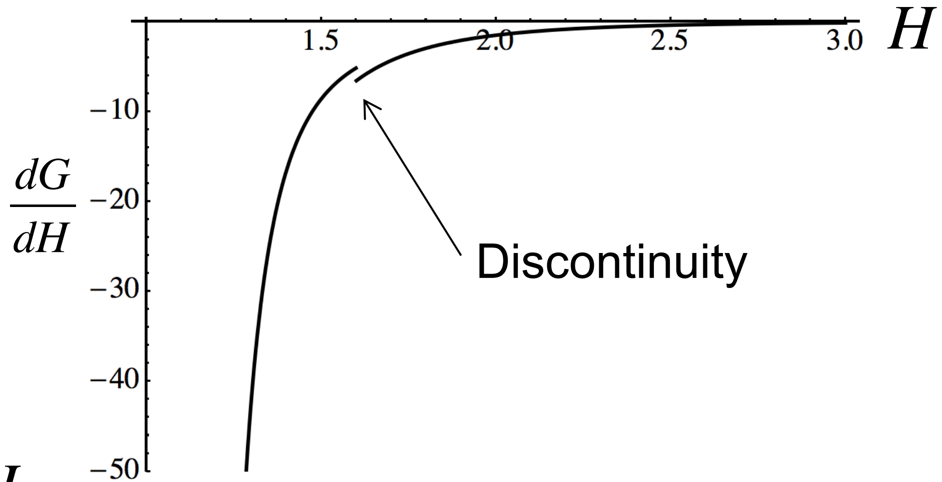
$H_1 = G(H)$ where

$$G(H) = \begin{cases} 0.8234(H - 1.1)^{-1.287} & H \leq 1.6 \\ 1.5501(H - 0.6778)^{-3.064} + 3.3 & H \geq 1.6 \end{cases} \qquad (6)$$

