A New Data-derived Wall Damping Function for the Universal Velocity Profile

AIAA AVIATION Conference, San Diego June 8-12, 2026

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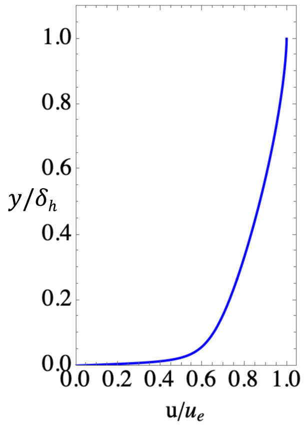
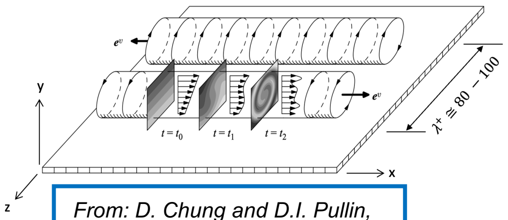
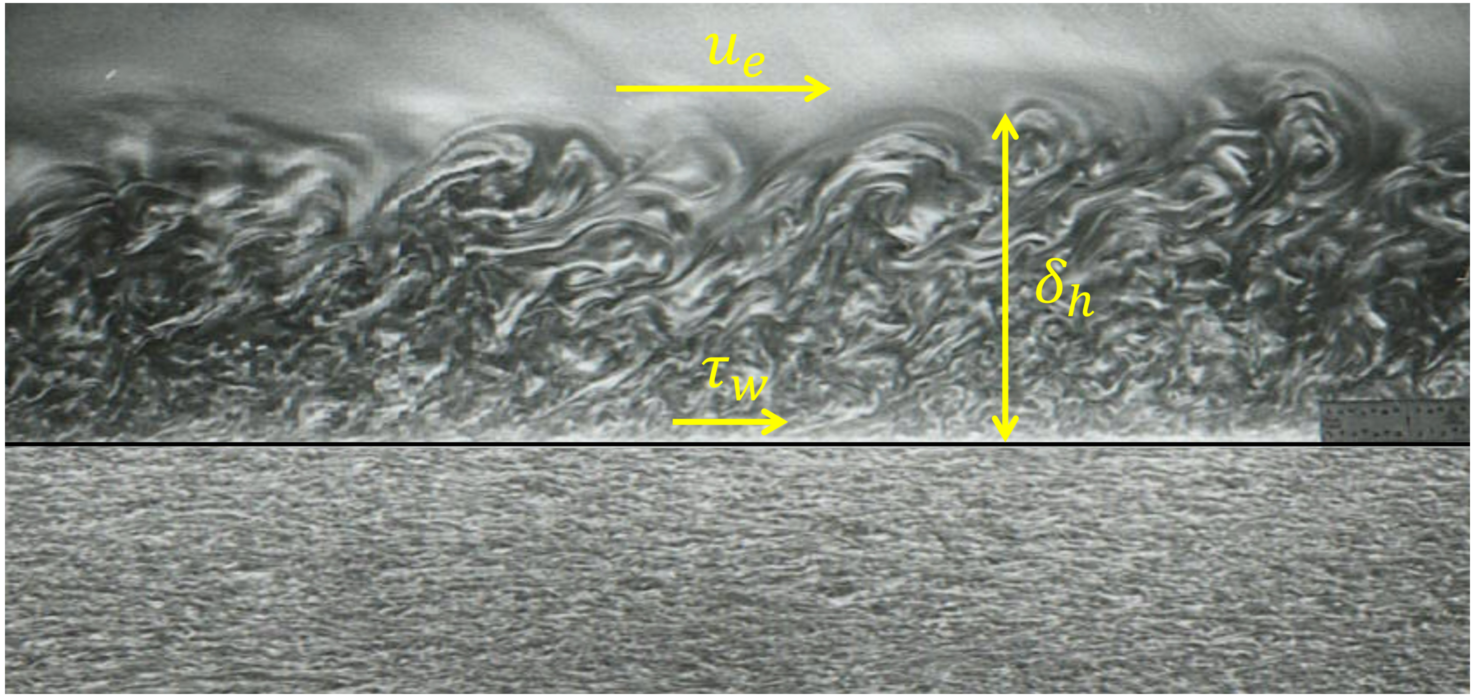
June 11, 2026

Turbulent boundary layer - notation

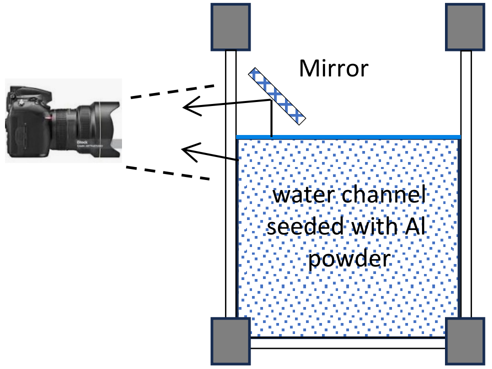




From: D. Chung and D.I. Pullin, JFM vol 631, 2009 – Fig 2



Boundary layer thickness is arbitrarily defined. Typically, $\delta_{0.99}$ is used where $u(\delta)=0.99u_e$

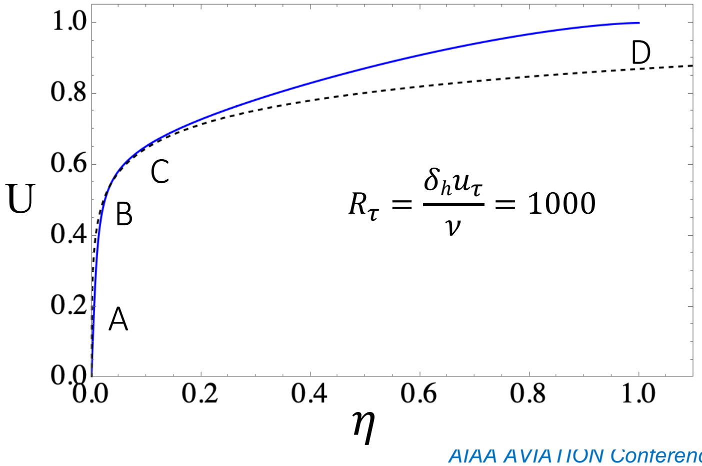


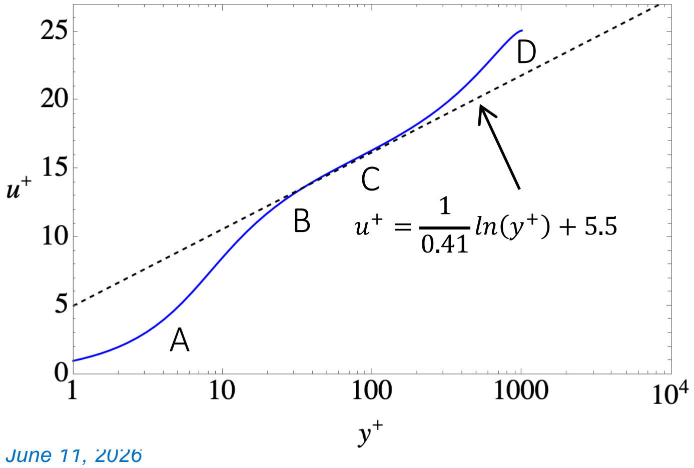
$$\tau_w = \mu \frac{\partial U}{\partial y}\bigg|_{y=0} \qquad u_\tau = \sqrt{\frac{\tau_w}{\rho}}$$

