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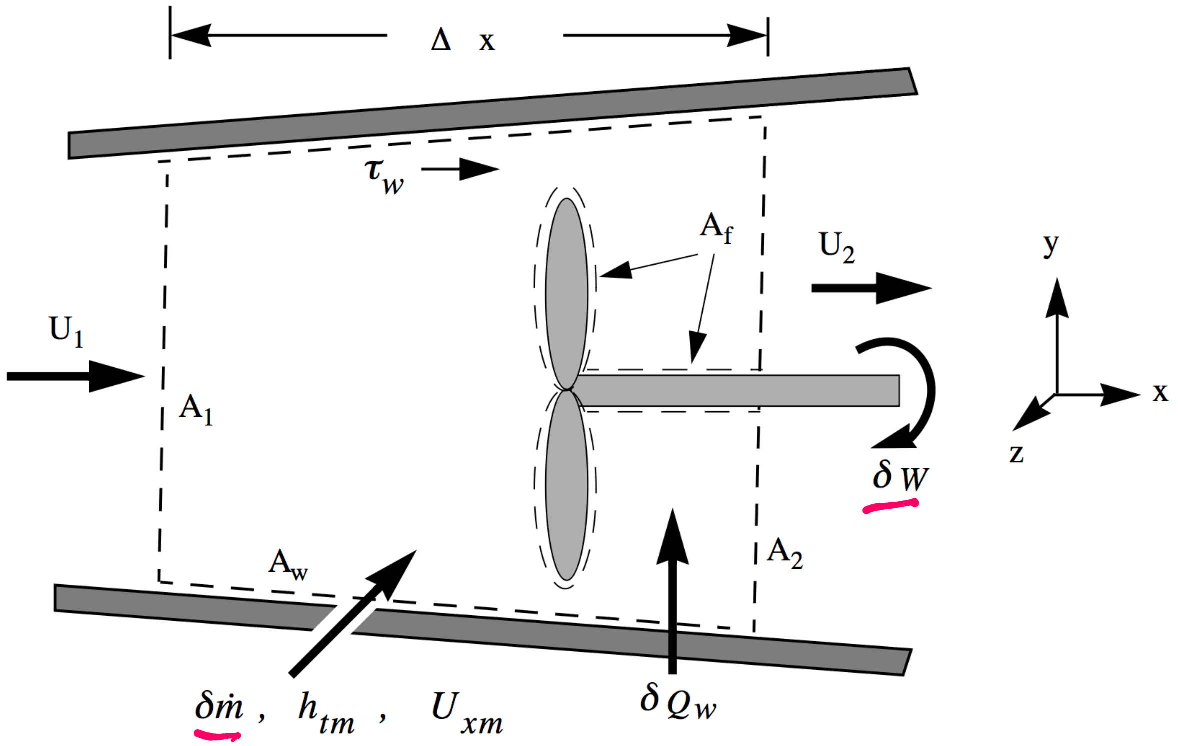
AA210A

Fundamentals of Compressible Flow

Chapter 9 - Quasi-one-dimensional flow

9.1 Control volume, integral conservation laws

Eulerian-Lagrangian control volume



Assume the time averaged flow is stationary. Note that the fan generates an unsteady, periodic flow. If we average flow variables over many full cycles of the fan rotation then the unsteady term in the conservations equations can be dropped. **The time mean is said to be stationary, that is, independent of the averaging interval.**

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9.1.1 Conservation of mass

$$\int_{A(t)} \rho \bar{U} \bullet \bar{n} dA = \int_{A_2} \rho_2 U_2 dA - \int_{A_1} \rho_1 U_1 dA + \int_{A_w} \rho \bar{U} \bullet \bar{n} dA + \int_{A_f(t)} \rho \bar{U} \bullet \bar{n} dA = 0$$

Total mass added to the control volume

$$\int_{A_w} \rho \bar{U} \bullet \bar{n} dA = -\delta \dot{m} .$$

No mass is added through the fan surface. The integrated form of the law of mass conservation is

$$\int_{A_2} \rho_2 U_2 dA - \int_{A_1} \rho_1 U_1 dA = \delta \dot{m} .$$

9.1.2 Conservation of x-momentum

$$\left.\int_A (\rho\bar{U}\bar{U} + P\bar{\bar{\delta}} - \bar{\bar{\tau}}) \bullet \bar{n}dA\right|_x =$$

$$\int_{A_2} (\rho_2 U_2 U_2 + P_2 - \tau_{xx2})dA - \int_{A_1} (\rho_1 U_1 U_1 + P_1 - \tau_{xx1})dA + \qquad (8.4)$$

$$\left.\int_{A_w} (\rho\bar{U}\bar{U} + P\bar{\bar{\delta}} - \bar{\bar{\tau}}) \bullet \bar{n}dA\right|_x + \left.\int_{A_f(t)} (\rho\bar{U}(\underbrace{\bar{U} - \bar{U}_A}_{0}) + P\bar{\bar{\delta}} - \bar{\bar{\tau}}) \bullet \bar{n}dA\right|_x = 0$$

The added mass may carry x-momentum

$$U_{xm}\delta\dot{m}$$

The fan exerts a force on the flow.

$$\left.\int_A (\rho\bar{U}\bar{U} + P\bar{\bar{\delta}} - \bar{\bar{\tau}}) \bullet \bar{n}dA\right|_x =$$

$$\int_{A_2} (\rho_2 U_2 U_2 + P_2 - \tau_{xx2})dA - \int_{A_1} (\rho_1 U_1 U_1 + P_1 - \tau_{xx1})dA +$$

$$\left.\int_{A_w} (P\bar{\bar{\delta}} - \bar{\bar{\tau}}) \bullet \bar{n}dA\right|_x - U_{xm}\delta\dot{m} + \delta F_x = 0$$

Where the force by the flow on the fan is.

$$\delta F_x = \int_{A_f(t)} (P\bar{\bar{\delta}} - \bar{\bar{\tau}}) \bullet \bar{n}\, dA \Bigg|_x .$$

9.1.3 Conservation of energy

$$\int_A (\rho h_t \bar{U} - \bar{\bar{\tau}} \bullet \bar{U} + \bar{Q}) \bullet \bar{n} dA =$$

$$\int_{A_2} (\rho_2 h_{t2} U_2 - \tau_{xx2} U_2 + Q_2) dA - \int_{A_1} (\rho_1 h_{t1} U_1 - \tau_{xx1} U_1 + Q_1) dA +$$

$$\int_{A_w} (\rho h_t \bar{U} - \bar{\bar{\tau}} \bullet \bar{U} + \bar{Q}) \bullet \bar{n} dA +$$

$$\int_{A_f(t)} (\rho(e + k)(\bar{U} - \cancelto{0}{\bar{U}_A}) + P\bar{U} - \bar{\bar{\tau}} \bullet \bar{U}) \bullet \bar{n} dA = 0$$

The injected mass carries its stagnation enthalpy with it.

$$\int_{A_w} (\rho h_t \bar{U} - \bar{\bar{\tau}} \bullet \bar{U} + \bar{Q}) \bullet \bar{n} dA = -\delta Q - h_{tm} \delta \dot{m} .$$

The last term in the energy equation is the work done by the flow on the fan.

$$\delta W = \int_{A_f(t)} (P\bar{U} - \bar{\bar{\tau}} \bullet \bar{U}) \bullet \bar{n} dA$$

The integrated energy conservation equation is

$$\int_{A_2} (\rho_2 h_{t_2} U_2 - \tau_{xx2} U_2 + Q_2) dA - \int_{A_1} (\rho_1 h_{t_1} U_1 - \tau_{xx1} U_1 + Q_1) dA =$$

$$\delta Q + h_{tm} \delta \dot{m} - \delta W$$

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9.2 Area averaged flow

Average the flow across the channel.

$$\hat{\rho}(x) = \frac{1}{A(x)}\int \rho(x, y, z)dydz .$$

Define

$$\hat{T}(x) = \frac{1}{A(x)}\int T(x, y, z)dydz$$

$$\hat{P}(x) = \frac{1}{A(x)}\int P(x, y, z)dydz$$

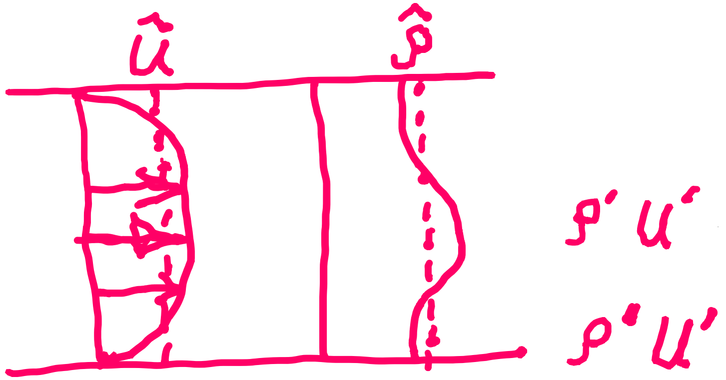
$$\hat{s}(x) = \frac{1}{A(x)}\int s(x, y, z)dydz$$

$$\hat{U}(x) = \frac{1}{A(x)}\int U(x, y, z)dydz$$

$$\hat{\tau}_{xx}(x) = \frac{1}{A(x)}\int \tau_{xx}(x, y, z)dydz$$

$$\hat{Q}_x(x) = \frac{1}{A(x)}\int Q_x(x, y, z)dydz$$

Every variable in the flow can be written as a mean plus a fluctuation.



$$\rho(x,y,z) = \hat{\rho}(x) + \rho'(x,y,z)$$

$$T(x,y,z) = \hat{T}(x) + T'(x,y,z)$$

$$P(x,y,z) = \hat{P}(x) + P'(x,y,z)$$

$$s(x,y,z) = \hat{s}(x) + s'(x,y,z)$$

$$U(x,y,z) = \hat{U}(x) + U'(x,y,z)$$

$$\tau_{xx}(x,y,z) = \hat{\tau}_{xx}(x) + \tau_{xx}'(x,y,z)$$

$$Q_x(x,y,z) = \hat{Q}_x(x) + Q_x'(x,y,z)$$

Express the mass flux integral in terms of means and fluctuations.

$$\int_A \rho U dA = \int_A (\hat{\rho} + \rho')(\hat{U} + U')dA = \int_A \hat{\rho}\hat{U}dA + \int_A \rho' U' dA =$$

$$\hat{\rho}(x)\hat{U}(x)A(x) + \overline{\rho' U'}A(x)$$

As long as nonlinear correlations are small, the mean is an accurate approximation.