Cebeci - Schlichting

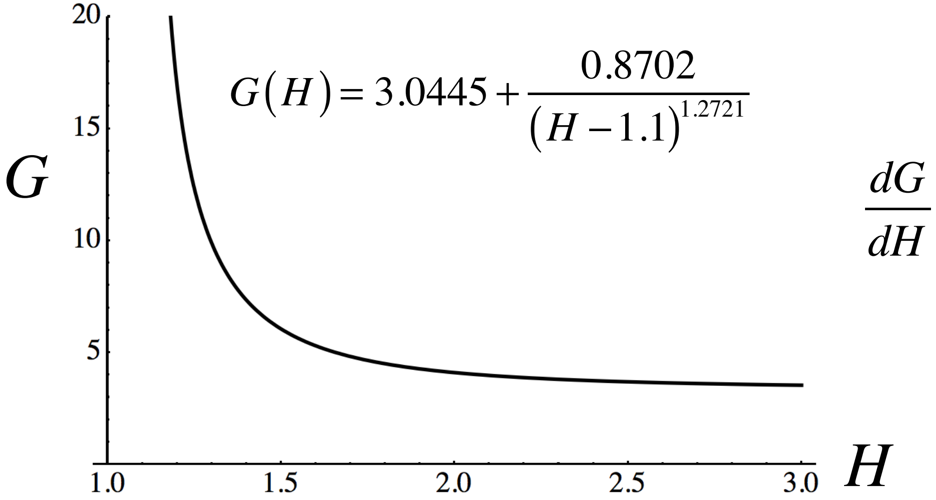


Gap

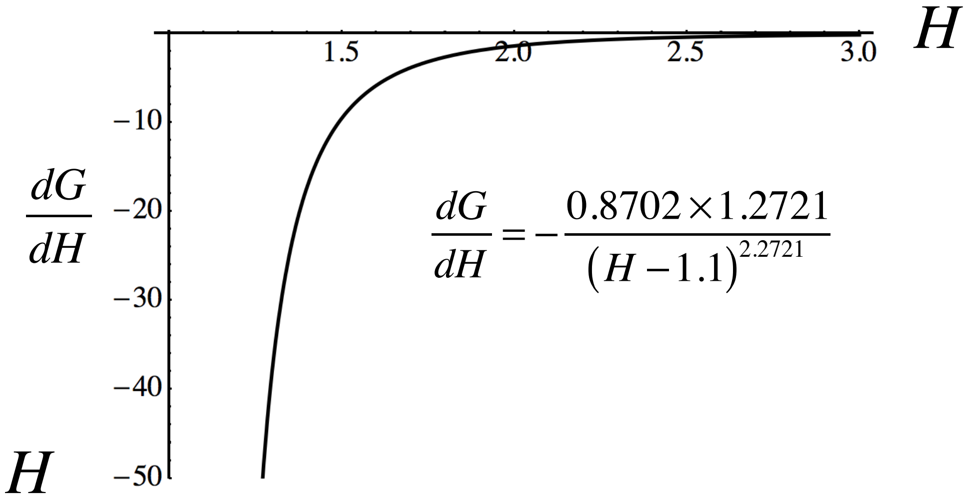


Discontinuity

I prefer a single smooth function

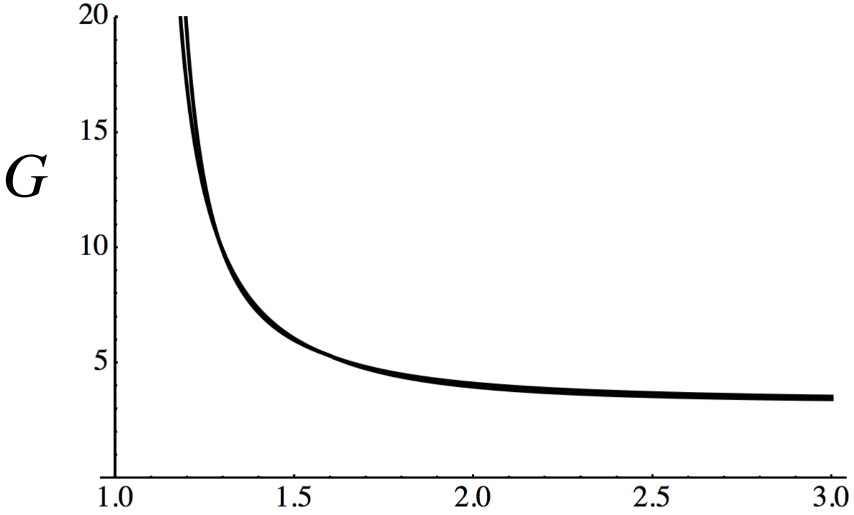


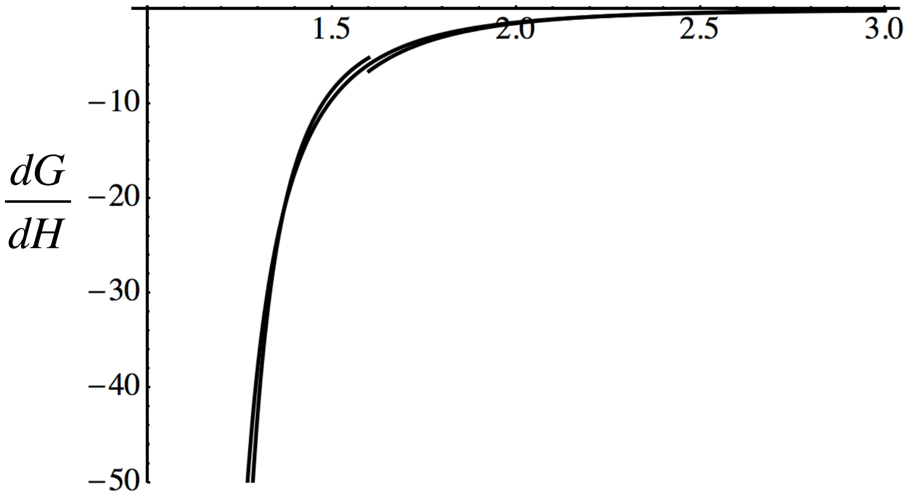
$$G(H) = 3.0445 + \frac{0.8702}{(H-1.1)^{1.2721}}$$



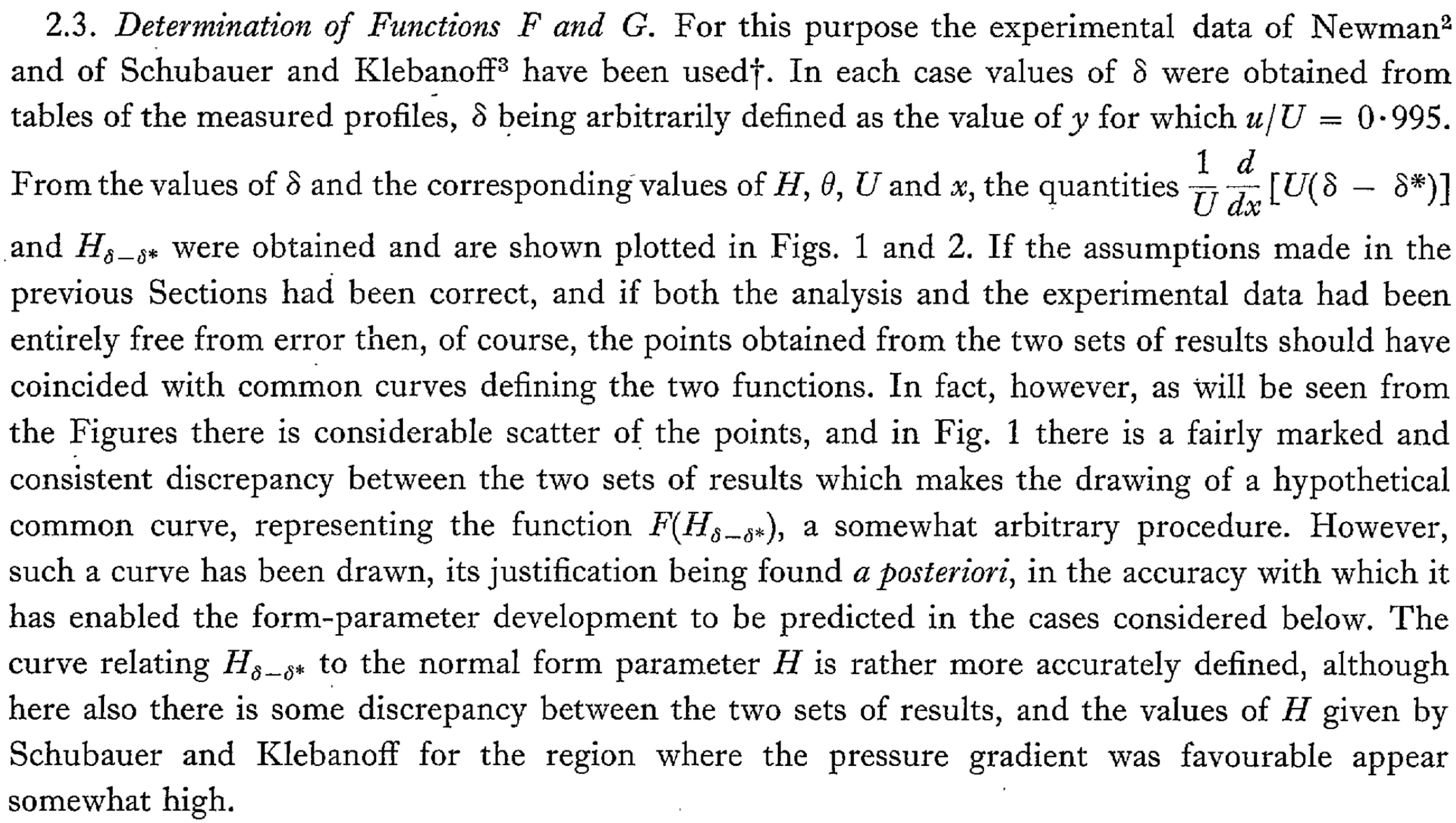
$$\frac{dG}{dH} = -\frac{0.8702 \times 1.2721}{(H-1.1)^{2.2721}}$$