$$\eta = \frac{y}{\delta_h}$$

$$U = \frac{u}{u_e}$$

$$R_{\delta_h} = \frac{\delta_h u_e}{\nu}$$





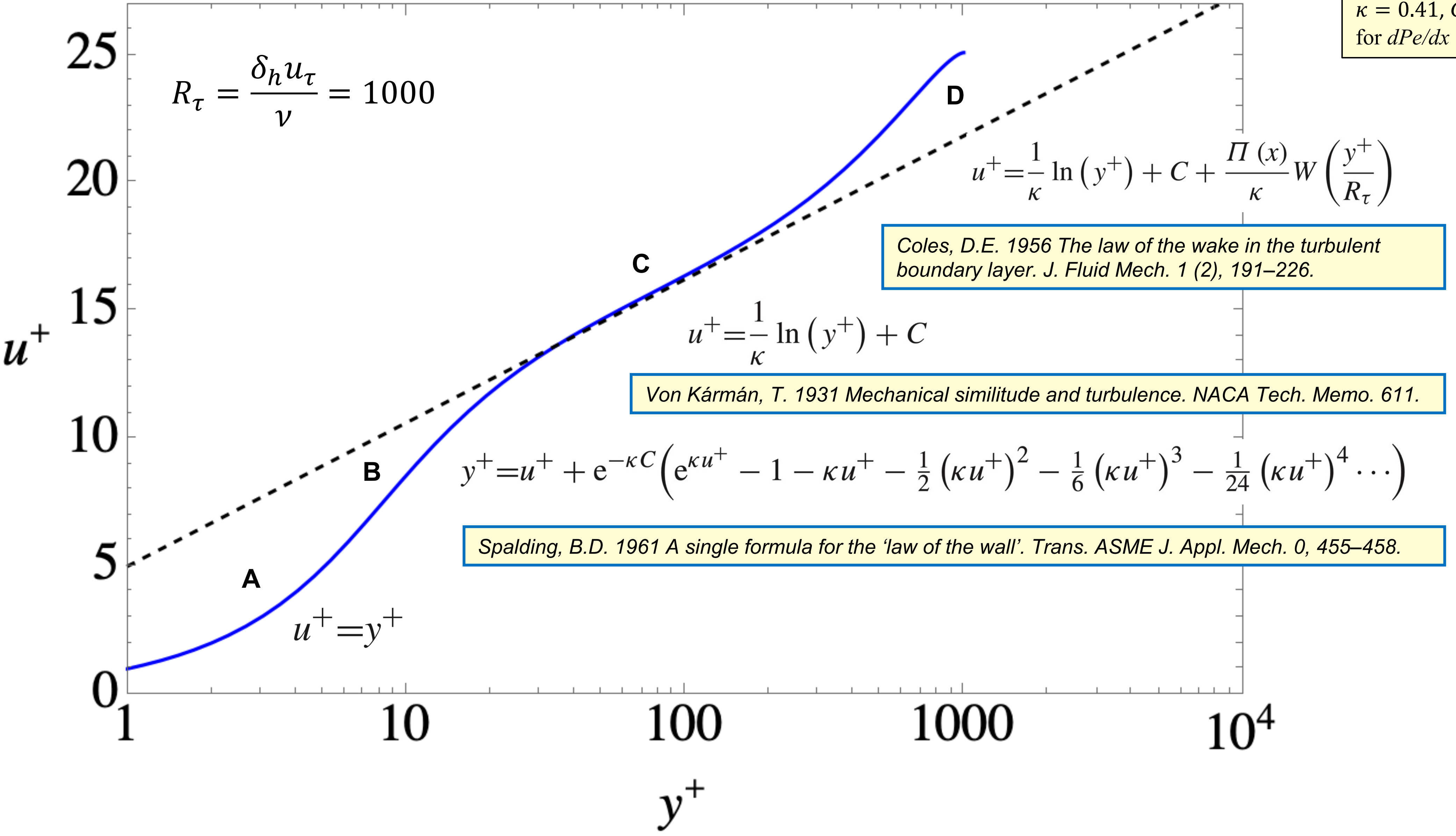
$$y^+ = \frac{y u_\tau}{\nu}$$

$$u^+ = \frac{u}{u_\tau}$$

$$R_\tau = \frac{\delta_h u_\tau}{\nu}$$

Classical wall-wake formulation

Coles recommends
$W = 2\sin^2\left(\frac{\pi}{2}\frac{y}{\delta}\right)$
$\kappa = 0.41$, $C = 5.5$
for $dPe/dx = 0$, $\Pi = 0.62$



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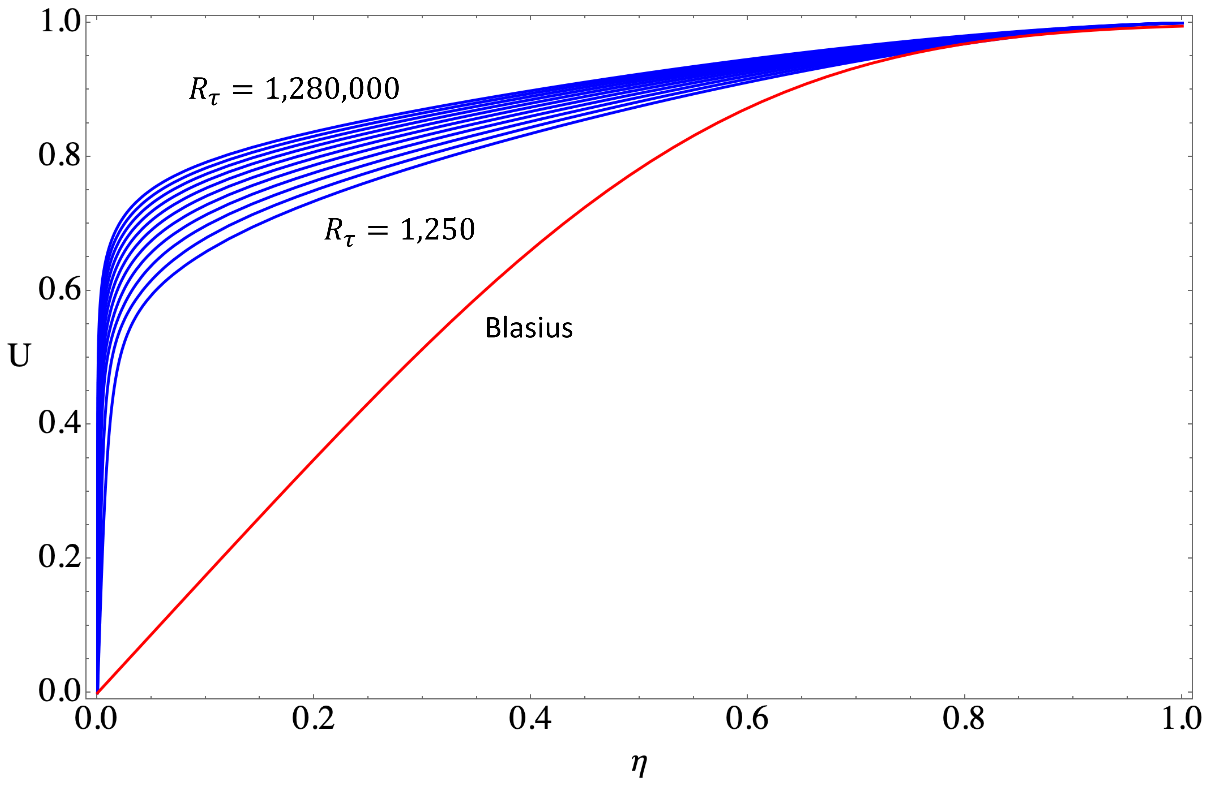
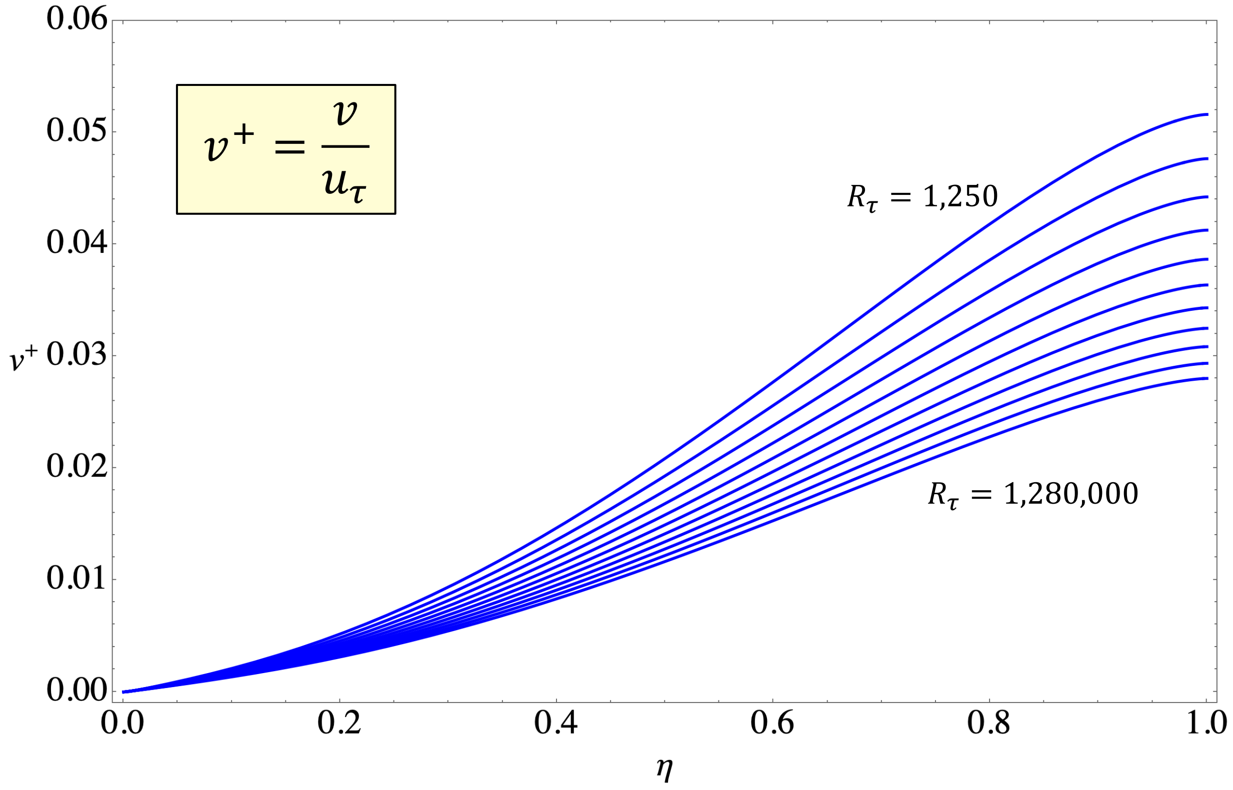
Reynolds number dependence





The velocity gradient at the wall increases rapidly with Reynolds number.