In terms of area averaged variables, the integral equations of motion are as follows.

$$\hat{\rho}_2 \hat{U}_2 A_2 - \hat{\rho}_1 \hat{U}_1 A_1 = \delta \dot{m}$$

$$(\hat{\rho}_2 \hat{U}_2 \hat{U}_2 + \hat{P}_2 - \hat{\tau}_{xx2}) A_2 - (\hat{\rho}_1 \hat{U}_1 \hat{U}_1 + \hat{P}_1 - \hat{\tau}_{xx1}) A_1 +$$

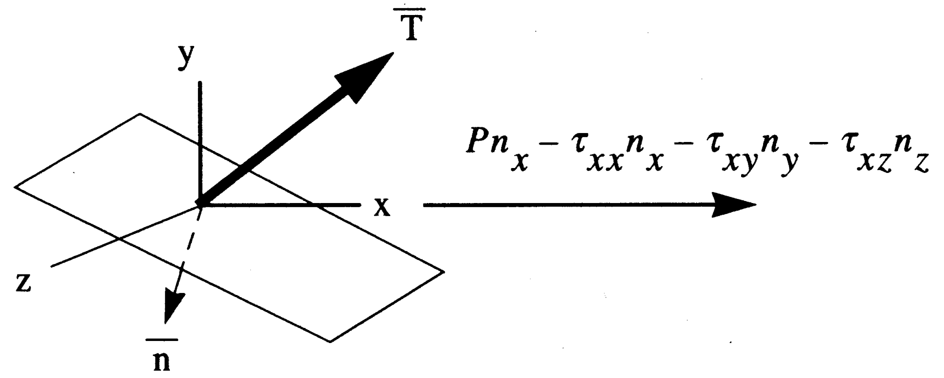
$$\left. \int_{A_w} (P\bar{\bar{\delta}} - \bar{\bar{\tau}}) \bullet \bar{n}\, dA \right|_x - U_{xm} \delta \dot{m} + \delta F_x = 0$$

$$(\hat{\rho}_2 \hat{H}_2 \hat{U}_2 - \hat{\tau}_{xx2} \hat{U}_2 + \hat{Q}_2) A_2 - (\hat{\rho}_1 \hat{H}_1 \hat{U}_1 - \hat{\tau}_{xx1} \hat{U}_1 + \hat{Q}_1) A_1 =$$

$$\delta Q + h_{tm} \delta \dot{m} - \delta W$$

9.2.1 The traction vector

The pressure-stress integral on the wall.



The traction vector

$$\bar{T} = (P\bar{\bar{\delta}} - \bar{\bar{\tau}}) \bullet \bar{n} = \begin{bmatrix} Pn_x - \tau_{xx}n_x - \tau_{xy}n_y - \tau_{xz}n_z \\ -\tau_{xy}n_x + Pn_y - \tau_{yy}n_y - \tau_{yz}n_z \\ -\tau_{zx}n_z - \tau_{zy}n_y + Pn_z - \tau_{zz}n_z \end{bmatrix}$$

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Imagine the length of the control volume made very small.

$$\left.\int_{A_w}(P\bar{\bar{\delta}}-\bar{\bar{\tau}})\bullet\bar{n}\,dA\right|_x = \int_{A_w}(Pn_x-\tau_{xx}n_x-\tau_{xy}n_y-\tau_{xz}n_z)dA \cong$$

$$\left(\frac{P_1+P_2}{2}\right)(A_1-A_2)-\left(\frac{\tau_{xx1}+\tau_{xx2}}{2}\right)(A_1-A_2)+\tau_w\pi\left(\frac{D_1+D_2}{2}\right)\Delta x$$

where

$$\int_{A_w} n_x dA = (A_1-A_2)$$

and

$$\int_{A_w}(\tau_{xy}n_y+\tau_{xz}n_z)dA \equiv -\tau_w\pi\left(\frac{D_1+D_2}{2}\right)\Delta x.$$

Introduce the hydraulic diameter

$$D = \left(\frac{4A}{\pi}\right)^{1/2}.$$

The integrated equations of motion now take the form

$$\rho_2 U_2 A_2 - \rho_1 U_1 A_1 = \delta\dot{m}$$

$$(\rho_2 U_2 U_2 + P_2 - \tau_{xx2})A_2 - (\rho_1 U_1 U_1 + P_1 - \tau_{xx1})A_1 -$$
$$\left(\frac{P_1 + P_2}{2}\right)(A_2 - A_1) + \left(\frac{\tau_{xx1} + \tau_{xx2}}{2}\right)(A_2 - A_1) + \tau_w \pi \left(\frac{D_1 + D_2}{2}\right)\Delta x =$$
$$U_{xm}\delta\dot{m} - \delta F_x$$

$$(\rho_2 h_{t2} U_2 - \tau_{xx2} U_2 + Q_{x2})A_2 - (\rho_1 h_{t1} U_1 - \tau_{xx1} U_1 + Q_{x1})A_1 =$$
$$\delta Q + h_{tm}\delta\dot{m} - \delta W$$

where the “hat” has been dropped.

Let the length of the control volume go to zero.

$$\rho_2 U_2 A_2 - \rho_1 U_1 A_1 \quad \Rightarrow \quad d(\rho U A)$$

$$\rho_2 U_2^2 A_2 - \rho_1 U_1^2 A_1 \quad \Rightarrow \quad d(\rho U^2 A)$$

$$P_2 A_2 - P_1 A_1 \quad \Rightarrow \quad d(PA)$$

$$\tau_{xx2} A_2 - \tau_{xx1} A_1 \quad \Rightarrow \quad d(\tau_{xx} A)$$

$$\left(\frac{P_1 + P_2}{2}\right)(A_2 - A_1) \quad \Rightarrow \quad PdA$$

$$\left(\frac{\tau_{xx1} + \tau_{xx2}}{2}\right)(A_2 - A_1) \quad \Rightarrow \quad \tau_{xx} dA$$

$$\tau_w \pi \left(\frac{D_1 + D_2}{2}\right)\Delta x \quad \Rightarrow \quad \tau_w \pi D dx$$

$$\rho_2 U_2 h_{t2} A_2 - \rho_1 U_1 h_{t1} A_1 \quad \Rightarrow \quad d(\rho U h_t A)$$

$$\tau_{xx2} U_2 A_2 - \tau_{xx1} U_1 A_1 \quad \Rightarrow \quad d(\tau_{xx} U A)$$

$$Q_{x2} A_2 - Q_{x1} A_1 \quad \Rightarrow \quad d(Q_x A)$$

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The integrated equations are now expressed in terms of differentials.

$$d(\rho UA) = \delta\dot{m}$$

$$d(\rho U^2 A) + d(PA) - d(\tau_{xx}A) - PdA + \tau_{xx}dA =$$
$$-\tau_w \pi D dx + U_{xm}\delta\dot{m} - \delta F_x$$

$$d(\rho U h_t A) + (-d(\tau_{xx}UA)) + d(Q_x A) = \delta Q + h_{tm}\delta\dot{m} - \delta W$$

Use continuity to simplify the momentum and energy equations.

$$U\delta\dot{m} + \rho UAdU + AdP - Ad\tau_{xx} = -\tau_w \pi D dx + U_{xm}\delta\dot{m} - \delta F_x$$

$$h_t\delta\dot{m} + \rho UAdh_t - \frac{\tau_{xx}}{\rho}\delta\dot{m} - \rho UAd\left(\frac{\tau_{xx}}{\rho}\right) + \frac{Q_x}{\rho U}\delta\dot{m} + \rho UAd\left(\frac{Q_x}{\rho U}\right) =$$
$$\delta Q + h_{tm}\delta\dot{m} - \delta W$$

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The 1-D (area averaged) equations of motion.

$$d(\rho U A) = \delta\dot{m}$$

$$d(P-\tau_{xx}) + \rho U dU = -\tau_w\left(\frac{\pi D dx}{A}\right) + \frac{(U_{xm}-U)\delta\dot{m}}{A} - \frac{\delta F_x}{A}$$

$$d\left(h_t - \frac{\tau_{xx}}{\rho} + \frac{Q_x}{\rho U}\right) = \frac{\delta Q}{\rho U A} - \frac{\delta W}{\rho U A} + \left(h_{tm} - \left(h_t - \frac{\tau_{xx}}{\rho} + \frac{Q_x}{\rho U}\right)\right)\frac{\delta\dot{m}}{\rho U A}$$

Introduce the friction coefficient.

$$C_f = \frac{\tau_w}{\frac{1}{2}\rho U^2}$$

and the heat and work per unit mass flow

$$\delta q = \frac{\delta Q}{\rho U A} \quad ; \qquad \delta w = \frac{\delta W}{\rho U A}.$$

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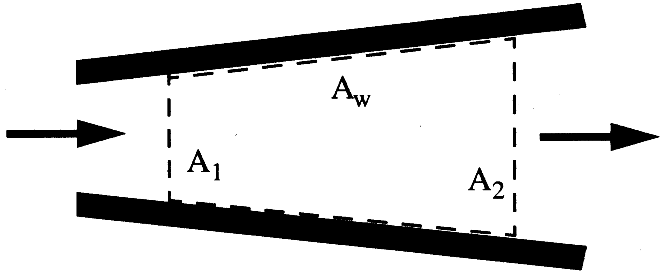
Finally the area averaged equations of motion take the concise form

$$\boxed{\begin{aligned} d(\rho U A) &= \delta\dot{m} \\ d(P-\tau_{xx}) + \rho U dU &= -\frac{1}{2}\rho U^2\left(4C_f\frac{dx}{D}\right) + \frac{(U_{xm}-U)\delta\dot{m}}{A} - \frac{\delta F_x}{A} \\ d\left(h_t - \frac{\tau_{xx}}{\rho} + \frac{Q_x}{\rho U}\right) &= \delta q - \delta w + \left(h_{tm} - \left(h_t - \frac{\tau_{xx}}{\rho} + \frac{Q_x}{\rho U}\right)\right)\frac{\delta\dot{m}}{\rho U A} \end{aligned}}$$

9.2.2 Steady, gravity-free, adiabatic flow of a compressible fluid in a channel



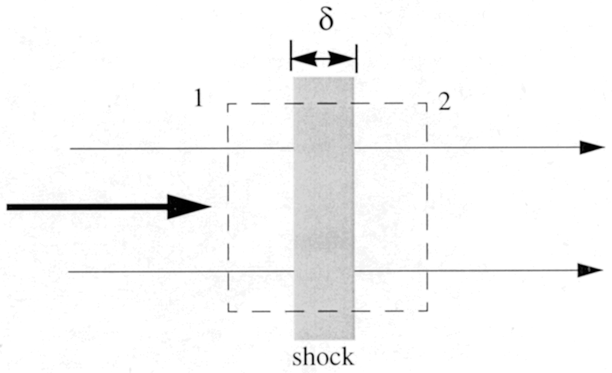
For this case the energy equation takes the form of a perfect differential.

$$d\left(h_t - \frac{\tau_{xx}}{\rho} + \frac{Q_x}{\rho U}\right) = 0.$$

For most flow situations (outside of shock waves) the stress and heat conduction terms can be neglected. Thus

$$h_{t2} = h_{t1}.$$

9.3 Normal shock waves



The equations of motion reduce to a set of perfect differentials.

$$\left.\begin{aligned} d(\rho U) &= 0 \\ d(P - \tau_{xx} + \rho U^2) &= 0 \\ d\left(h_t - \frac{\tau_{xx}}{\rho} + \frac{Q_x}{\rho U}\right) &= 0 \end{aligned}\right\}$$

Each equation generates a conserved quantity.

$$\left.\begin{aligned} \rho U &= constant1 \\ P - \tau_{xx} + \rho U^2 &= constant2 \\ h_t - \frac{\tau_{xx}}{\rho} + \frac{Q_x}{\rho U} &= constant3 \end{aligned}\right\}$$

Equate conditions at states 1 and 2.

$$\left.\begin{aligned} \rho_1 U_1 &= \rho_2 U_2 \\ P_1 - \tau_{xx1} + \rho_1 {U_1}^2 &= P_2 - \tau_{xx2} + \rho_2 {U_2}^2 \\ h_{t1} - \frac{\tau_{xx1}}{\rho_1} + \frac{Q_{x1}}{\rho_1 U_1} &= h_{t2} - \frac{\tau_{xx2}}{\rho_2} + \frac{Q_{x2}}{\rho_2 U_2} \end{aligned}\right\}$$

For a Newtonian fluid

$$\tau_{xx} = \left(\frac{4}{3}\mu + \mu_v\right)\frac{\partial U}{\partial x}$$

and

$$Q_x = -k\frac{\partial T}{\partial x}.$$

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Now assume uniform flow at stations 1 and 2. That is assume that the velocity and temperature gradients are zero ahead of and behind the shock wave.