Comparison





From Head's paper



2.3. *Determination of Functions F and G.* For this purpose the experimental data of Newman[2] and of Schubauer and Klebanoff[3] have been used†. In each case values of δ were obtained from tables of the measured profiles, δ being arbitrarily defined as the value of y for which $u/U = 0{\cdot}995$. From the values of δ and the corresponding values of H, θ, U and x, the quantities $\frac{1}{U}\frac{d}{dx}[U(\delta - \delta^*)]$ and $H_{\delta-\delta^*}$ were obtained and are shown plotted in Figs. 1 and 2. If the assumptions made in the previous Sections had been correct, and if both the analysis and the experimental data had been entirely free from error then, of course, the points obtained from the two sets of results should have coincided with common curves defining the two functions. In fact, however, as will be seen from the Figures there is considerable scatter of the points, and in Fig. 1 there is a fairly marked and consistent discrepancy between the two sets of results which makes the drawing of a hypothetical common curve, representing the function $F(H_{\delta-\delta^*})$, a somewhat arbitrary procedure. However, such a curve has been drawn, its justification being found *a posteriori*, in the accuracy with which it has enabled the form-parameter development to be predicted in the cases considered below. The curve relating $H_{\delta-\delta^*}$ to the normal form parameter H is rather more accurately defined, although here also there is some discrepancy between the two sets of results, and the values of H given by Schubauer and Klebanoff for the region where the pressure gradient was favourable appear somewhat high.

Typical range of H vs R_{ex} for turbulent boundary layers

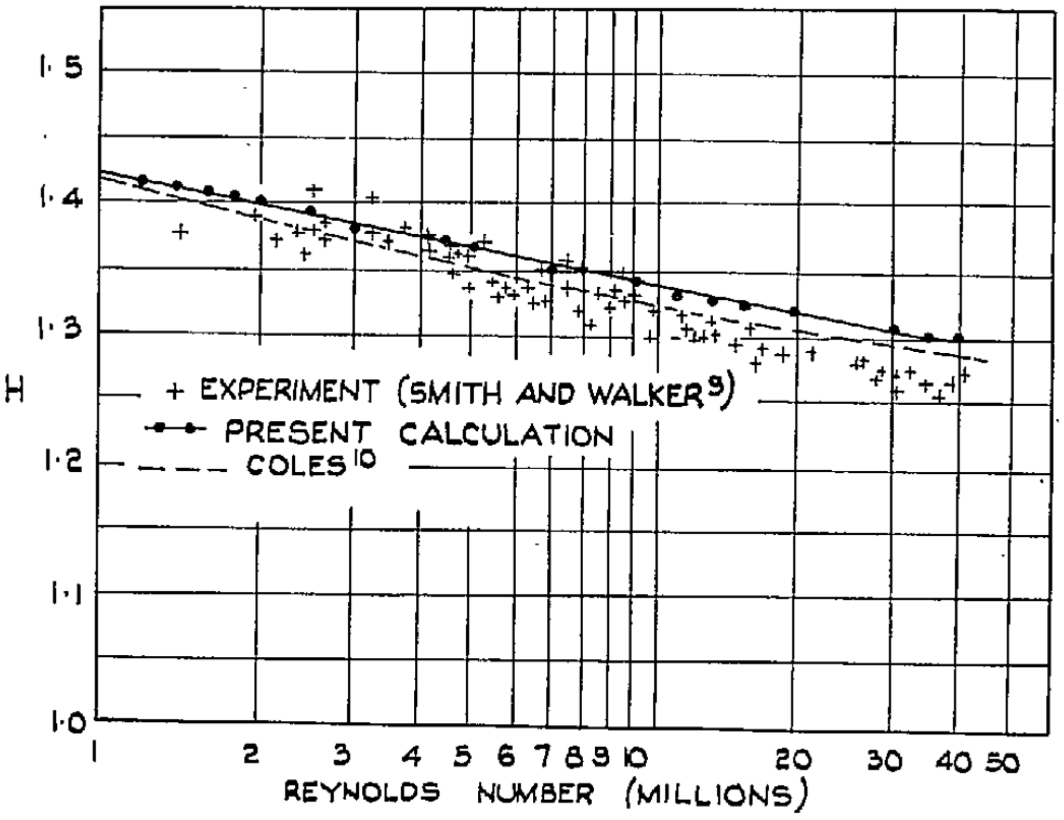


FIG. 3. Flat-plate results compared with experiment.