$$\lim_{y\to 0}\frac{u}{y} = \frac{{u_\tau}^2}{\nu} \longrightarrow \lim_{\eta\to 0}\frac{U}{\eta} \approx \frac{\kappa R_\tau}{ln\,(R_\tau)}$$

$$\lim_{\eta\to 0}\frac{U}{\eta}\bigg|_{1250} \approx 70$$

$$\lim_{\eta\to 0}\frac{U}{\eta}\bigg|_{1280000} \approx 37000$$

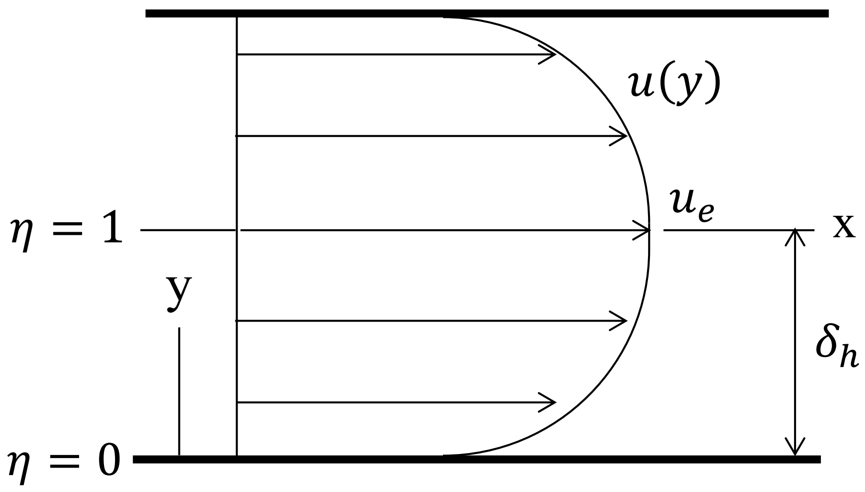
The Universal Velocity Profile

The UVP is derived from the channel/pipe flow equation

$$\eta = \frac{y}{\delta_h}$$

$$U = \frac{u}{u_e}$$

$$R_{\delta_h} = \frac{\delta_h\, u_e}{\nu}$$



δ_h - channel half-height

$$y^+ = \frac{y u_\tau}{\nu}$$

$$u^+ = \frac{u}{u_\tau}$$

$$R_\tau = \frac{\delta_h\, u_\tau}{\nu}$$

Balance between shear stress and the pressure gradient

viscous shear stress

$$\frac{\partial}{\partial y}\overline{u'v'} - \nu\frac{\partial^2 u}{\partial y^2} + \frac{1}{\rho}\frac{dp_e(x)}{dx} = 0$$

Reynolds shear stress

pressure gradient

Neglect possible dependence of the mean pressure on y.

Channel flow equation, contd.

$$\frac{\partial}{\partial y}\overline{u'v'} - \nu\frac{\partial^2 u}{\partial y^2} + \frac{1}{\rho}\frac{dp_e(x)}{dx} = 0$$

$$y^+ = \frac{y u_\tau}{\nu}$$

$$u^+ = \frac{u}{u_\tau}$$

$$R_\tau = \frac{\delta_h\, u_\tau}{\nu}$$

Express in terms of friction variables and integrate once

viscous shear stress

$$\tau^+ + \frac{du^+}{dy^+} - \left(1 - \frac{y^+}{R_\tau}\right) = 0$$

$$\tau^+ = -\frac{\overline{u'v'}}{{u_\tau}^2}$$

Reynolds shear stress

pressure gradient

Mixing length model for the turbulent shear stress

$$\tau^{+} = \left(\lambda(y^{+})\frac{\mathrm{d}u^{+}}{\mathrm{d}y^{+}}\right)^{2}$$

Prandtl 1925

$$\left(\frac{\mathrm{d}u^{+}}{\mathrm{d}y^{+}}\right)^{2} + \frac{1}{\lambda(y^{+})^{2}}\frac{\mathrm{d}u^{+}}{\mathrm{d}y^{+}} - \frac{1}{\lambda(y^{+})^{2}}\left(1-\frac{y^{+}}{R_{\tau}}\right) = 0$$

Take the positive root and remove the singularity at $\lambda = 0$

$$\frac{du^{+}}{dy^{+}} = \frac{2\left(1-\frac{y^{+}}{R_{\tau}}\right)}{1+\left(1+4\lambda\,(y^{+})^{2}\left(1-\frac{y^{+}}{R_{\tau}}\right)\right)^{1/2}}$$

$$y^{+} = \frac{yu_{\tau}}{\nu}$$

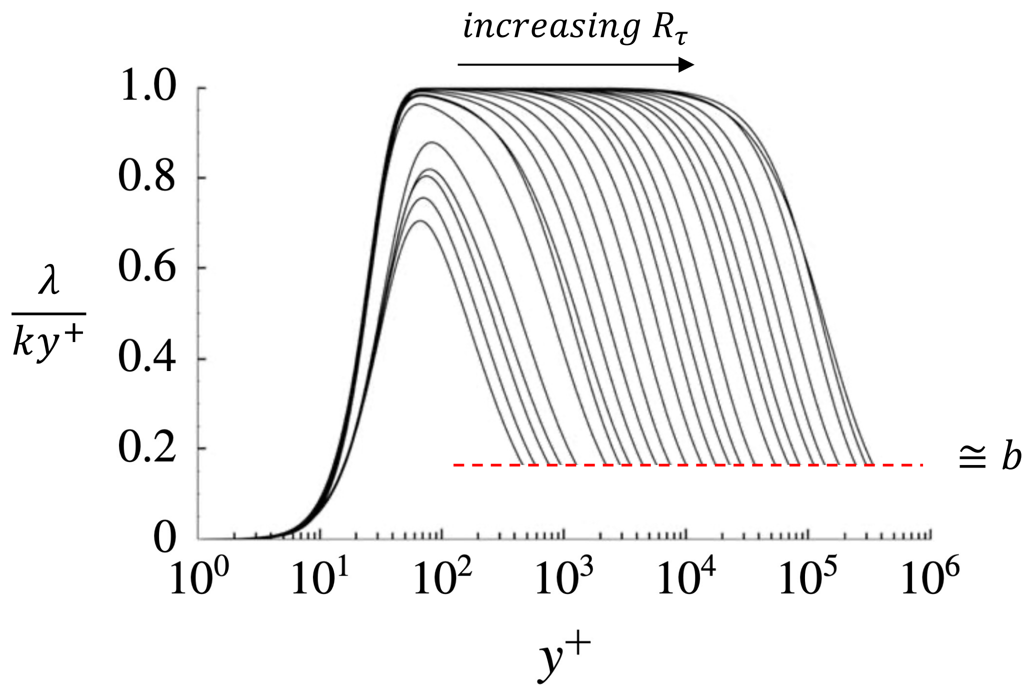
$$u^{+} = \frac{u}{u_{\tau}}$$

$$R_{\tau} = \frac{\delta_{h}\, u_{\tau}}{\nu}$$

Define a mixing length function

Exponential Damping Function

$$\lambda(k, a, m, b, n, R_\tau, y^+) = \frac{k y^+ \left(1 - e^{-(y^+/a)^m}\right)}{\left(1 + \left(\frac{y^+}{b R_\tau}\right)^n\right)^{1/n}}$$



k - essentially the Kármán constant.

a - wall damping length scale similar to the van Driest length.

m - exponent that, along with *a*, governs the shape and thickness of the near wall profile.

b - length scale proportional to the distance above the wall to the beginning of the outer layer.

n - exponent that, along with *b*, controls the transition of the profile to the wake function.

The Universal Velocity Profile (UVP) - Integrate the velocity derivative from the wall

$$u^+(k, a, m, b, n, R_\tau, y^+) = \int_0^{y^+} \frac{2\left(1 - \frac{s}{R_\tau}\right)}{1 + \left(1 + 4\lambda^2\left(1 - \frac{s}{R_\tau}\right)\right)^{1/2}} ds$$

The velocity profile is uniformly valid from the wall to the channel centerline at all Reynolds numbers.

The friction law is generated by evaluating the UVP at $y^+ = R_\tau$

$$\frac{u_e}{u_\tau} = \int_0^{R_\tau} \frac{2\left(1 - \frac{s}{R_\tau}\right)}{1 + \left(1 + 4\lambda^2\left(1 - \frac{s}{R_\tau}\right)\right)^{1/2}} ds = \left(\frac{2}{C_f}\right)^{1/2}$$

UVP versus TBL data - exponential damping function

The UVP fits Turbulent Boundary Layer data quite well

Minimize
$$G = \sum_{i=1}^{N} \left(u^+(k,a,m,b,n,R_\tau,y_i^+) - u_i^+(y_i^+)\right)^2$$

Results – 7 numerical simulation data sets, 4 experimental data sets

R_τ	$(u_e/u_\tau)_{data}$	$(u_e/u_\tau)_{uvp}$	k	a	m	b	n	u^+_{rms}	u_e/U
1343	25.5088	25.4939	0.4222	24.7756	1.1820	0.1828	2.3298	0.03617	0.993
1475	25.9305	25.8994	0.4205	24.7786	1.1732	0.1787	2.3622	0.03332	0.993
1616	26.2722	26.2365	0.4200	24.7834	1.1720	0.1764	2.3548	0.03390	0.993
1779	26.5926	26.5818	0.4187	24.3610	1.2032	0.1757	2.2932	0.03215	0.994
1962	26.8226	26.8512	0.4191	24.6388	1.1752	0.1747	2.2833	0.03298	0.994
2088	27.0332	27.0255	0.4289	25.9290	1.1480	0.1696	2.2516	0.03143	0.994
2652 2571	27.4177	27.4073	0.4221	25.1424	1.1130	0.1724	2.3087	0.03150	0.993 0.995
Exp. 2109	26.8104	26.9239	0.4361	26.0709	1.1410	0.1665	2.1993	0.05453	0.996
Exp. 4374	28.8876	29.0940	0.4338	26.3286	1.1060	0.1664	1.8792	0.09473	0.996
Exp. 9090	30.5483	30.7301	0.4214	25.0804	1.1216	0.1829	1.7753	0.13364	0.996
Exp. 17 207	32.0670	32.4649	0.4136	24.6549	1.0846	0.1816	1.8397	0.16864	0.996

Table source

M. A. Subrahmanyam, B. J. Cantwell, and J. J. Alonso, "A universal velocity profile for turbulent wall flows including adverse pressure gradient boundary layers," J. Fluid Mech. 933, A16–1 to A16–31 (2022)

Data sources

G. Eitel-Amor, Ö. Ramis, and P. Schlatter, "Simulation and validation of a spatially evolving turbulent boundary layer up to re = 8300," International Journal of Heat and Fluid Flow 47, 57 – 69 (2014).