The classical shock jump conditions are:

$$\boxed{\begin{aligned} &\rho_1 U_1 = \rho_2 U_2 \\ &P_1 + \rho_1 {U_1}^2 = P_2 + \rho_2 {U_2}^2 \\ &h_{t1} = h_{t2} \end{aligned}}$$

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9.3.1 The Rankine Hugoniot relations

The jump conditions can be combined to produce a relationship between pressure and density in which the velocity does not appear.

Note that this relationship makes no assumption about the material in which the shock is propagating. It could be a gas, it could be water, it could be rock or some other continuous material.

Combine mass and momentum.

$$U_1 U_2 = \frac{P_2 - P_1}{\rho_2 - \rho_1}$$

The energy jump condition for a calorically perfect gas is

Equate

$$\left(\frac{\gamma}{\gamma - 1}\right)\frac{P_1}{\rho_1} + \frac{1}{2}{U_1}^2 = \left(\frac{\gamma}{\gamma - 1}\right)\frac{P_2}{\rho_2} + \frac{1}{2}{U_2}^2$$

Combine mass and energy

$$U_1 U_2 = \left(\frac{2\gamma}{\gamma - 1}\right)\frac{P_2\rho_1 - P_1\rho_2}{(\rho_2 - \rho_1)(\rho_2 + \rho_1)}$$

The Rankine-Hugoniot equation.

$$\boxed{\frac{P_2}{P_1} = \frac{\frac{\gamma+1}{\gamma-1}\left(\frac{\rho_2}{\rho_1}\right) - 1}{\frac{\gamma+1}{\gamma-1} - \left(\frac{\rho_2}{\rho_1}\right)}}$$

9.3.2 Shock property ratios in a calorically perfect gas

The velocity ratio

Define a reference flow state where the flow velocity equals the speed of sound.

$$C_pT_1 + \frac{1}{2}{U_1}^2 = \frac{\gamma+1}{2(\gamma-1)}{a^*}^2$$

$$C_pT_2 + \frac{1}{2}{U_2}^2 = \frac{\gamma+1}{2(\gamma-1)}{a^*}^2$$

Use the ideal gas law to write the last pair of relations as

$$\frac{\gamma}{\gamma-1}P_1 + \frac{1}{2}\rho_2U_2U_1 = \frac{\gamma+1}{2(\gamma-1)}\rho_1{a^*}^2$$

$$\frac{\gamma}{\gamma-1}P_2 + \frac{1}{2}\rho_1U_2U_1 = \frac{\gamma+1}{2(\gamma-1)}\rho_2{a^*}^2$$

Subtract

$$\frac{\gamma}{\gamma-1}(P_2-P_1) - \frac{1}{2}(\rho_2-\rho_1)U_1U_2 = \frac{\gamma+1}{2(\gamma-1)}{a^*}^2(\rho_2-\rho_1)$$

$$\boxed{U_1U_2 = \frac{P_2-P_1}{\rho_2-\rho_1}}$$

The velocity ratio (continued)

Replace the pressure and cancel the density. The result is the Prandtl relation.

$$U_1U_2 = a^{*2}$$

Now.

$$C_pT_1 + \frac{1}{2}{U_1}^2 = \frac{\gamma+1}{2(\gamma-1)}U_1U_2$$

Or

$$\frac{U_2}{U_1} = \frac{2}{\gamma+1}\left(\frac{\gamma RT_1}{{U_1}^2}\right) + \left(\frac{\gamma-1}{\gamma+1}\right)$$

Finally

$$\boxed{\frac{U_2}{U_1} = \frac{1+\left(\frac{\gamma-1}{2}\right){M_1}^2}{\left(\frac{\gamma+1}{2}\right){M_1}^2} = \frac{\rho_1}{\rho_2}}$$

With this result, all of the important properties of a shock wave can be expressed in terms of the upstream Mach number.

Velocity ratio

$$\boxed{\frac{U_2}{U_1} = \frac{1 + \left(\frac{\gamma-1}{2}\right) {M_1}^2}{\left(\frac{\gamma+1}{2}\right) {M_1}^2} = \frac{\rho_1}{\rho_2}}$$

Shock strength

$$\frac{P_2}{P_1} = \frac{\frac{2\gamma}{\gamma-1} {M_1}^2 - 1}{\frac{\gamma+1}{\gamma-1}}$$

Temperature jump

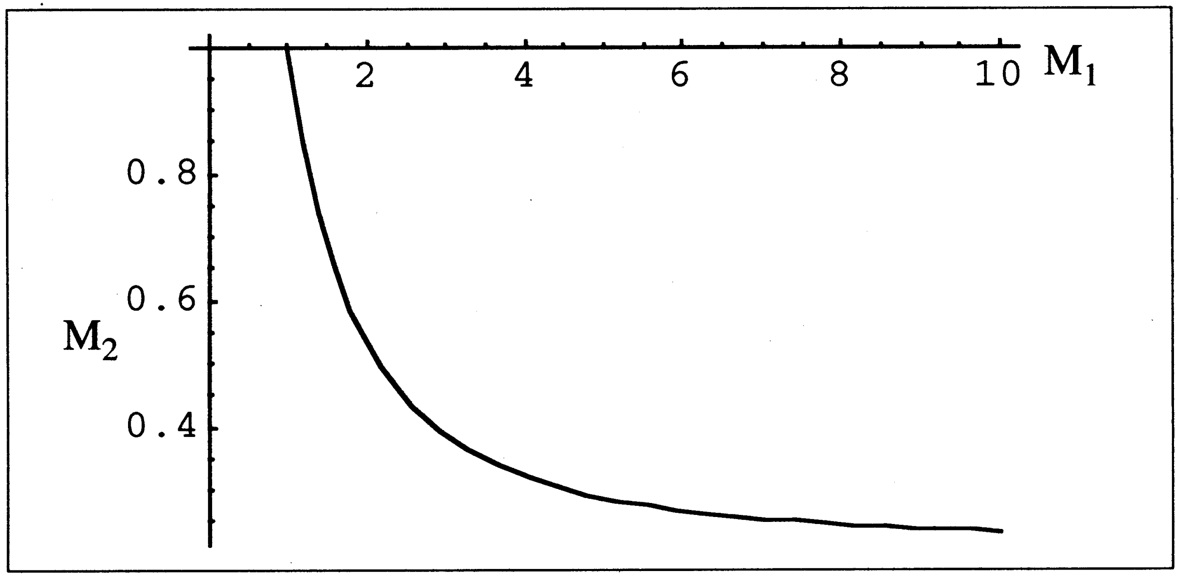
$$\frac{T_2}{T_1} = \left(\frac{P_2}{P_1}\right)\left(\frac{\rho_1}{\rho_2}\right) = \left(\frac{\gamma {M_1}^2 - \left(\frac{\gamma-1}{2}\right)}{\frac{\gamma+1}{2}}\right)\left(\frac{1 + \left(\frac{\gamma-1}{2}\right) {M_1}^2}{\left(\frac{\gamma+1}{2}\right) {M_1}^2}\right)$$

Downstream Mach number

$$\left(\frac{M_2}{M_1}\right)^2 = \left(\frac{U_2}{U_1}\right)^2 \frac{T_1}{T_2}$$

The downstream Mach number

$$M_2{}^2 = \left(\frac{1 + \left(\frac{\gamma-1}{2}\right) M_1{}^2}{\gamma M_1{}^2 - \left(\frac{\gamma-1}{2}\right)} \right)$$



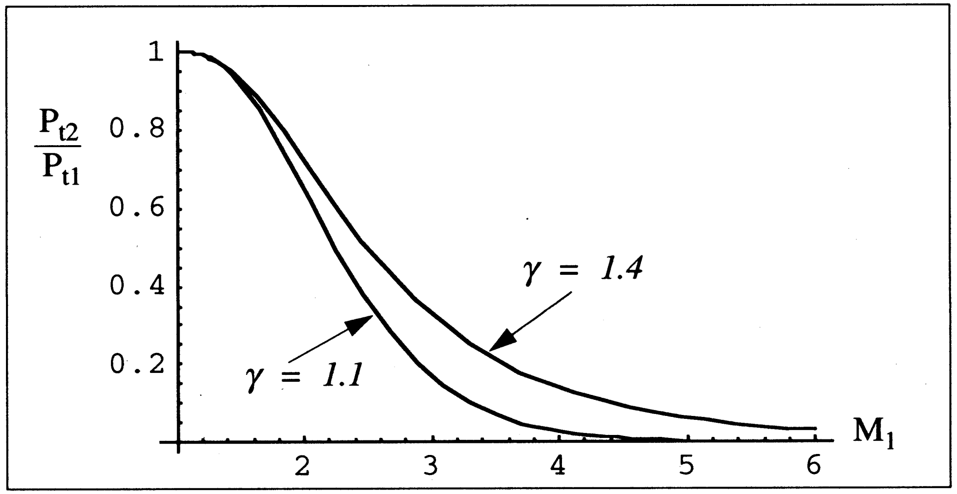
$$\lim_{M_1 \to \infty} M_2 = \sqrt{(\gamma - 1)/(2\gamma)}$$

Stagnation pressure ratio

$$\frac{P_{t2}}{P_{t1}} = \left(\frac{\frac{\gamma+1}{\gamma-1}}{\frac{2\gamma}{\gamma-1} {M_1}^2 - 1} \right)^{\frac{1}{\gamma-1}} \left(\frac{\left(\frac{\gamma+1}{2}\right) {M_1}^2}{1 + \left(\frac{\gamma-1}{2}\right) {M_1}^2} \right)^{\frac{\gamma}{\gamma-1}}$$

At high Mach numbers

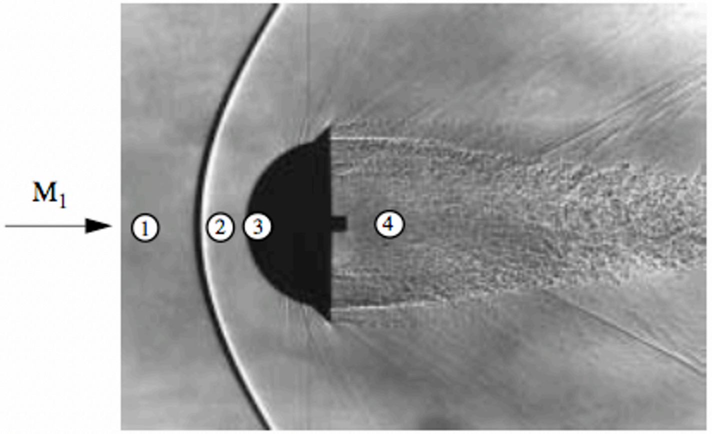
$$\lim_{M_1 \to \infty} \frac{P_{t2}}{P_{t1}} = \left(\frac{\gamma + 1}{2\gamma M_1^2} \right)^{\frac{1}{\gamma - 1}}.$$



Entropy change

$$\frac{P_{t2}}{P_{t1}} = e^{-\left(\frac{s_2 - s_1}{R}\right)}.$$

Problem 10 - The figure below shows supersonic flow of air over a model of a re-entry body at a free stream Mach number, $M_1 = 2$.