Recall the incompressible von Karman integral momentum equation

$$\boxed{\frac{d\theta}{dx}+(2+H)\frac{\theta}{U_e}\frac{dU_e}{dx}=\frac{C_f}{2}} \qquad (9.82)$$

For given initial conditions on theta and H and known free stream velocity distribution $U_e(x)$ this equation is solved along with the auxiliary equations

$$C_f=\frac{0.246}{10^{0.678H}\left(\frac{U_e}{U_\infty}R_\theta\right)^{0.268}} \qquad R_\theta=\frac{U_\infty\theta}{\nu}$$

$$\frac{1}{U_e}\frac{d}{dx}\left(U_e\theta H_1\right)=F\left(H_1\right)=\frac{0.0306}{\left(H_1-3.0\right)^{0.6169}}$$

$$H_1=G\left(H\right)=3.0445+\frac{0.8702}{\left(H-1.1\right)^{1.2721}}$$

Expand the derivative of H

$$\frac{dH}{dx}=\left(\frac{-\frac{G(H)}{\theta}\frac{d\theta}{dx}-G\left(H\right)\frac{1}{U_e}\frac{dU_e}{dx}+\frac{0.0299}{\theta(G(H)-3.0)^{0.6169}}}{\frac{dG}{dH}}\right)$$

Recall the von Karman integral equation

$$\frac{d\theta}{dx}+(2+H)\frac{\theta}{U_e}\frac{dU_e}{dx}=\frac{C_f}{2}$$

Substitute into the equation for H.

$$\frac{dH}{dx}=\left(\frac{(1+H)\frac{G(H)}{U_e}\frac{dU_e}{dx}-\frac{G(H)}{\theta}\frac{C_f}{2}+\frac{0.0299}{\theta(G(H)-3.0)^{0.6169}}}{\frac{dG}{dH}}\right)$$

Express the equations in dimensionless terms using R_{ex}, R_θ, and C_p.

$$C_p = \frac{P_e - P_\infty}{\frac{1}{2}\rho {U_\infty}^2} = 1 - \left(\frac{U_e}{U_\infty}\right)^2$$

$$P_\infty + \frac{1}{2}\rho {U_\infty}^2 = P_e + \frac{1}{2}\rho {U_e}^2$$

$$R_{ex} = \frac{U_\infty x}{\nu} \qquad R_\theta = \frac{U_\infty \theta}{\nu}$$

Substitute the Ludweig-Tillman relation for C_f. Solve the resulting pair of ODEs for $R_\theta\left(R_{ex}\right)$ and $H\left(R_{ex}\right)$.

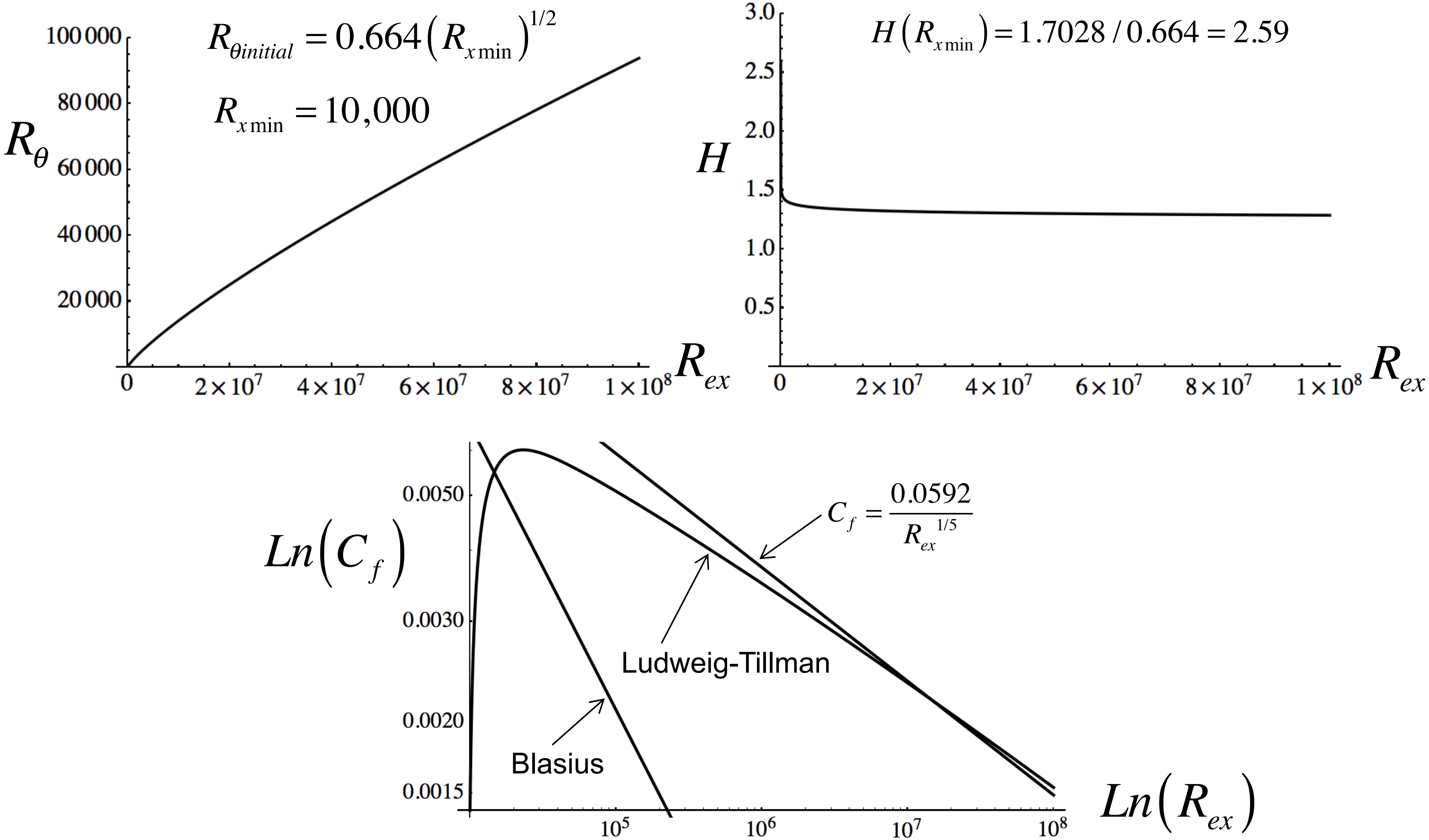
$$\frac{dR_\theta}{dR_{ex}} = \frac{1}{2}\left(\frac{2+H}{1-C_p}\right) R_\theta \frac{dC_p}{dR_{ex}} + \frac{0.246}{10^{0.678H}(1-C_p)^{0.268} {R_\theta}^{0.268}}$$

$$\frac{dH}{dR_{ex}} = \frac{\left(\frac{G(H)}{2}\left(\frac{1+H}{1-C_p}\right)\frac{dC_p}{dR_{ex}} + \frac{0.246\times G(H)}{10^{0.678H}(1-C_p)^{0.268} {R_\theta}^{1.268}} - \frac{0.0299}{(G(H)-3.0)^{0.6169} R_\theta}\right)}{\frac{0.8702\times 1.2721}{(H-1.1)^{0.2721}}}$$