M. P. Simens, J. Jiménez, S. Hoyas, and Y. Mizuno, "A high-resolution code for turbulent boundary layers," Journal of Computational Physics 228, 4218 – 4231 (2009).

G. Borrell, J. A. Sillero, and J. Jiménez, "A code for direct numerical simulation of turbulent boundary layers at high Reynolds numbers in bg/p supercomputers," Computers and Fluids 80, 37 – 43 (2013),

J. A. Sillero, J. Jiménez, and R. D. Moser, "One-point statistics for turbulent wall-bounded flows at Reynolds numbers up to δ + = 2000," Physics of Fluids 25, 105102 (2013).

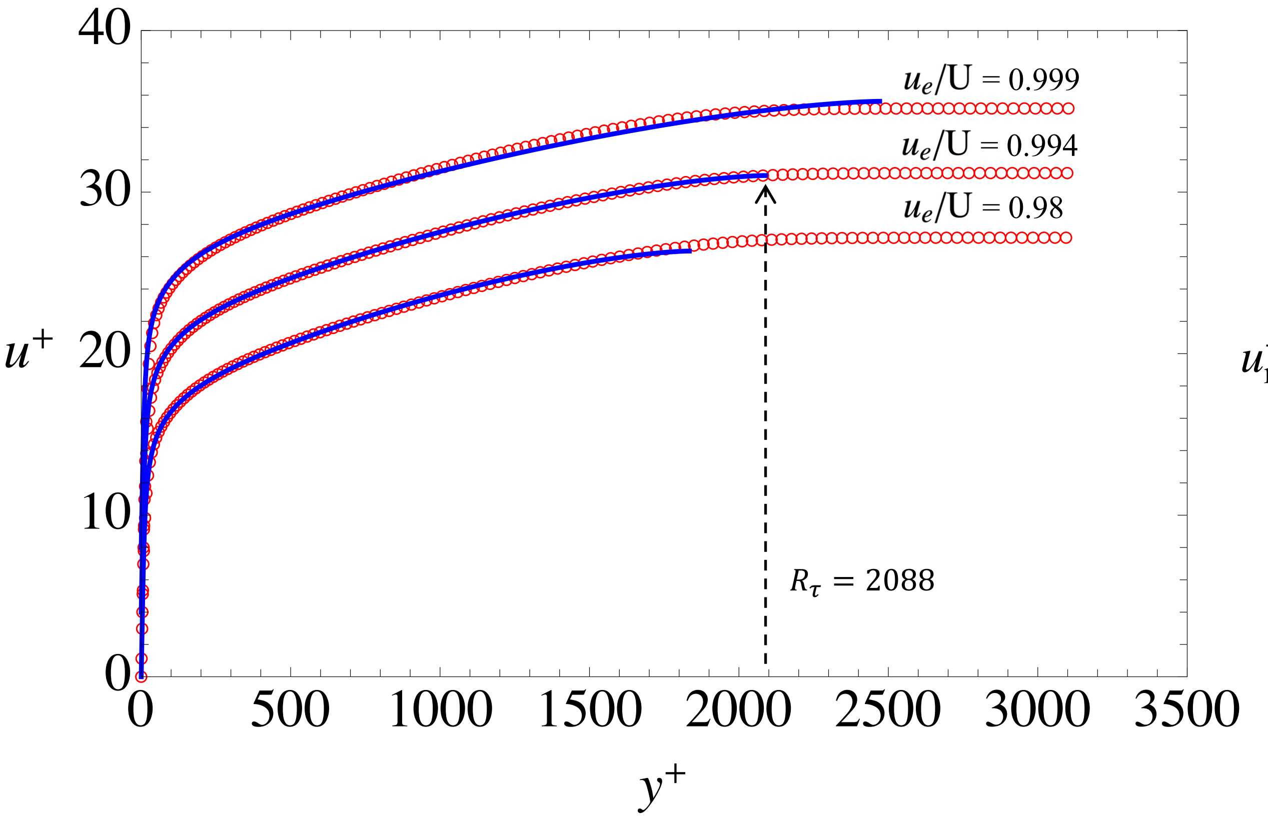
R. Baidya, J. Phillip, N. Hutchins, J. P. Monty, and I. Marusic, "Distance from the wall scaling of turbulent motions in wall-bounded flows," Physics of Fluids 29, 020712 (2017).

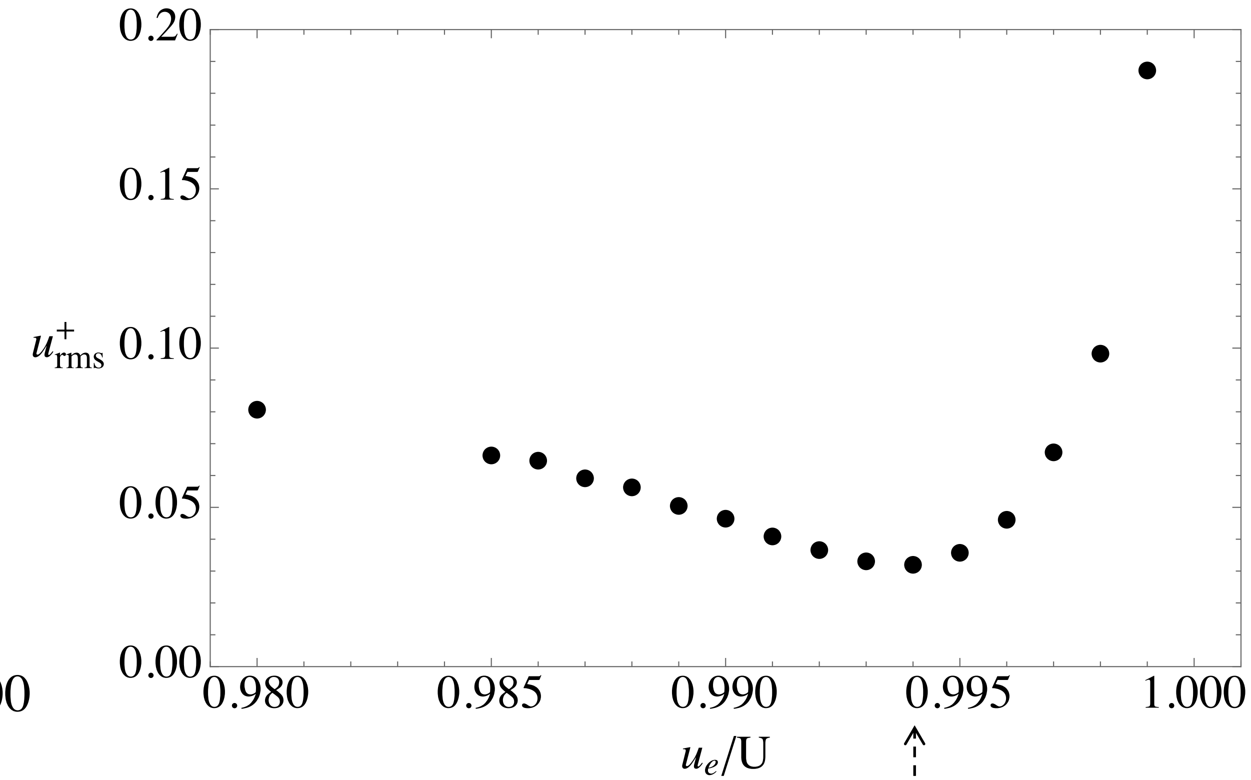
A. E. Perry and I. Marusic, "A wall-wake model for the turbulence structure of boundary layers. part 2. further experimental support," Journal of Fluid Mechanics 298, 389 – 407 (1995).

M. Jones, I. Marusic, and A. E. Perry, "Evolution and structure of sink-flow turbulent boundary layers," Journal of Fluid Mechanics 428, 1 – 27 (2001).

Turbulent Boundary Layer equivalent channel half height thickness δ_h

δ_h is defined as the thickness that minimizes the error between a specific data set and the UVP





J. A. Sillero, J. Jiménez, and R. D. Moser, "One-point statistics for turbulent wall-bounded flows at Reynolds numbers up to δ + = 2000," Physics of Fluids 25, 105102 (2013).