The temperature of the free stream is $300°K$ and the pressure is one atmosphere.

1) Determine the stagnation temperature and pressure of a fluid element located at stations 1 (free stream), 2 (just behind the shock) and 3 (at the stagnation point on the body). State the assumptions used to solve the problem. Express your answer in °K and atmospheres.

Solution - The stagnation temperature of the free stream is determined from

$$\frac{T_{t1}}{T_1} = 1 + \left(\frac{\gamma - 1}{2}\right) M_1^2 = 1 + 0.2 \times 4 = 1.8 \text{ (1 point)}$$

Therefore

$$T_{t1} = 1.8 \times 300 = 540°K \text{ (1 point)}$$

If we assume the flow up to the stagnation point is adiabatic and the heat capacities are constant then the stagnation temperature is the same at stations 1, 2 and 3,

$$T_{t1} = T_{t2} = T_{t3}. \text{ (1 point)}$$

The stagnation pressure is derived from

$$\frac{P_{t1}}{P_1} = \left(\frac{T_{t1}}{T_1}\right)^{\frac{\gamma}{\gamma - 1}} = 1.8^{3.5} = 7.824 \text{ (1 point)}$$

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The freestream stagnation pressure is $P_{t1} = 7.824\ atmospheres$.

Across the shock the stagnation pressure drops according to

$$\frac{P_{t2}}{P_{t1}} = \left(\frac{\frac{\gamma+1}{\gamma-1}}{\frac{2\gamma}{\gamma-1}M_1^2 - 1}\right)^{\frac{1}{\gamma-1}}\left(\frac{\left(\frac{\gamma+1}{2}\right)M_1^2}{1+\frac{\gamma-1}{2}M_1^2}\right)^{\frac{\gamma}{\gamma-1}} = 0.721 \ \textbf{(1 point)}$$

The stagnation pressure behind the shock is

$$P_{t2} = 0.721 \times 7.824 = 5.64\ atmospheres\,.\ \textbf{(1 point)}$$

It is reasonable to assume that the flow from 2 to 3 is adiabatic and isentropic and so one may expect

$$P_{t3} = P_{t2} = 5.64\ atmospheres \ \textbf{(1 point)}$$

2) What can you say about the state of the gas at point 4?

Solution - The flow near the back of the re-entry body is at a low pressure and nearly zero velocity. The path from state 1 to state 4 involves large thermal and velocity gradients leading to an entropy increase and loss of stagnation pressure and very likely a drop in stagnation enthalpy. So we would expect both the pressure and temperature at station 4 to be lower than the free stream stagnation values. **(2 points)**

3) Refer the stagnation temperatures at 1, 2 and 3 to an observer at rest with respect to the upstream gas. To such an observer the body is moving to the left at a Mach number of 2.0.

Solution - In a frame of reference fixed with respect to the upstream gas, the gas velocity in region 1 is zero, $U_1' = 0$ and the stagnation temperature is

$$T_{t1}' = 300\,°K \text{. } \textbf{(1 point)}$$

The velocity of the body in this frame is

$$U_{body} = -M_1 \times \sqrt{\gamma R T_1} = -2 \times \sqrt{1.4 \times 287 \times 300} = -694.4\ M/sec$$

In the rest frame of the shock the velocity ratio across the shock is determined from

$$\frac{U_2}{U_1} = \frac{1 + \left(\frac{\gamma - 1}{2}\right) M_1^2}{\left(\frac{\gamma + 1}{2}\right) M_1^2} = 0.375$$

The gas velocity behind the shock is

$$U_2 = 0.375 \times 694.4 = 260.4\ M/sec \text{. } \textbf{(1 point)}$$

The gas velocity at station 2 referred to the rest frame of the upstream gas is

$$U_2' = -694.4 + 260.4 = -434\ M/sec$$

The temperature ratio across the shock is

$$\frac{T_2}{T_1} = 1.6875$$

The stagnation temperature at station 2 referred to the rest frame of the upstream gas is

$$T_{t2}' = T_2 + \frac{(U_2')^2}{2C_p} = 1.6875 \times 300 + \frac{(434)^2}{2 \times 1005} = 600K$$

and the stagnation temperature at station 3 referred to the rest frame of the upstream gas is

$$T_{t3}' = T_3 + \frac{(U_3')^2}{2C_p} = 540 + \frac{(694.4)^2}{2 \times 1005} = 779.9K$$

9.3.4 Example - stagnation at a leading edge in supersonic flow

The figure below shows a supersonic flow of Helium (atomic weight equals 4) over the leading edge of a thick flat plate at a free stream Mach number $M_1 = 2.0$.

The temperature of the free stream is $300\,K$ and the pressure is one atmosphere.

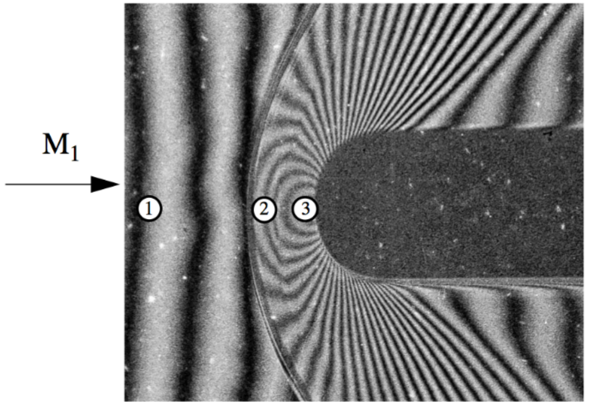

Figure 9.7: *Supersonic flow of helium over a leading edge.*

1) Determine the *energy* per unit mass of a fluid element located at points 1 (free stream), 2 (just behind the shock) and 3 (at the stagnation point on the body). State the assumptions used to solve the problem. Express your answer in Joules/kg.

Solution

The energy per unit mass of a flowing gas is the sum of internal energy and kinetic energy per unit mass, e + k .

a) Assume the gas is calorically perfect - constant heat capacities.

b) Assume the flow is adiabatic from station 1 to station 3.

c) Assume the body is adiabatic.

For Helium the number of degrees of freedom equals 3 and at the conditions of the free stream we have the following values.

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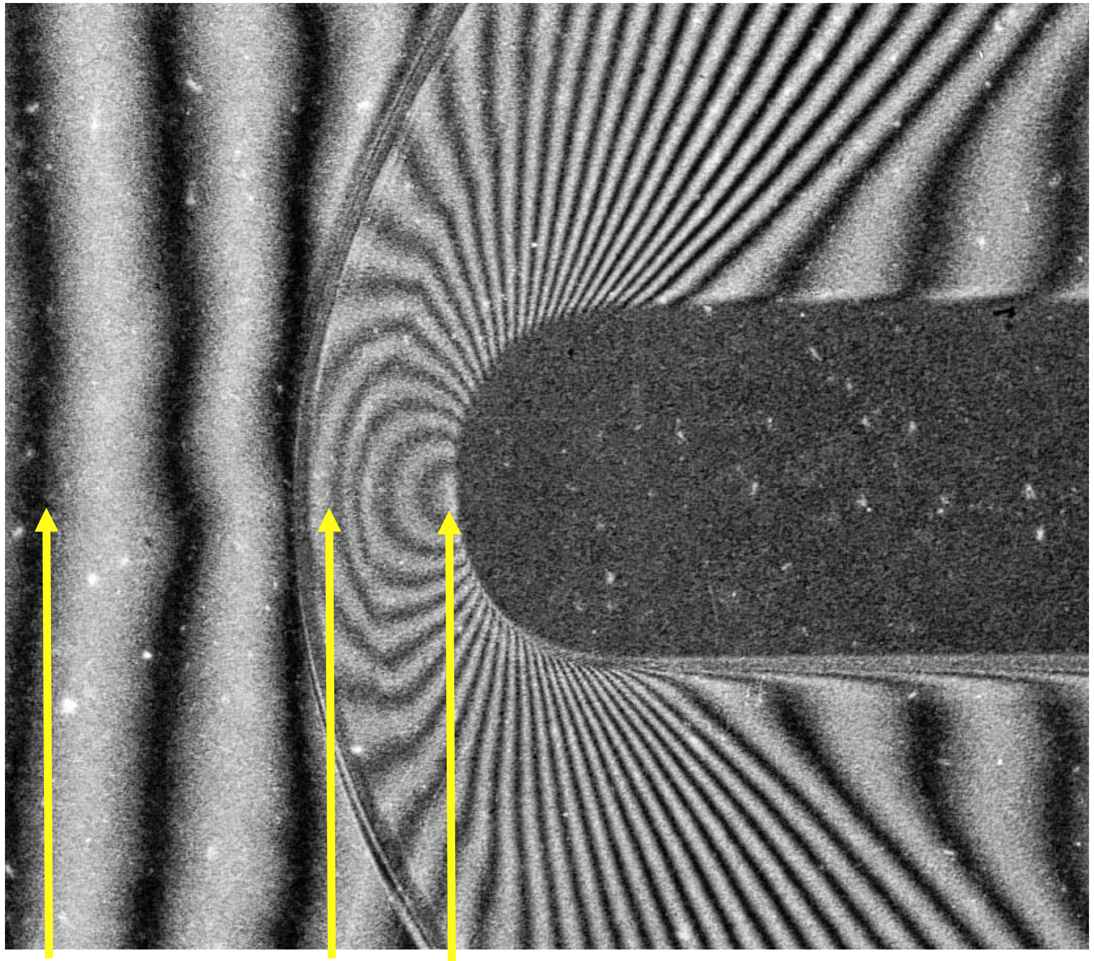
Figure 9.7 Helium at Mach 2.0