Where

$$G\left(H\right) = 3.0445 + \frac{0.8702}{(H-1.1)^{1.2721}}$$

Zero pressure gradient turbulent boundary layer $C_p = 0$



Potential flow about a circular cylinder

$$\frac{U_e}{U_\infty} = 2Sin\left(\frac{x}{R}\right)$$

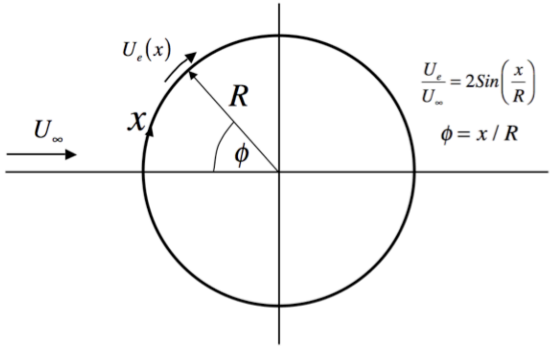


Figure 9.19 Example for Thwaites' method.

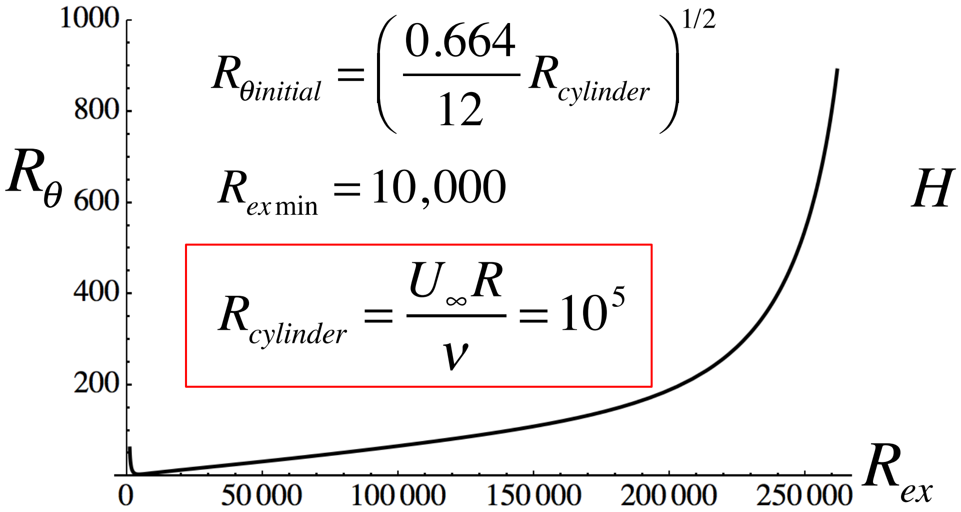
$$\left(\frac{\theta}{R}\right)^2 R_e = \frac{0.441}{Sin^6(\phi)} \int_0^\phi Sin^5(\phi')d\phi'$$

$$R_e = \frac{U_\infty 2R}{\nu}$$

Thwaites' method gives a finite momentum thickness at the forward stagnation point. This is useful in a wing leading edge calculation.

$$\lim_{\phi \to 0} \left(\frac{\theta}{R}\right)^2 R_e = \frac{0.441}{\phi^6} \int_0^\phi \phi^5 d\phi' = \frac{0.441}{6}$$

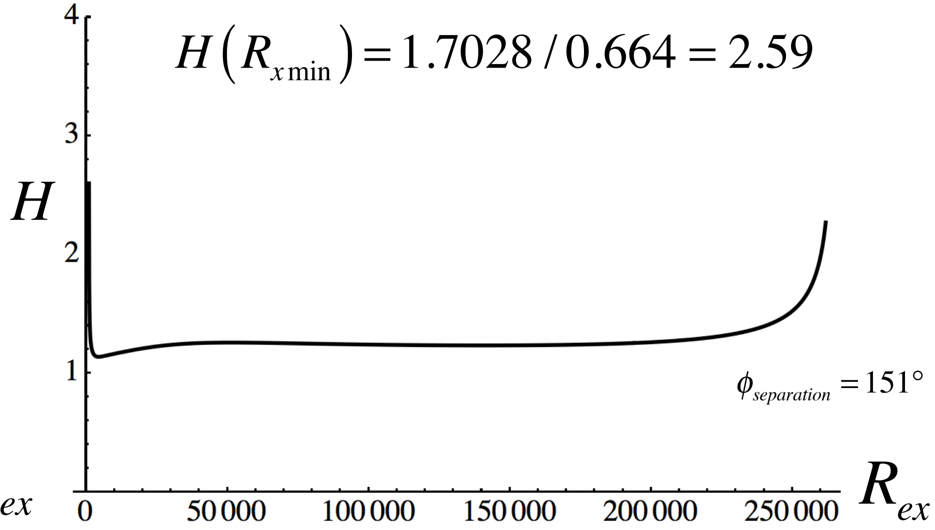
Heads method applied to flow about a circular cylinder