Data source

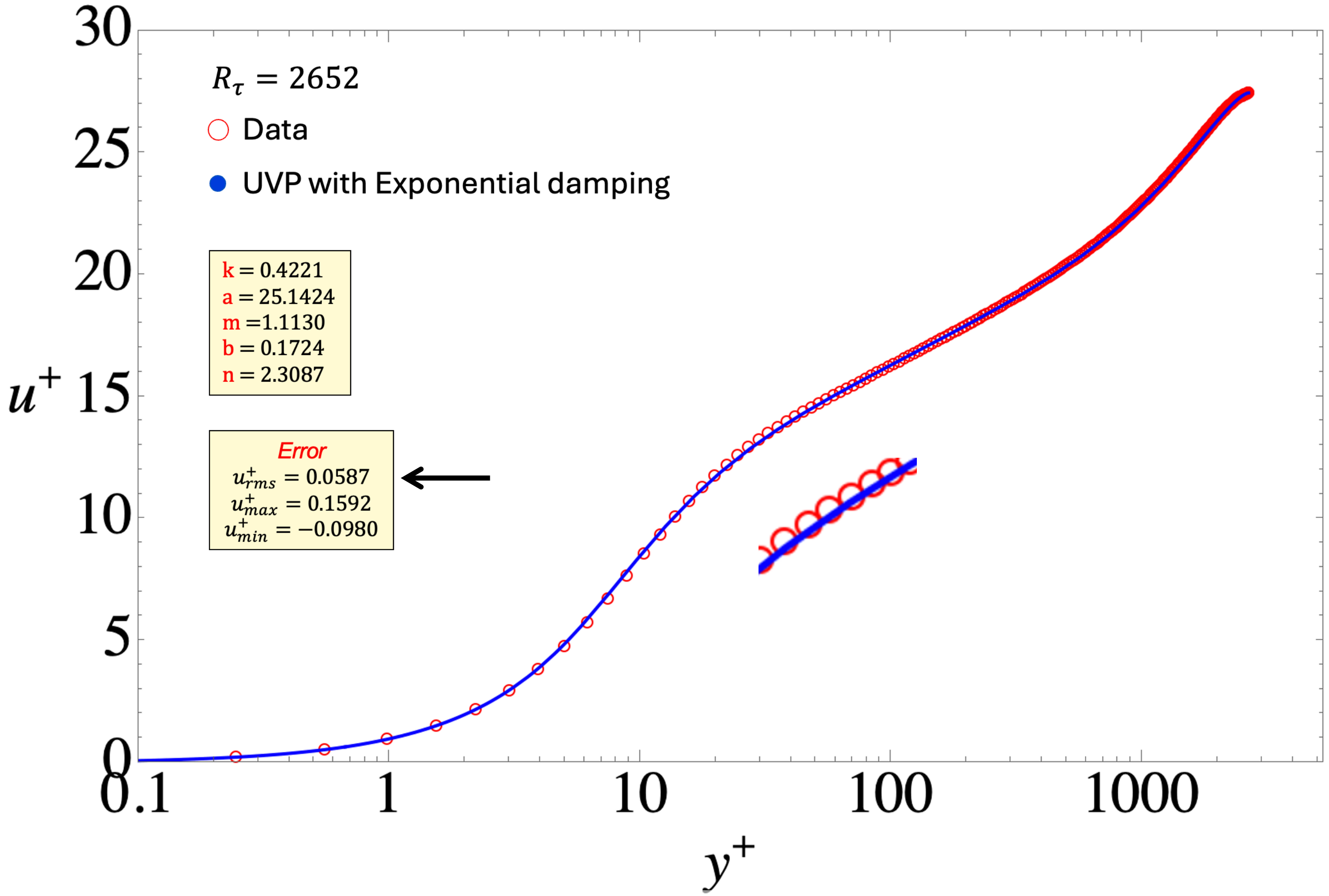
The choice δ_{995} works well in most cases.
The choice δ_{99} is generally too small.

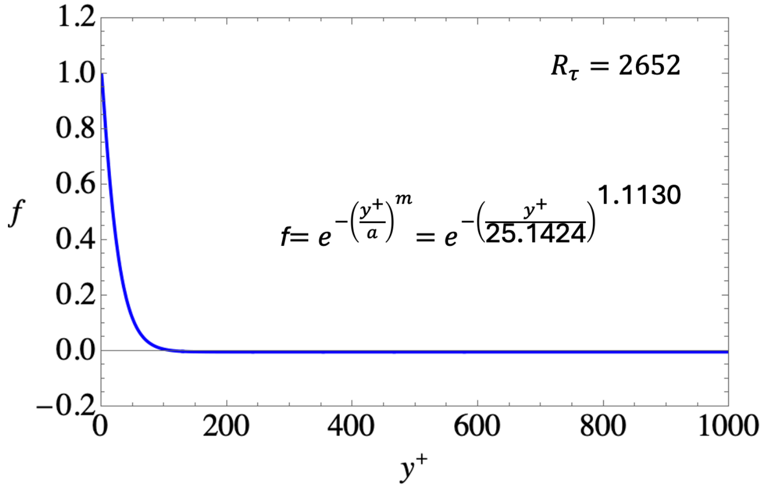
$\delta_h = \delta_{994}$

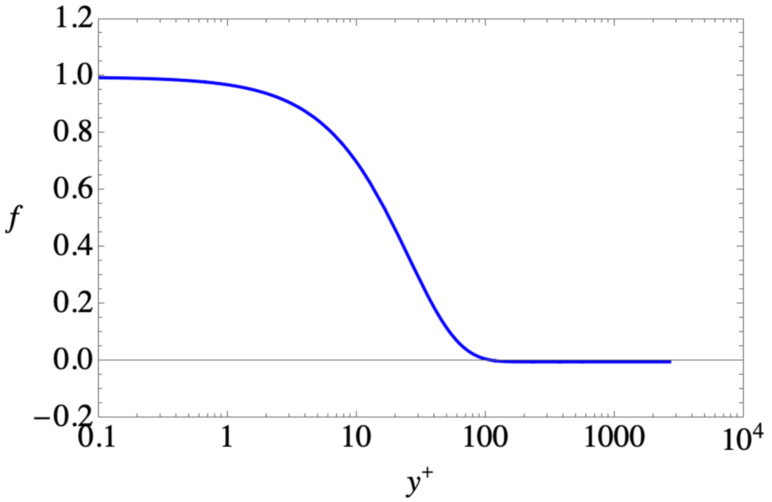
The case $R_\tau = 2652$ using $\delta_{h_{995}}$

Damping Function

$$\lambda(y^+) = \frac{ky^+(1 - e^{-(y^+/a)^m})}{\left(1 + \left(\frac{y^+}{bR_\tau}\right)^n\right)^{1/n}}$$