1 2 3

$$
\begin{aligned}
R &= \frac{8314.472}{4} = 2078.62\, m^2/\text{sec}^2 - K \\
C_v &= \frac{3}{2} R = 3117.93\, m^2/\text{sec}^2 - K \\
C_p &= \frac{3+2}{2} R = 5196.55\, m^2/\text{sec}^2 - K \\
\gamma &= 5/3 \\
a &= \sqrt{\gamma R T} = \sqrt{\frac{5}{3} 2078.62\,(300)} = 1019.46\, m^2/\text{sec}^2 \\
U_1 &= 2\,(1019.46) = 2038.92\, m/\text{sec} \\
e_1 + k_1 &= C_v T_1 + \frac{1}{2} {U_1}^2 = 3117.93\,(300) + 0.5 (2038.92)^2 = 935379 + 2078597 \\
e_1 + k_1 &= 3013976\, J/kg
\end{aligned}
\tag{9.70}
$$

The stagnation temperature of the free stream is determined from

$$
\frac{T_t}{T} = 1 + \left(\frac{\gamma - 1}{2}\right) M^2. \tag{9.71}
$$

Thus

$$
T_{t1} = 300 \left(1 + \frac{4}{3}\right) = 700\, K \tag{9.72}
$$

Across a normal shock at Mach 2 the temperature ratio is

$$
\frac{T_2}{T_1} = \frac{\left(1 + \left(\frac{\gamma-1}{2}\right) {M_1}^2\right) \left(\gamma {M_1}^2 - \left(\frac{\gamma-1}{2}\right)\right)}{\left(\frac{\gamma+1}{2}\right)^2 {M_1}^2} \tag{9.73}
$$

which gives

$$
\frac{T_2}{T_1} = \frac{\left(1 + \frac{4}{3}\right) \left(\frac{5}{3}(4) - \left(\frac{1}{3}\right)\right)}{\left(\frac{4}{3}\right)^2 (4)} = \frac{\left(\frac{7}{3}\right)\left(\frac{19}{3}\right)}{\left(\frac{4}{3}\right)\left(\frac{16}{3}\right)} = \frac{7}{4}\left(\frac{19}{16}\right) = 2.078. \tag{9.74}
$$

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Assume the flow is adiabatic from the free stream to the stagnation point.

$$h_1 + \frac{1}{2}{U_1}^2 = h_2 + \frac{1}{2}{U_2}^2 = h_3 + \frac{1}{2}{U_3}^2 \tag{9.75}$$

We can rewrite this equation as follows.

$$RT_1 + (e_1 + k_1) = RT_2 + (e_2 + k_2) = RT_3 + (e_3 + k_3) \tag{9.76}$$

The temperatures at stations 1, 2 and 3 are respectively

$$\begin{aligned} T_1 &= 300\,K \\ T_2 &= 2.078\,(300) = 623.44\,K \\ T_3 &= T_{t1} = 700\,K. \end{aligned} \tag{9.77}$$

Now

$$\begin{aligned} &e_1 + k_1 = 3.014 \times 10^6\;J/kg \\ &(e_2 + k_2) = (e_1 + k_1) - R\,(T_2 - T_1) = 3.014 \times 10^6 - 2078.62\,(623.44 - 300) \\ &= 3.014 \times 10^6 - 0.6723 \times 10^6 = 2.3417 \times 10^6\;J/kg \\ &(e_3 + k_3) = (e_1 + k_1) - R\,(T_3 - T_1) = 3.014 \times 10^6 - 2078.62\,(700 - 300) \\ &= 3.014 \times 10^6 - 0.8314 \times 10^6 = 2.1826 \times 10^6\;J/kg. \end{aligned} \tag{9.78}$$

The energy of a fluid element decreases considerably across the shock and then decreases further to the stagnation point.

2) Describe the mechanism by which the energy of the fluid element changes as it moves from station 1 to station 3.

The work done by the pressure and viscous normal force field on the fluid element is the mechanism by which the energy decreases in moving from station 1 to station 3. The flow energy decreases across the shock wave through a combination of pressure and viscous normal stress forces of roughly equal magnitude that act to compress the fluid element increasing its internal energy while decelerating it and reducing its kinetic energy. The loss of kinetic energy dominates the increase in internal energy.

Between stations 2 and 3 the flow further decelerates as the pressure increases toward the stagnation point. Viscous normal forces also act in region 2 to 3 but because the streamwise velocity gradients are small (compared to the shock) viscous forces are generally much smaller than the pressure forces.

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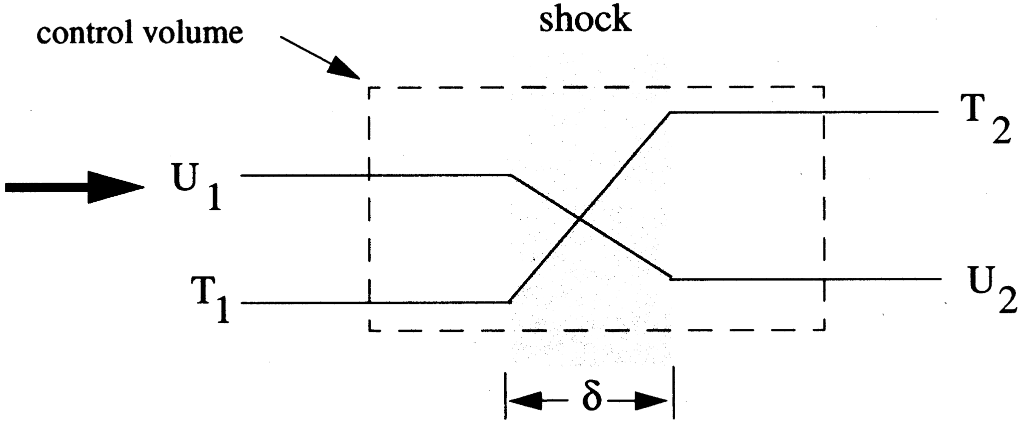
9.4 Shock wave thickness



Transport equation for the entropy

$$\frac{d}{dt}\int_{V(t)} \rho s dV + \int_{A(t)} \left(\rho \bar{U} s - \frac{k}{T}\nabla T\right) \bullet \bar{n} dA = \int_{V(t)} \left(\frac{\Upsilon + \Phi}{T}\right) dV$$

For a Newtonian heat conducting fluid

$$\Phi = 2\mu\left(S_{ij} - \frac{1}{3}\delta_{ij}S_{kk}\right)\left(S_{ij} - \frac{1}{3}\delta_{ij}S_{kk}\right) + \mu_v(S_{ii}S_{kk})$$

and

$$\Upsilon = \frac{\kappa}{T}\left(\frac{\partial T}{\partial x_j}\frac{\partial T}{\partial x_j}\right)$$

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The stress tensor

$$\tau_{ij} = 2\mu S_{ij} - ((2/3)\mu - \mu_v)\delta_{ij}S_{kk}$$

Within the shock wave

$$\tau_{ij} = \begin{bmatrix} \left(\frac{4}{3}\mu + \mu_v\right)\frac{dU}{dx} & 0 & 0 \\ 0 & \left(-\frac{2}{3}\mu + \mu_v\right)\frac{dU}{dx} & 0 \\ 0 & 0 & \left(-\frac{2}{3}\mu + \mu_v\right)\frac{dU}{dx} \end{bmatrix}.$$

Modified rate-of-strain tensor

$$S_{ij} - \frac{1}{3}\delta_{ij}S_{kk} = \begin{bmatrix} \frac{2}{3}\frac{dU}{dx} & 0 & 0 \\ 0 & -\frac{1}{3}\frac{dU}{dx} & 0 \\ 0 & 0 & -\frac{1}{3}\frac{dU}{dx} \end{bmatrix}$$

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Kinetic energy dissipation within the shock

$$\Phi = \left(\frac{4}{3}\mu + \mu_v\right)\left(\frac{dU}{dx}\right)^2 .$$

Temperature "dissipation"

$$\Upsilon = \frac{\kappa}{T}\left(\frac{dT}{dx_j}\right)^2 .$$

Now integrate the entropy equation.

$$\left(\rho UsA - \frac{\kappa}{T}\frac{dT}{dx}A\right)_2 - \left(\rho UsA - \frac{\kappa}{T}\frac{dT}{dx}A\right)_1 = \left(\int_0^{\delta}\left(\frac{\Upsilon + \Phi}{T}\right)dx\right)A .$$

The areas cancel on both sides and the temperature gradients at 1 and 2 are zero.

$$\rho U(s_2 - s_1) = \int_0^{\delta}\left(\left(\frac{\frac{4}{3}\mu + \mu_v}{T}\right)\left(\frac{dU}{dx}\right)^2 + \frac{\kappa}{T^2}\left(\frac{dT}{dx}\right)^2\right)dx .$$

Now let's define a simple model of the flow. Let

$$\frac{dU}{dx} \approx \frac{U_2 - U_1}{\delta} \quad ; \quad \frac{dT}{dx} \approx \frac{T_2 - T_1}{\delta} \quad ; \quad T \approx \frac{T_2 + T_1}{2}.$$

Evaluate the viscosity and thermal conductivity at the mean temperature. Now the entropy balance is

$$\frac{\rho U(s_2 - s_1)(T_2 + T_1)\delta}{2} = \left(\left(\frac{4}{3}\mu + \mu_v\right)\left((U_2 - U_1)^2\right) + 2\kappa\frac{(T_2 - T_1)^2}{(T_2 + T_1)}\right)$$

which can be expressed as

$$\rho U(s_2 - s_1)\delta = \left(\frac{2\left(\frac{4}{3}\mu + \mu_v\right)\frac{U_1^{\ 2}}{T_1}\left(\frac{U_2}{U_1} - 1\right)^2 + 4\kappa\frac{\left(\frac{T_2}{T_1} - 1\right)^2}{\left(\frac{T_2}{T_1} + 1\right)}}{\left(\frac{T_2}{T_1} + 1\right)}\right)$$

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Shock Reynolds number

$$\frac{\rho U \delta}{\mu} = \left(\frac{2(\gamma - 1)\left(\frac{4}{3} + \frac{\mu_v}{\mu}\right) M_1^2 \left(\frac{\rho_1}{\rho_2} - 1\right)^2 + 4\frac{\kappa}{C_p \mu} \frac{\left(\frac{T_2}{T_1} - 1\right)^2}{\left(\frac{T_2}{T_1} + 1\right)}}{\left(\frac{1}{\gamma}\right)\left(\frac{T_2}{T_1} + 1\right)\frac{(s_2 - s_1)}{C_v}} \right).$$

Prandtl number

$$P_r = \frac{C_p \mu}{\kappa}$$

Entropy jump

$$\frac{(s_2 - s_1)}{C_v} = ln\left(\frac{T_2}{T_1}\left(\frac{\rho_1}{\rho_2}\right)^{(\gamma-1)}\right)$$