$$R_{\theta initial} = \left(\frac{0.664}{12} R_{cylinder} \right)^{1/2}$$

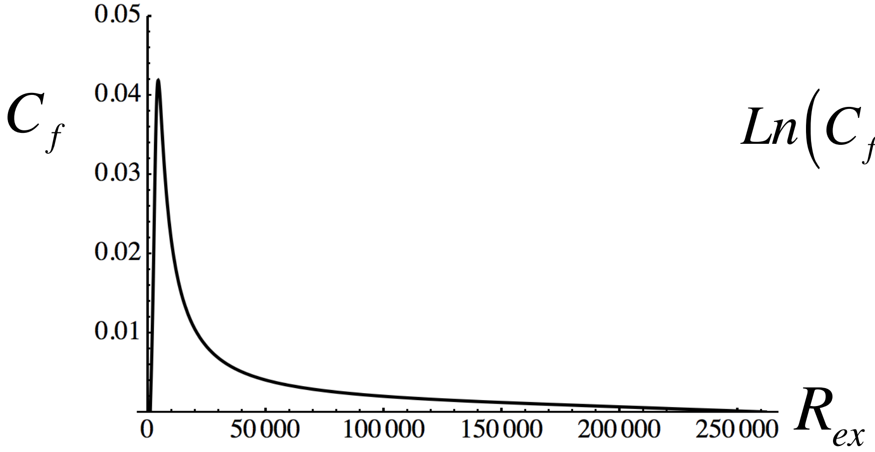
$$R_{ex\min} = 10{,}000$$

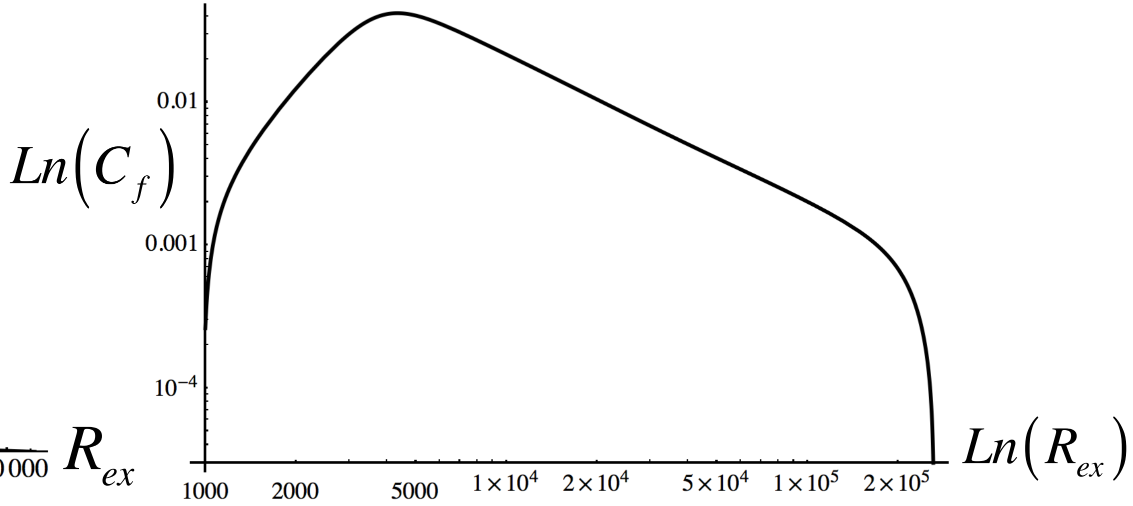
$$\boxed{R_{cylinder} = \frac{U_\infty R}{\nu} = 10^5}$$