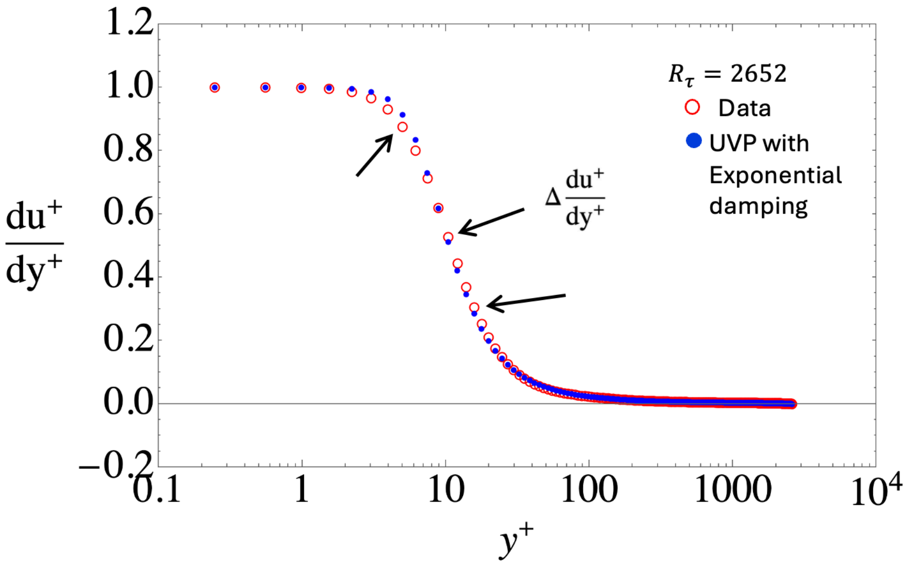


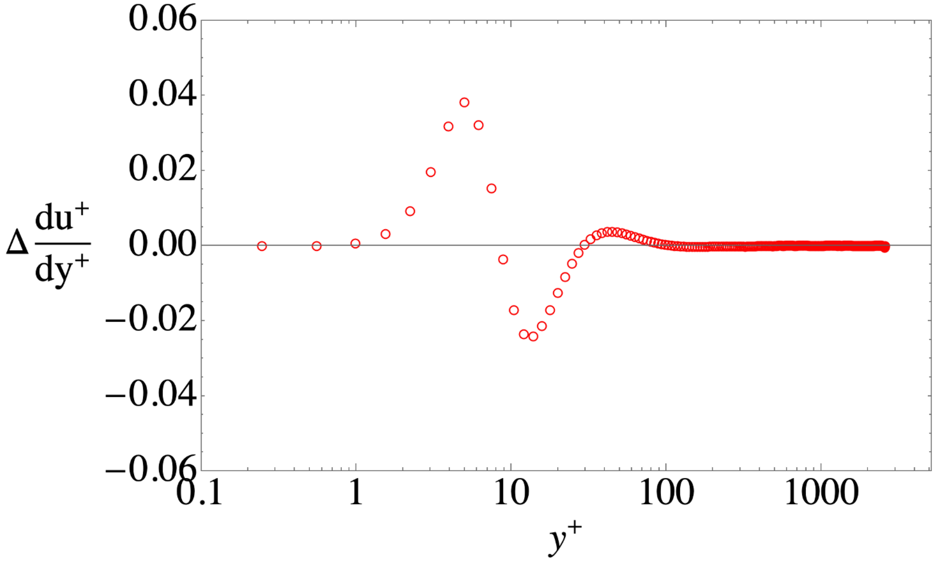


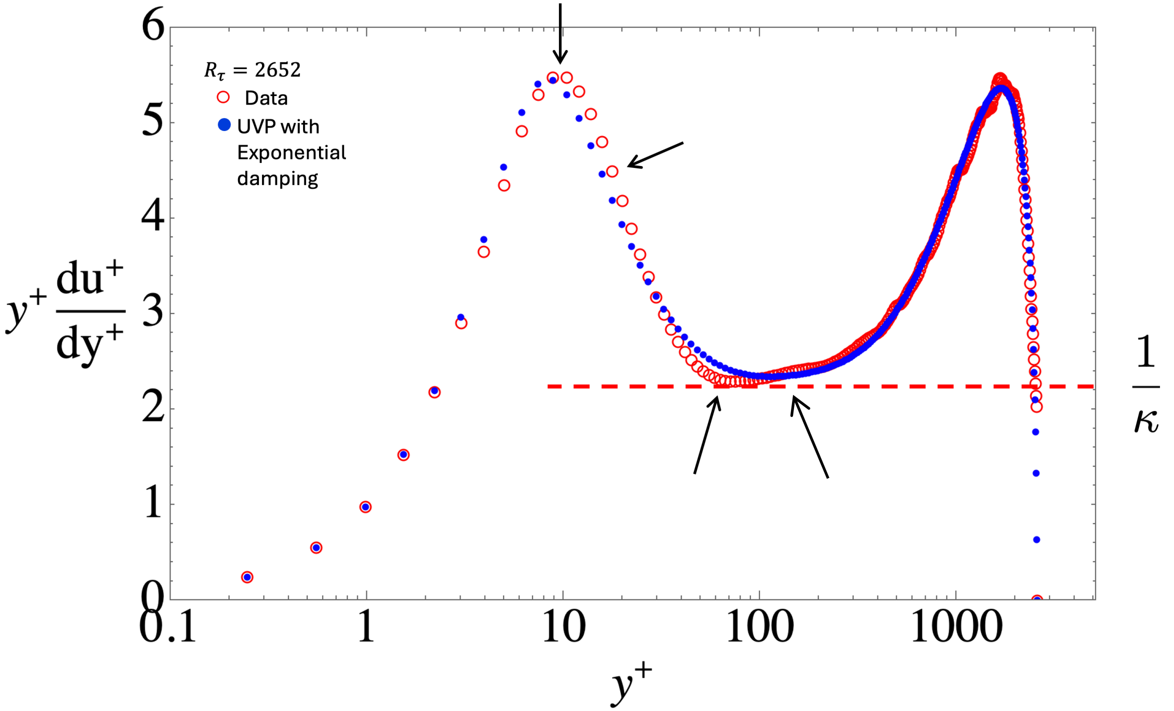
Data source

G. Eitel-Amor, Ö. Ramis, and P. Schlatter, "Simulation and validation of a spatially evolving turbulent boundary layer up to re = 8300," International Journal of Heat and Fluid Flow 47, 57 – 69 (2014).

Inaccuracy in the velocity derivative and log indicator function







Differentiate the law of the wall

$$u^+ = \frac{1}{\kappa} ln(y^+) + C \qquad y^+ \frac{du^+}{dy^+} = \frac{1}{\kappa}$$

New data-derived wall damping function

Inaccuracy of the UVP near the wall calls for a new wall damping function.

Exponential Damping function

$$\lambda = \frac{ky^+\left(1 - e^{-(y^+/a)^m}\right)}{\left(1 + \left(\frac{y^+}{bR_\tau}\right)^n\right)^{1/n}} \longrightarrow \lambda = \frac{ky^+(1 - \sigma)}{\left(1 + \left(\frac{y^+}{bR_\tau}\right)^n\right)^{1/n}}$$

Replacement

How do we determine σ ?

Ask:

What value of wall damping σ_i would be needed at each y_i^+ in the velocity profile to produce a perfect match between the velocity derivative data $\left(\frac{du^+}{dy^+}\right)_i$ and the UVP?

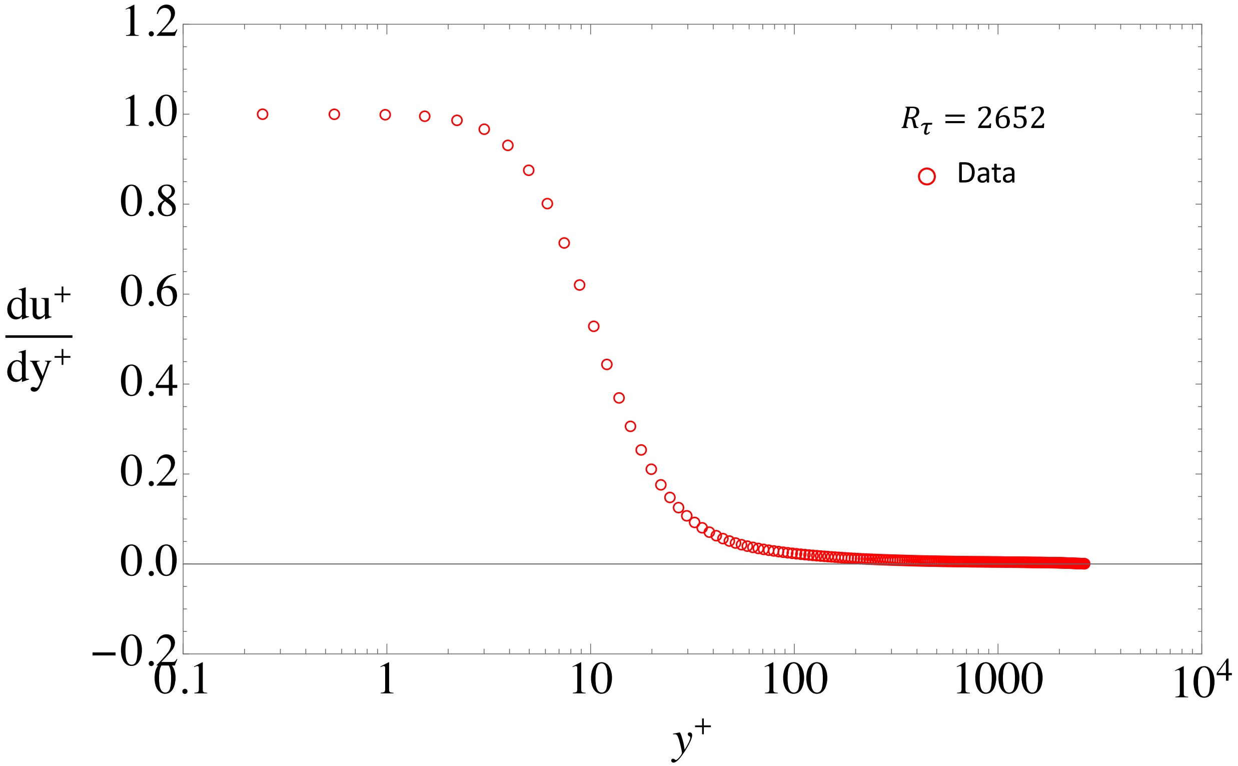
Step 1 - Specify data for y_i^+ and $\left(\frac{du^+}{dy^+}\right)_i$