Finally

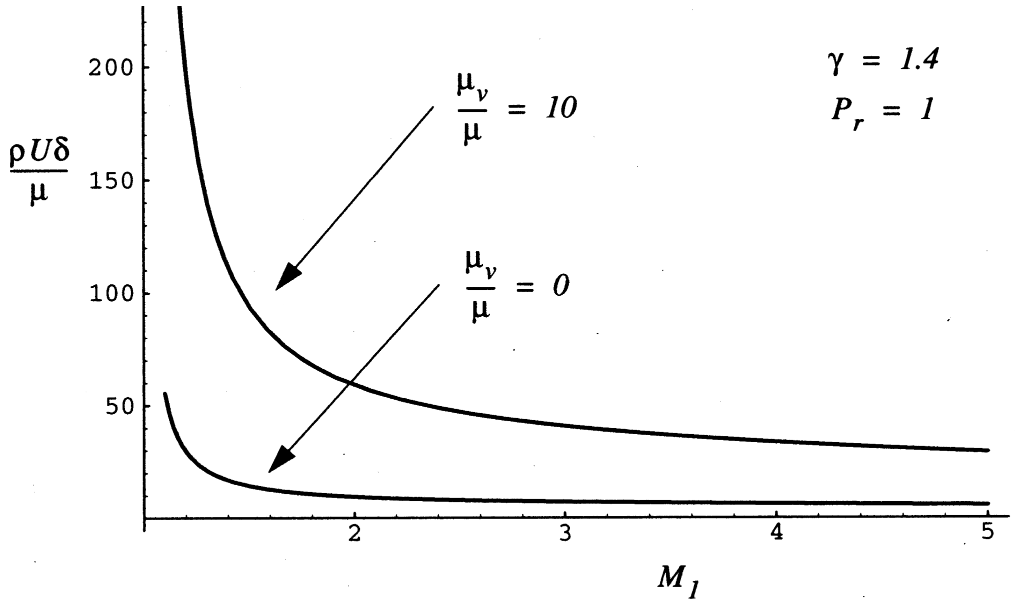
$$\frac{\rho U \delta}{\mu} = \left(\frac{2\gamma(\gamma-1)\left(\frac{4}{3} + \frac{\mu_v}{\mu}\right)M_1^2\left(\frac{\rho_1}{\rho_2} - 1\right)^2 + \frac{4\gamma}{P_r}\frac{\left(\frac{T_2}{T_1} - 1\right)^2}{\left(\frac{T_2}{T_1} + 1\right)}}{\left(\frac{T_2}{T_1} + 1\right) ln\left(\frac{T_2}{T_1}\left(\frac{\rho_1}{\rho_2}\right)^{(\gamma-1)}\right)}\right)$$

The right hand side can be written in terms of the upstream Mach number using the shock jump relations.

$$\frac{\rho_1}{\rho_2} = \frac{(\gamma-1)M_1^2+2}{(\gamma+1)M_1^2} \quad ; \quad \frac{T_2}{T_1} = \frac{(2\gamma M_1^2-(\gamma-1))((\gamma-1)M_1^2+2)}{(\gamma+1)^2 M_1^2}$$

Thus

$$\frac{\rho U \delta}{\mu} = F(M_1^2, \gamma, P_r, \mu_v/\mu)$$



The thickness can also be related to the mean free path in the gas

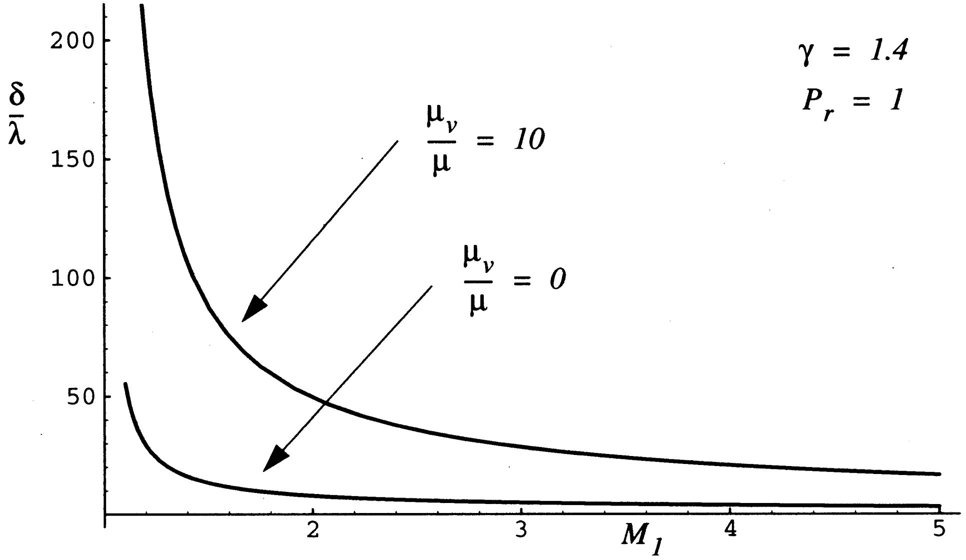
$$\frac{\rho U \delta}{\rho a \lambda} \approx F(M_1^2, \gamma, P_r, \mu_v/\mu)$$

Let

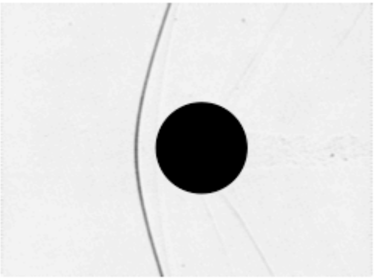
$$\frac{U}{a} = \left(\frac{U_1 + U_2}{2}\right)\left(\frac{2}{a_1 + a_2}\right) = M_1\left(\left(\frac{\rho_1}{\rho_2} + 1\right)\Big/\left(\sqrt{\frac{T_2}{T_1}} + 1\right)\right)$$

Now

$$\frac{\delta}{\lambda} \approx \frac{F(M_1^2, \gamma, P_r, \mu_v/\mu)}{M_1}\left(\left(\sqrt{\frac{T_2}{T_1}} + 1\right)\Big/\left(\frac{\rho_1}{\rho_2} + 1\right)\right)$$



Problem 11 - The photo below shows the flow of Helium gas past a sphere at a Mach number of 1.05. The pressure is one atmosphere ($1.01325 \times 10^5 N/m^2$) and the temperature is $300°K$. The viscosity of Helium at that temperature is $\mu_1 = 1.98 \times 10^{-5}\ kg/m\text{-}Sec$. Consider a fluid element that passes through the shock on the flow centerline.



Estimate the acceleration of the fluid element as it traverses the shock wave. Express your answer in m/Sec^2.

SOLUTION

The acceleration can be estimated as the velocity change of the fluid element over the time required to traverse the shock.

$$\frac{\Delta U}{\Delta t} \cong \frac{U_2 - U_1}{\Delta t} \cong (U_2 - U_1)\left(\frac{U_2 + U_1}{2\delta}\right) = \frac{1}{\delta}\left(\frac{U_2^2 - U_1^2}{2}\right) = \frac{C_p}{\delta}(T_1 - T_2) \quad \textbf{(5 points)}$$

The weak shock relations can be used in this case.

$$U_2 - U_1 = -\frac{4}{\gamma + 1}\varepsilon a_1 \quad \textbf{(2 points)}$$

The result from HW#3 is

$$\frac{\rho U \delta}{\mu} = 3\left(\left(\frac{4}{3} + \frac{\mu_v}{\mu}\right) + \frac{4(\gamma - 1)}{P_r}\right)\frac{1}{\varepsilon} \quad \textbf{(3 points)}$$

Let $(U_1 + U_2)/2 \cong M_1 a_1$. Now

$$\frac{\Delta U}{\Delta t} \cong \left(-\frac{4}{\gamma + 1}\varepsilon a_1\right)(M_1 a_1)\frac{\rho U}{\mu}\frac{1}{3\left(\left(\frac{4}{3} + \frac{\mu_v}{\mu}\right) + \frac{4(\gamma - 1)}{P_r}\right)\frac{1}{\varepsilon}} \quad \textbf{(2 points)}$$

Simplify the above

$$\frac{\Delta U}{\Delta t} \cong \left(-\frac{\rho_1 U_1}{\mu_1}\right)\frac{4M_1 a_1^2 \varepsilon^2}{3(\gamma+1)\left(\left(\frac{4}{3}+\frac{\mu_v}{\mu}\right)+\frac{4(\gamma-1)}{P_r}\right)}$$

Use the ideal gas law to replace the density

$$\frac{\Delta U}{\Delta t} \cong \left(-\frac{\gamma P_1 U_1}{\mu_1 \gamma R T_1}\right)\frac{4M_1 a_1^2 \varepsilon^2}{3(\gamma+1)\left(\left(\frac{4}{3}+\frac{\mu_v}{\mu}\right)+\frac{4(\gamma-1)}{P_r}\right)}$$

Cancel the speed of sound

$$\frac{\Delta U}{\Delta t} \cong \left(-\frac{\gamma P_1 U_1}{\mu_1}\right)\frac{4M_1 \varepsilon^2}{3(\gamma+1)\left(\left(\frac{4}{3}+\frac{\mu_v}{\mu}\right)+\frac{4(\gamma-1)}{P_r}\right)}$$

The flow speed is

$$U_1 = M_1\sqrt{\gamma\frac{R_u}{Mw}T_1} = 1.05\sqrt{\frac{5}{3}\frac{(8314)}{4}300} = 1070.4 \text{ m/Sec}$$

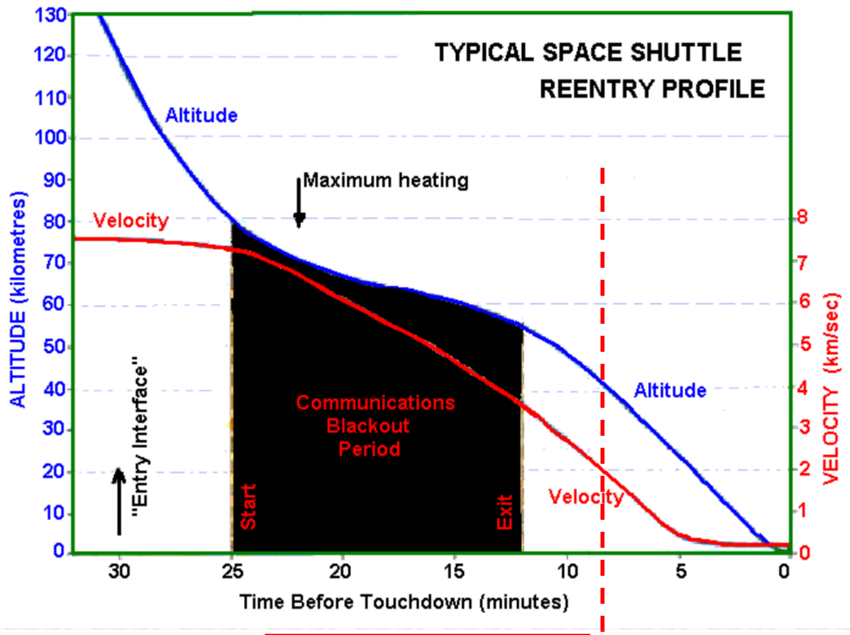
Now substitute the numbers.

$$\frac{\Delta U}{\Delta t} \cong \left(-\frac{1.66(1.01325\times 10^5)1070.4}{1.98\times 10^{-5}}\right)\frac{4(1.05)(0.05)^2}{3(2.66)\left(1.33+\frac{4(0.66)}{0.67}\right)}$$