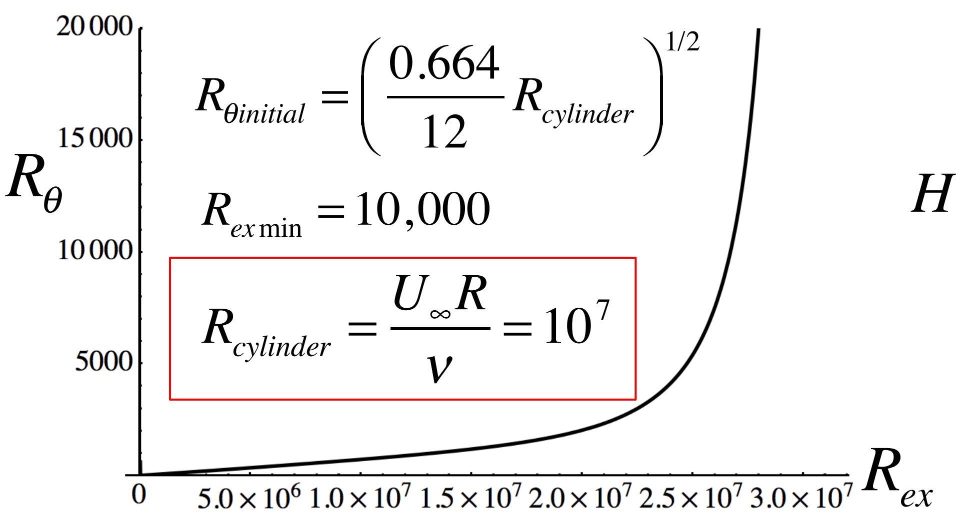


$$H\left(R_{x\min}\right) = 1.7028 / 0.664 = 2.59$$

$$\phi_{separation} = 151°$$



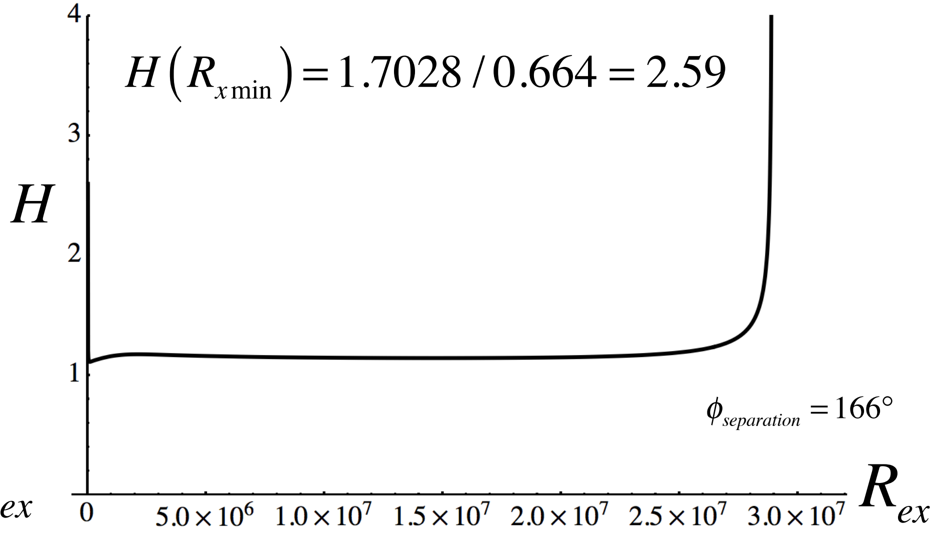




$$R_{\theta initial}=\left(\frac{0.664}{12}R_{cylinder}\right)^{1/2}$$

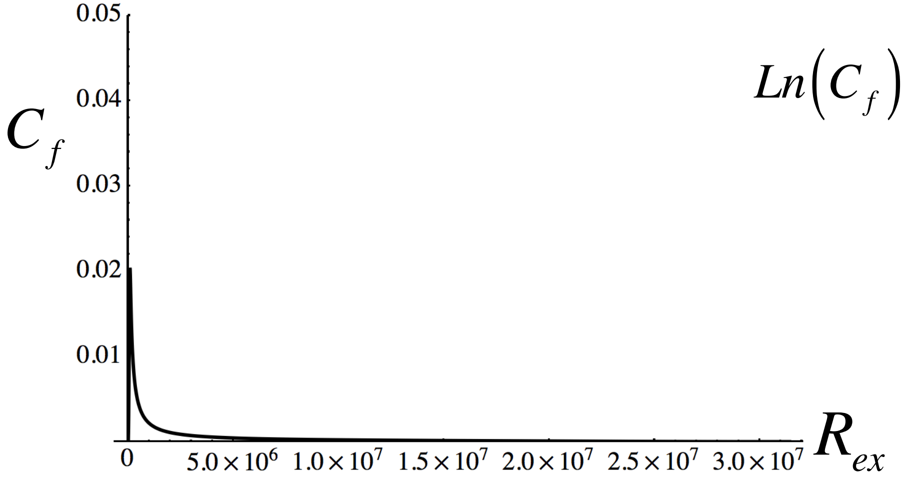
$$R_{ex\min}=10{,}000$$

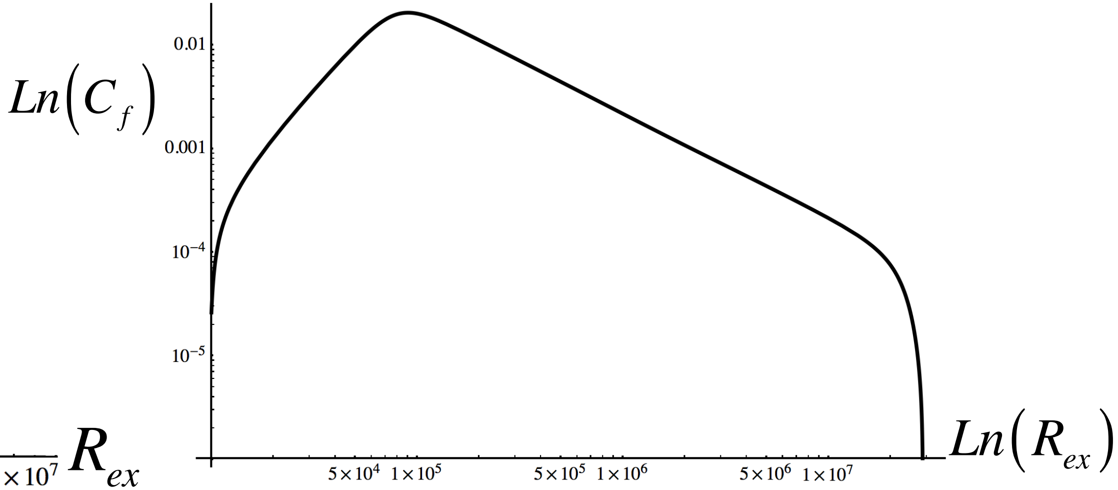
$$\boxed{R_{cylinder}=\frac{U_\infty R}{\nu}=10^7}$$



$$H\left(R_{x\min}\right)=1.7028/0.664=2.59$$

$$\phi_{separation}=166°$$





8.14 Problems

Problem 1 - Figure 8.45 depicts Couette flow of an ideal gas between two infinite parallel plates. The lower wall is adiabatic. Determine the entropy difference between the lower and upper walls.

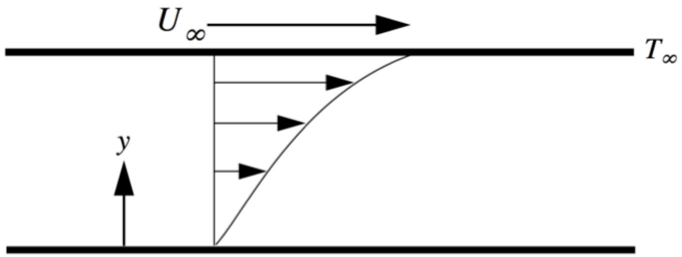


Figure 8.45: *Couette flow, adiabatic lower wall.*

Problem 2 - Figure 8.46 depicts Couette flow of helium gas between two infinite parallel walls spaced 1 cm apart. The lower wall is adiabatic and the speed of the upper wall is 400 meters/sec. The temperature of the upper wall is $300K$.