Step 2 - Solve for σ_i

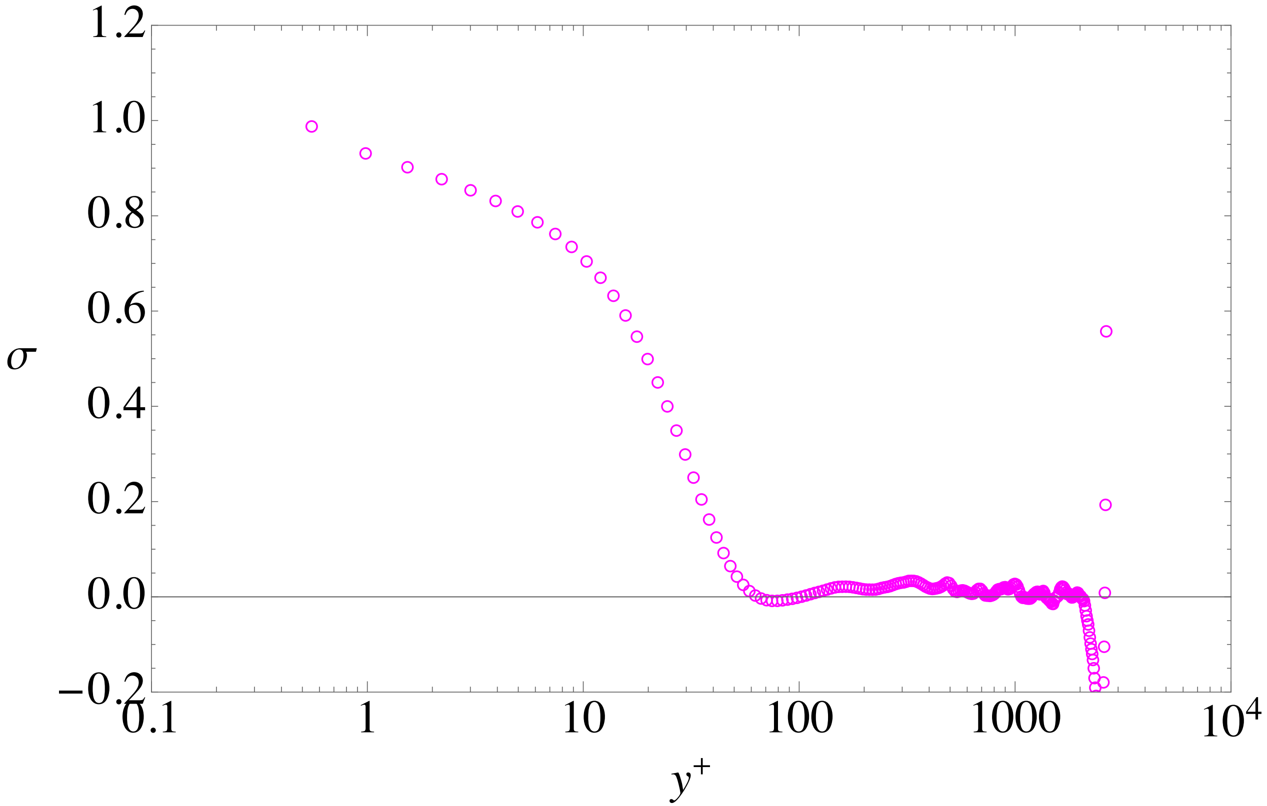
Solve $$\left(\frac{du^+}{dy^+}\right)_i = \frac{2\left(1-\frac{{y_i}^+}{R_\tau}\right)}{1+\left(1+4\left(\frac{k{y_i}^+(1-\sigma_i)}{\left(1+\left(\frac{{y_i}^+}{bR_\tau}\right)^n\right)^{1/n}}\right)^2\left(1-\frac{{y_i}^+}{R_\tau}\right)\right)^{1/2}}$$ for σ_i

The case $R_\tau = 2652$

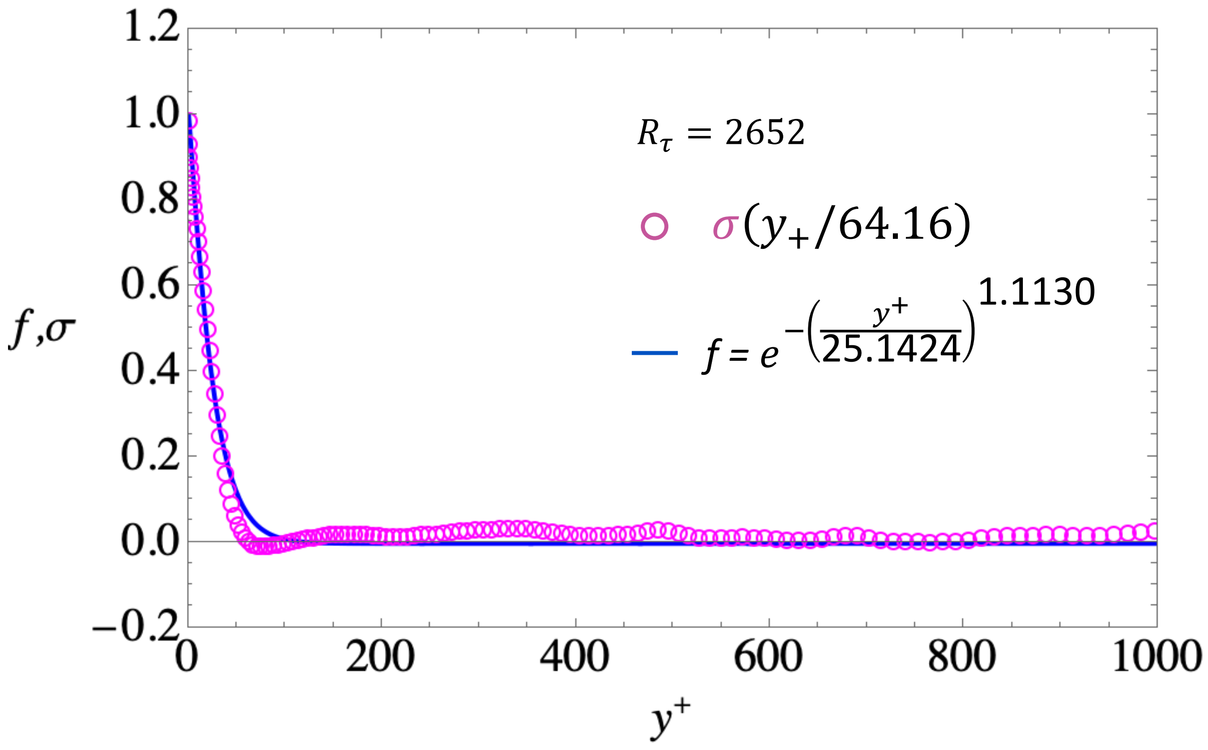
Data for y_i^+ and $\left(\frac{du^+}{dy^+}\right)_i$