The acceleration is estimated at

$$\frac{\Delta U}{\Delta t} \cong -2.27\times 10^9 \ m/Sec^2$$

Space shuttle re-entry
ignore heat capacity changes and real gas effects



8 minutes to touchdown
Altitude 40km
Speed 2000m/sec

$P_\infty = 277.522\ Pa$

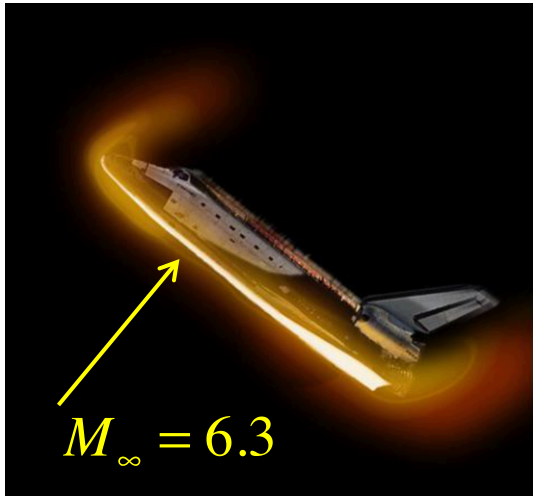
$\rho_\infty = 0.003851\ kg/m^3$

$T_\infty = 251.050\ K$

$a_\infty = 317.633\ m/\sec$

$M_\infty = 6.297$

Assume $\gamma = 1.4$



$M_2 = 0.402$

$$\frac{T_{t\infty}}{T_\infty} = 1 + \frac{\gamma-1}{2} M_\infty^{\ 2} = 8.930$$

$$\frac{P_{t\infty}}{P_\infty} = \left(1 + \frac{\gamma-1}{2} M_\infty^{\ 2}\right)^{\frac{\gamma}{\gamma-1}} = 8.930^{3.5} = 2128.41$$

Across the normal shock

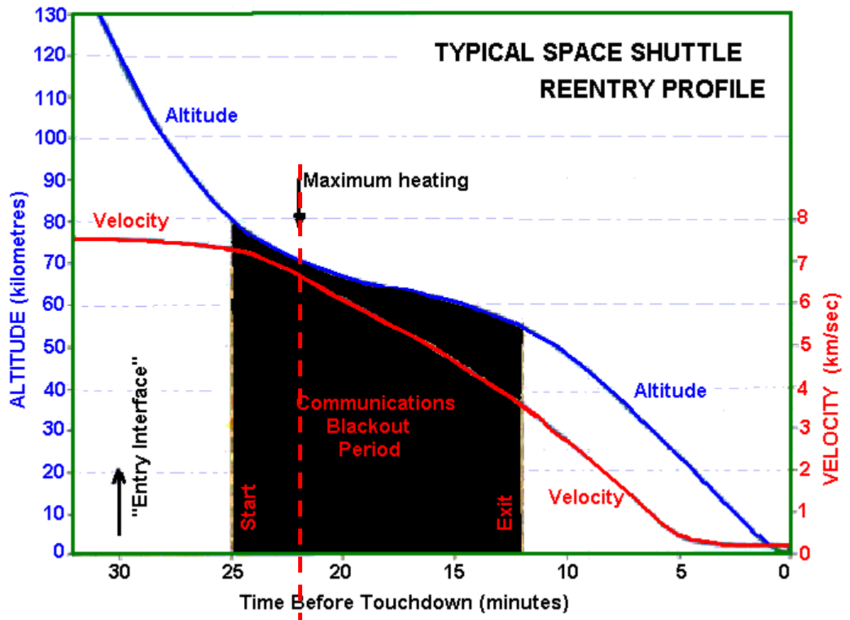
$$\frac{P_{t2}}{P_{t\infty}} = \left(\frac{\frac{\gamma+1}{\gamma-1}}{\frac{2\gamma}{\gamma-1} M_\infty^{\ 2} - 1}\right)^{\frac{1}{\gamma-1}} \left(\frac{\frac{\gamma+1}{2} M_\infty^{\ 2}}{1 + \frac{\gamma-1}{2} M_\infty^{\ 2}}\right)^{\frac{\gamma}{\gamma-1}} = 0.02416$$

$P_{t\infty} = 590{,}680\ Pa$

$P_{t2} = 14{,}270\ Pa$

$T_{t\infty} = 2241.87\ K$

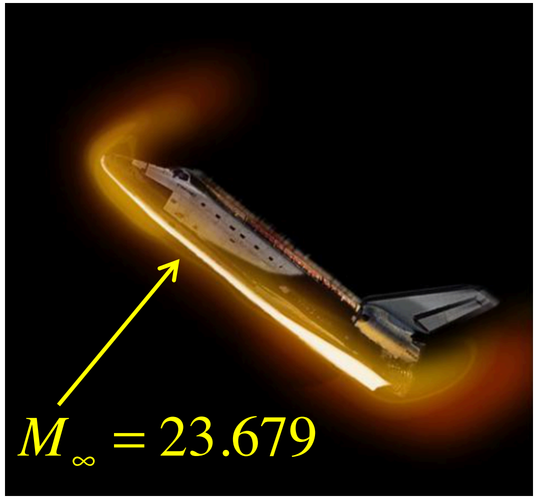
Space shuttle re-entry
ignore heat capacity changes and real gas effects



22 minutes to touchdown
Altitude 70km
Speed 7000m/sec

$$P_\infty = 4.63422\ Pa$$
$$\rho_\infty = 0.000074243\ kg/m^3$$
$$T_\infty = 217.45\ K$$
$$a_\infty = 295.614\ m/\sec$$
$$M_\infty = 23.679$$
$$Assume\ \gamma = 1.4$$

At such a high Mach number the flow is in fact totally dominated by real gas effects including dissociation. The temperatures reached are much lower, and the pressure behind the shock tends to be higher that predicted here.



$$M_2 = 0.379$$

$$\frac{T_{t\infty}}{T_\infty} = 1 + \frac{\gamma-1}{2}M_\infty^2 = 113.14$$

$$\frac{P_{t\infty}}{P_\infty} = \left(1 + \frac{\gamma-1}{2}M_\infty^2\right)^{\frac{\gamma}{\gamma-1}} = 113.14^{3.5} = 1.54048\times10^7$$

Across the normal shock

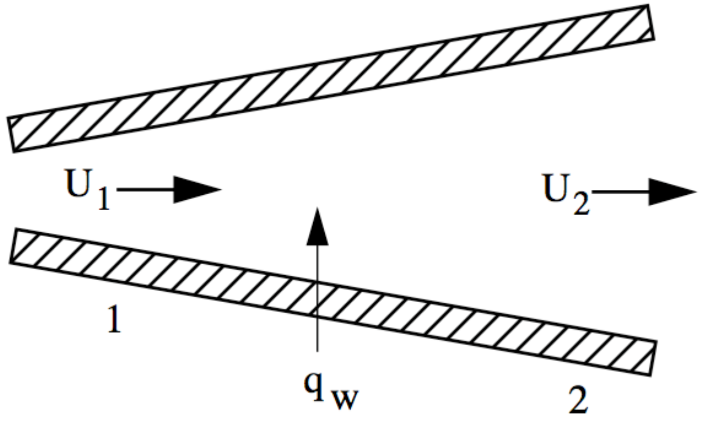
$$\frac{P_{t2}}{P_{t\infty}} = \left(\frac{\frac{\gamma+1}{\gamma-1}}{\frac{2\gamma}{\gamma-1}M_\infty^2 - 1}\right)^{\frac{1}{\gamma-1}}\left(\frac{\frac{\gamma+1}{2}M_\infty^2}{1+\frac{\gamma-1}{2}M_\infty^2}\right)^{\frac{\gamma}{\gamma-1}} = 4.68951\times10^{-5}$$

$$P_{t\infty} = 7.13893\times10^7\ Pa$$
$$P_{t2} = 3347.81\ Pa$$

$$?\ \longrightarrow T_{t\infty} = 24602.3\ K$$

9.5 Problems

Problem 1 - Heat in the amount of 10^6 *Joules/Kg* is added to a compressible flow of helium in a diverging channel. The heat is distributed so that the area averaged velocities at stations 1 and 2 are the same.



The temperature at station 1 is $1000K$ and the area ratio is $A_2/A_1 = 2$. Determine T_2/T_1, ρ_2/ρ_1, P_2/P_1, and $(s_2 - s_1)/C_p$.

Problem 2 - Recall Problem 5.3. Consider steady flow in one dimension where $\overline{U} = (U(x), 0, 0)$ and all velocity gradients are zero except

$$A_{11} = \frac{\partial U}{\partial x} \tag{8.102}$$

Work out the components of the Newtonian viscous stress tensor τ_{ij}. Note the role of the bulk viscosity. Inside a normal shock wave the velocity gradient can be as high as $10^{10}\ sec^{-1}$. Using values for Air at 300K and one atmosphere estimate the magnitude of the viscous normal stress inside a shock wave. Express your answer in atmospheres.

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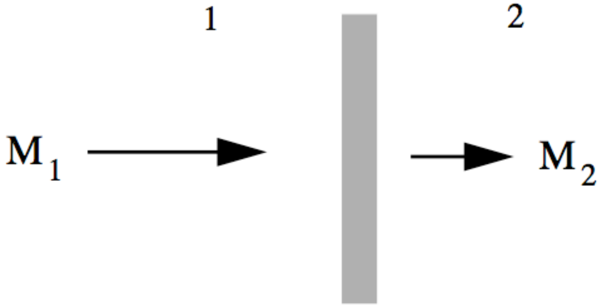
Problem 3 - Consider a normal shock wave in helium with Mach number $M_1 = 3$. The temperature of the upstream gas is $300K$ and the pressure is $10^5\ N/m^2$.



1) Determine the stagnation temperature in region 2 as measured by an observer at rest with respect to the upstream gas. This is an observer that sees the shock wave propagating to the left at Mach 3.

2) Determine the stagnation pressure in region 2 as measured by an observer at rest with respect to the upstream gas.

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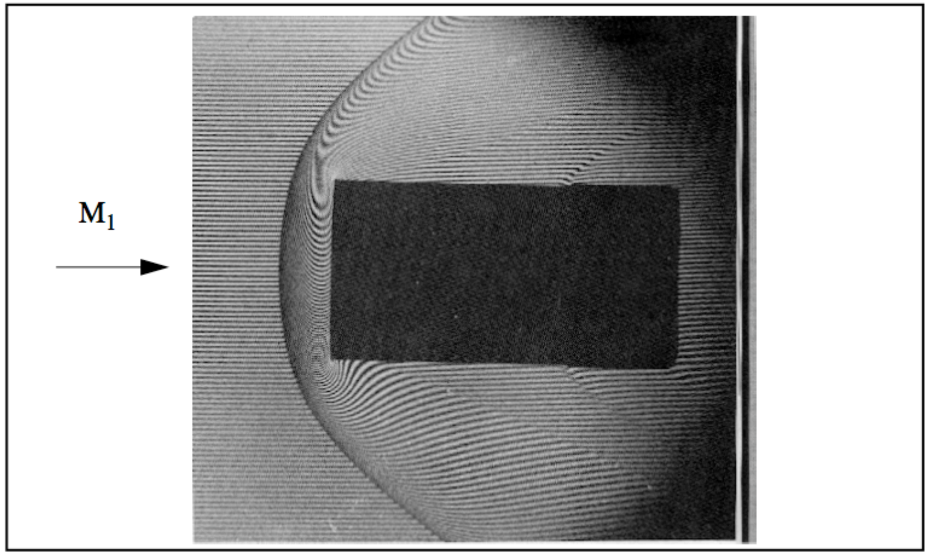
Problem 4 - The figure below shows supersonic flow of Carbon Dioxide, $\gamma = 4/3$, past a cylindrical bullet at a free stream Mach number, $M_1 = 2.77$.