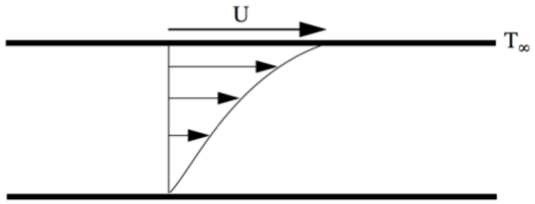


Figure 8.46: *Couette flow of helium, adiabatic lower wall.*

Assume the viscosity depends linearly on temperature.

$$\mu/\mu_\infty = T/T_\infty \tag{8.281}$$

Set up and solve the compressible flow equations for this simple flow. Note that the flow is assumed to be steady and all flow variables depend only on the coordinate normal to the wall.

1) Determine the speed of sound at the upper wall.

2) Determine the temperature of the lower wall.

3) Determine the shear stress.

4) Is there work done on the flow? How much?

5) Determine the heat flux through the upper wall.

6) Sketch the distribution of stagnation temperature across the channel.

7) Sketch the distribution of entropy across the channel.

Problem 3 - Figure 8.47 depicts Couette flow of a gas between two infinite parallel walls spaced a distance d apart. The lower wall is adiabatic. The reference Mach number is $M_\infty = U_\infty / \sqrt{\gamma R T_\infty}$. The viscosity is assumed to depend linearly on temperature $\mu/\mu_\infty = T/T_\infty$ and the reference Reynolds number is $R_e = \rho_\infty U_\infty d/\mu_\infty$.

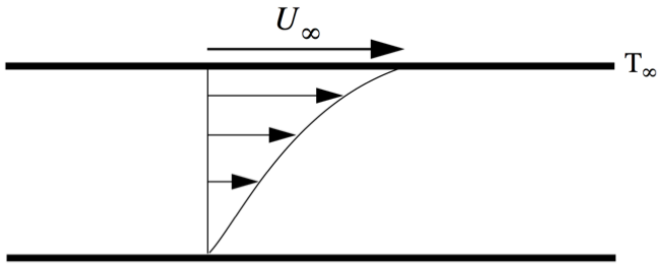


Figure 8.47: *Couette flow of an ideal gas, adiabatic lower wall.*

Sketch how the friction coefficient C_f depends on U_∞. At what Mach number is the friction coefficient an extremum? Is it a maximum or a minimum? Express your answer in terms of γ and the Prandtl number. What are the values for helium and air?

Problem 4 - Figure 8.48 shows the unsteady flow produced by a flat plate set into motion impulsively at velocity U_∞.

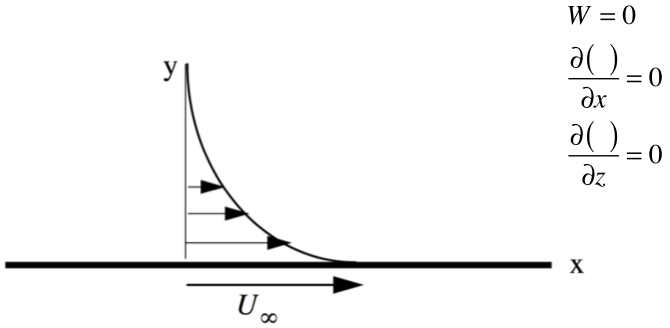


Figure 8.48: *Impulsively started flat plate.*

Note that in the compressible case there is frictional heating at the wall that will result in a non-zero V component at the wall.

The plate extends to infinity in both directions and the flow is perfectly parallel. Simplify the compressible flow equations. Solve for the velocity and vorticity in the incompressible case.

Problem 5 - In the discussion of boundary layers we considered several definitions of the thickness. How would you define a thickness based on the vorticity distribution? What might be the advantage of such a definition?

Problem 6 - Use the Howarth-Stewartson transformation to generate the velocity and temperature profiles in a laminar, compressible, zero pressure-gradient boundary layer at a free stream Mach number $M_\infty = 8$.

Rankine Oval

Problem 7 - The 2-D stream function for potential flow over an ~~elliptically shaped~~ body at zero angle of attack is produced by the superposition of a uniform flow plus a source and a sink of equal strength. The stream function and flow pattern are

$$\Psi = U_\infty y + \frac{Q}{2\pi} ArcTan\left(\frac{y}{x+a}\right) - \frac{Q}{2\pi} ArcTan\left(\frac{y}{x-a}\right) \qquad (8.282)$$

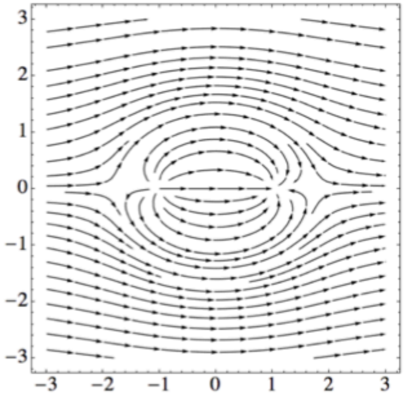


Figure 8.49: *Potential flow over a 2-D elliptical body.*

Choose two aspect ratios for the ellipse (length/width = 2, length/width = 20).

1) Use the potential flow solution to determine the pressure coefficient on the body.

2) Use Thwaites' method to calculate the properties of the laminar boundary layer up to separation. How does the separation point depend on the aspect ratio of the ellipse? Use the radius of curvature at the forward stagnation point to initiate the calculation.

3) Use Head's method to do the same for a turbulent boundary layer. For a given aspect ratio how does the separation point depend on the Reynolds number based on the length of the body?