See Van Dyke page 163.

The temperature of the free stream is $300°K$ and the pressure is one atmosphere.

a) Determine the temperature, pressure and Mach number of the gas on the centerline just downstream of the shock wave.

b) Estimate the temperature and pressure at the stagnation point on the upstream face of the cylinder.

c) Determine the entropy increase across the shock wave.

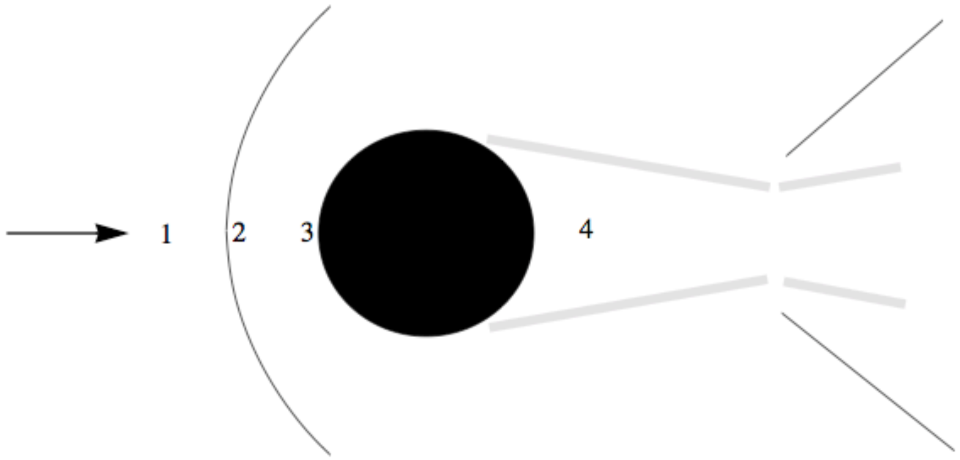
d) Estimate the thickness of the shock wave.

e) Estimate the acceleration of the fluid element as it traverses the shock wave.

Express your answer in m/Sec^2.

Problem 5 - Estimate the thickness of the shock wave in Helium discussed in Section 8.3.4.

Problem 6 - The sketch below shows supersonic flow of air, ($\gamma = 1.4$), past a sphere at a free stream Mach number, $M_1 = 1.53$. (cf. Van Dyke page 164)



a) Compare each of the following properties of the gas; stagnation enthalpy, h_t, stagnation pressure, P_t and entropy per unit mass, s at locations 1, 2 and 3 identified in the figure. State the assumptions needed to make your comparisons.

b) What can you say about the same properties of the gas at station 4? How certain is your answer? Why?

c) Determine the Mach number at station 2.

d) Is the energy per unit mass (internal plus kinetic) of a gas particle at 1 and 2 the same? Prove your answer.

Problem 7 - We often encounter practical situations involving weak shock waves where the Mach number upstream of the wave is very close to one.

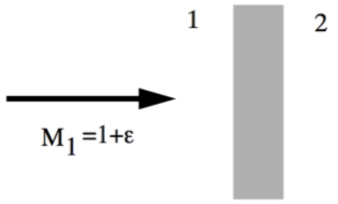


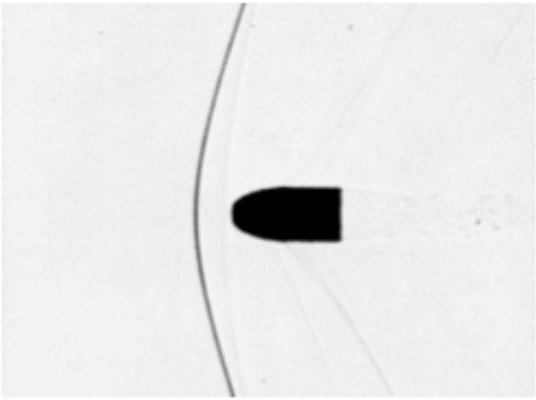
Figure 9.15: *Weak normal shock*

Let the Mach number ahead of the wave be $M_1 = 1 + \varepsilon$ where $\varepsilon < 1$. Derive the weak shock jump relations $M_2 \cong 1 - \varepsilon$ and

$$\begin{aligned} \frac{U_2 - U_1}{a_1} &\cong -\frac{4}{\gamma+1}\varepsilon \\ \frac{T_2 - T_1}{T_1} &\cong ? \\ \frac{P_2 - P_1}{P_1} &\cong ? \\ \frac{P_{t2} - P_{t1}}{P_{t1}} &\cong -\frac{16}{3}\frac{\gamma}{(\gamma+1)^2}\varepsilon^3. \end{aligned} \tag{9.103}$$

The last result in (9.103) is extremely important in that it shows that the stagnation loss across a weak shock is extremely small indeed. This fact is exploited in the design of supersonic inlets. Note that first and second order terms in ε have cancelled. I suggest you use symbol manipulation software such as Mathematica to derive this result.

Problem 8 - The photo below shows a rifle bullet moving in air at a Mach number of 1.1. The air temperature is 300°K. On the centerline the flow from the left passes through a normal shock wave and then stagnates on the nose of the bullet.



a) Determine the temperature, pressure and density change across the wave.

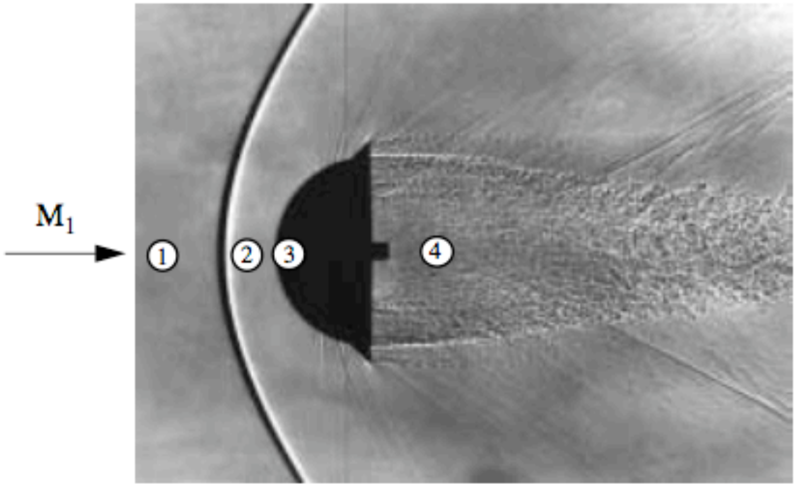
b) Compare the temperature, pressure and density of the gas at the nose of the bullet to values in the freestream.

c) Evaluate the entropy change.

d) State any assumptions used.

Problem 9 - Use the weak shock theory developed in problem 7 to estimate the thickness of the shock wave depicted in Problem 8. Develop an expression for estimating the thickness of a weak shock wave δ in terms of ε and γ.

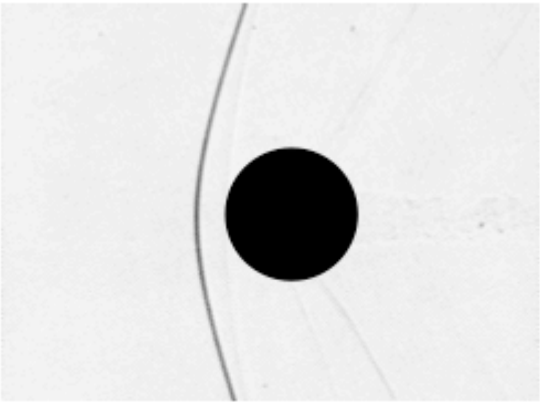
Problem 10 - The figure below shows supersonic flow of air over a model of a re-entry body at a free stream Mach number, $M_1 = 2$.



The temperature of the free stream is $300°K$ and the pressure is one atmosphere.
1) Determine the stagnation temperature and pressure of a fluid element located at stations 1 (free stream), 2 (just behind the shock) and 3 (at the stagnation point on the body). State the assumptions used to solve the problem. Express your answers in °K and atmospheres.
2) What can you say about the state of the gas at point 4?

3) Refer the stagnation temperatures at 1, 2 and 3 to an observer at rest with respect to the upstream gas. To such an observer the body is moving to the left at a Mach number of 2.0.

Problem 11 - The photo below shows the flow of Helium gas past a sphere at a Mach number of 1.05. The pressure is one atmosphere ($1.01325 \times 10^5 N/m^2$) and the temperature is $300°K$. The viscosity of Helium at that temperature is $\mu_1 = 1.98 \times 10^{-5}$ $kg/m\text{-}Sec$. Consider a fluid element that passes through the shock on the flow centerline.



Estimate the acceleration of the fluid element as it traverses the shock wave. Express your answer in m/Sec^2.

$$U_2' = U_2 + U_{body} = 260.4 - 694.4 = -434\ M/sec\ .\ \textbf{(1 point)}$$

The heat capacity of the gas is $C_p = 1005\ M^2/sec^2 - {}^\circ K$. The static temperature of the gas at station 2 is determined by the temperature jump across the shock.

$$\frac{T_2}{T_1} = \left(\frac{\gamma M_1^2 - \frac{\gamma - 1}{2}}{\frac{\gamma + 1}{2}}\right)\left(\frac{1 + \left(\frac{\gamma - 1}{2}\right)M_1^2}{\left(\frac{\gamma + 1}{2}\right)M_1^2}\right) = 1.6875$$

and the temperature at station 2 is

$$T_2 = 1.6875 \times 300 = 506.25\ {}^\circ K\ \textbf{(1 point)}$$

The stagnation temperature at station 2 in the rest frame of the upstream gas is

$$T_{t2}' = T_2 + \frac{1}{2}\frac{U_2'^2}{C_p} = 506.25 + \frac{434^2}{2 \times 1005} = 600\ {}^\circ K\ \textbf{(1 point)}$$

At station 3 the gas speed is equal to the speed of the body to the left. The stagnation temperature at station 3 in the rest frame of the upstream gas is

$$T_{t3}' = T_3 + \frac{1}{2}\frac{U_3'^2}{C_p} = 540 + \frac{694.4^2}{2 \times 1005} = 780\ {}^\circ K\ \textbf{(1 point)}$$

Note that in this frame of reference the stagnation temperatures at 2 and 3 are not equal and are considerably larger that the stagnation temperature in the rest frame of the shock.