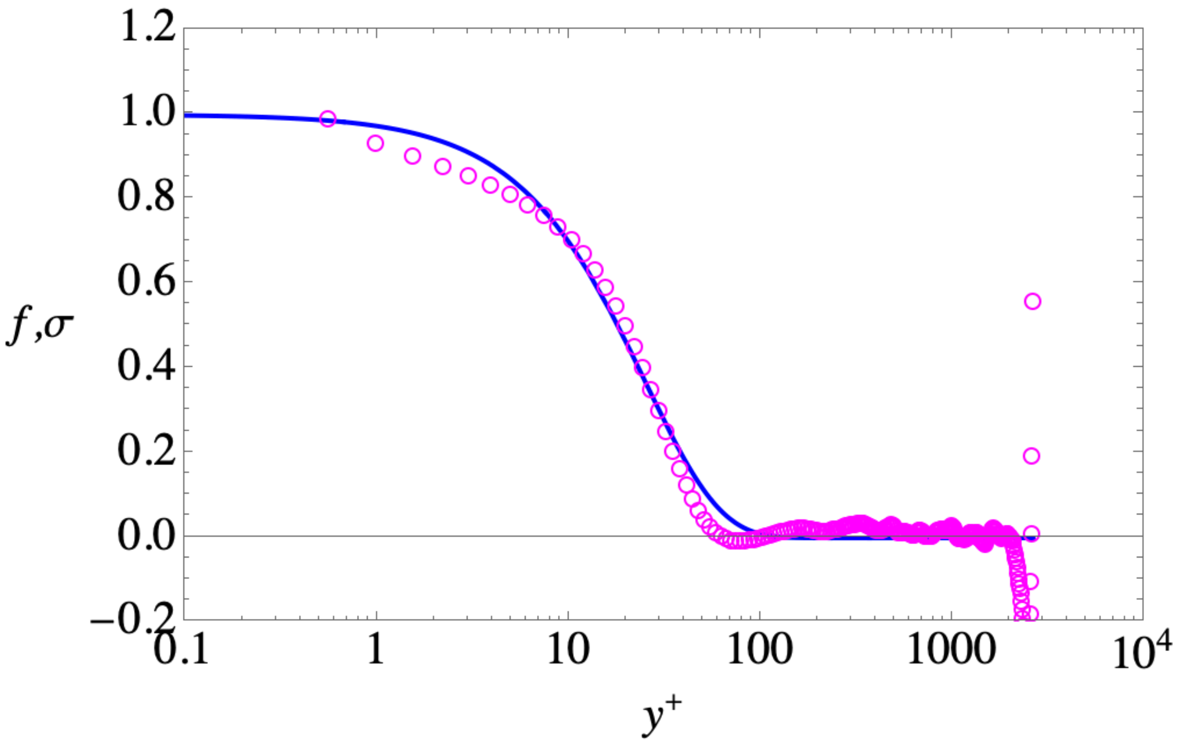


Solution for σ_i at each y_i^+



Comparison between the exponential damping function and σ

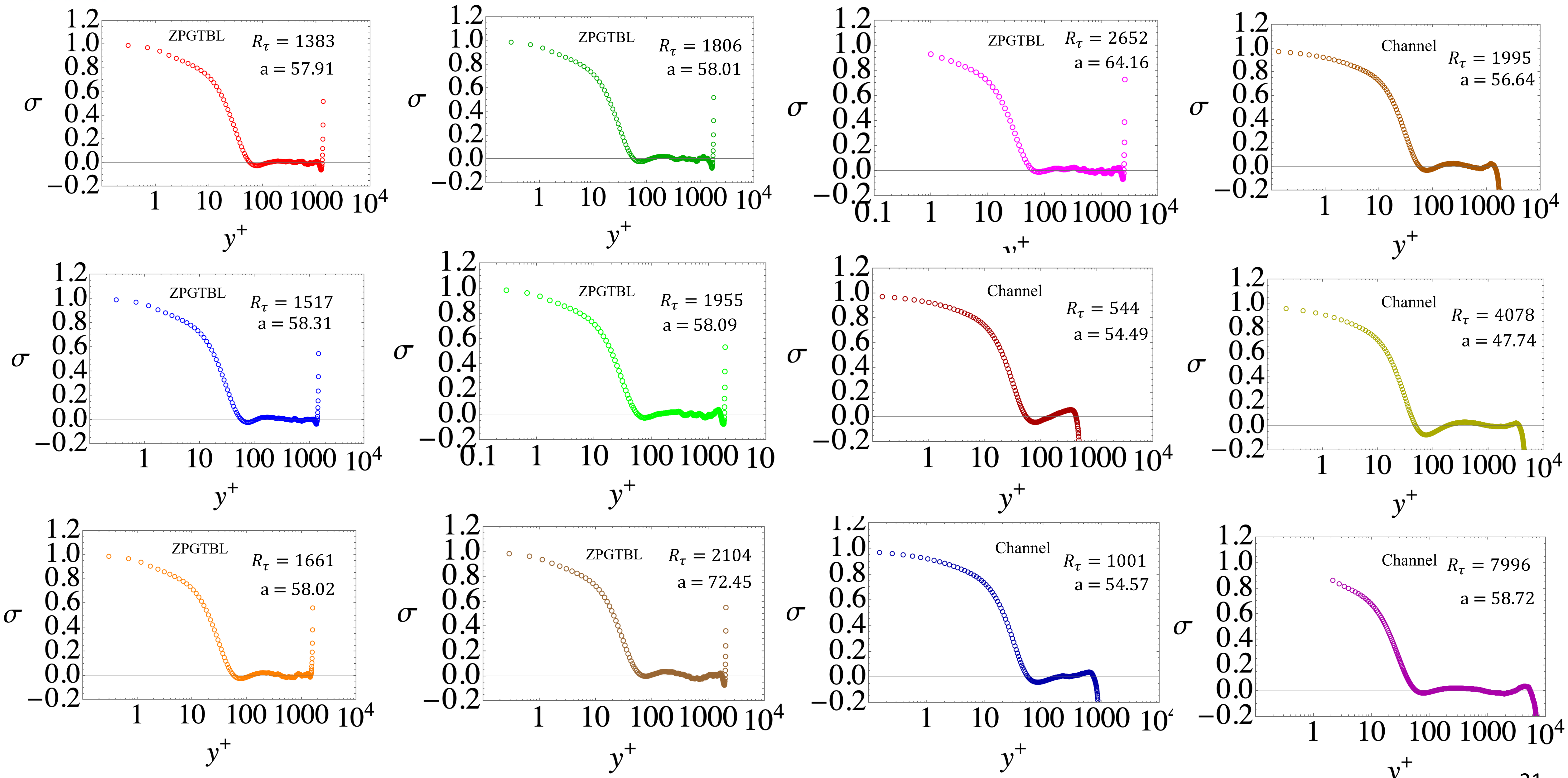




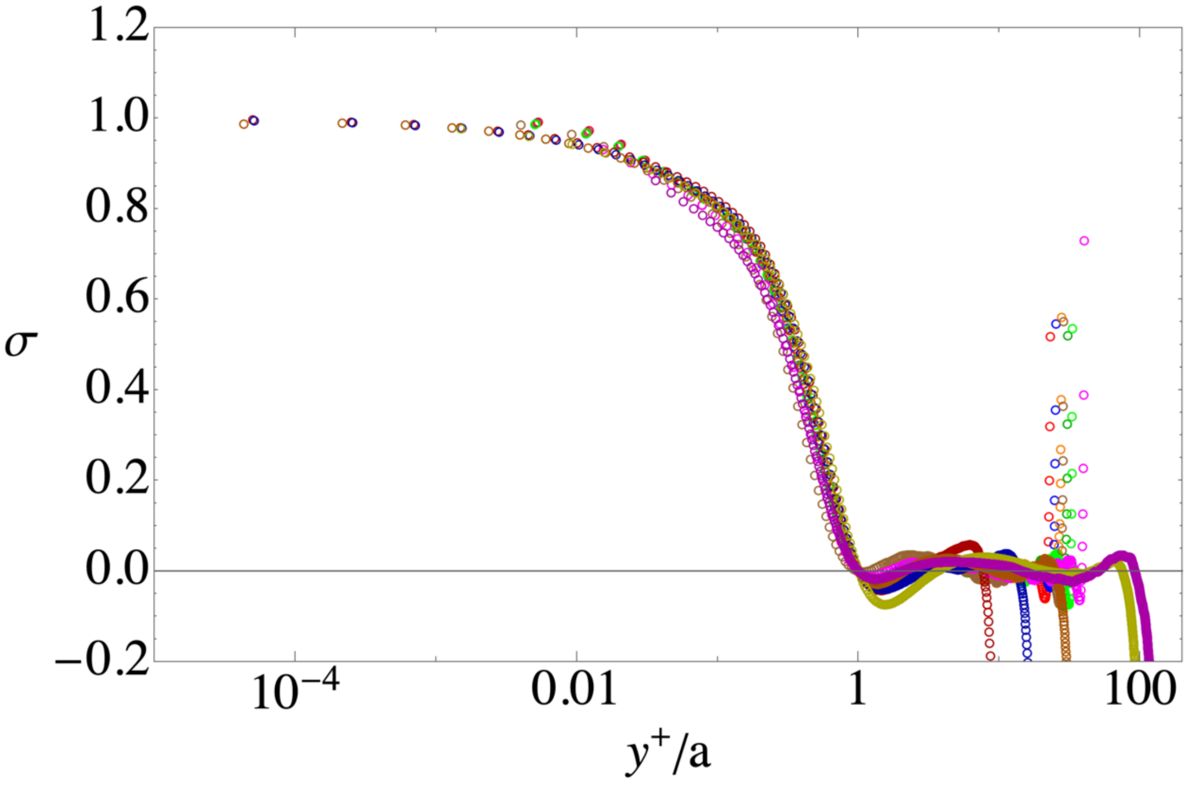
σ falls off faster than the exponential damping function

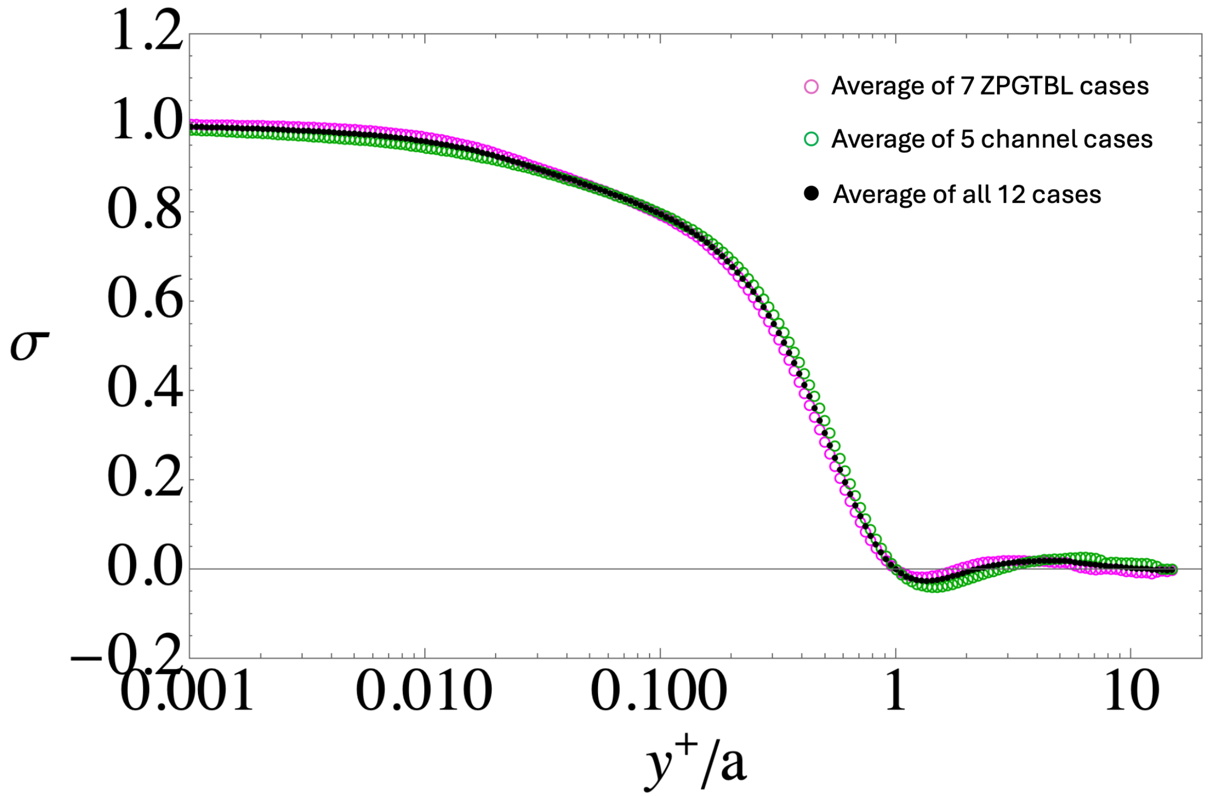
σ oscillates about zero with the first root at $y^+ = 64.16$

7 Zero Pressure Gradient simulations, 5 channel flow simulations



Normalize and average





Channel simulation data

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M. Bernardini, S. Pirozzoli, and P. Orlandi, “Velocity statistics in turbulent channel flow up to Rτ = 4000,” Journal of Fluid Mechanics 742, 171–191 (2014).

Y. Yamamoto and Y. Tsuji, “Numerical evidence of logarithmic regions in channel flow at Rτ = 8000,” Physical Review Fluids 3, 012602 (2018).

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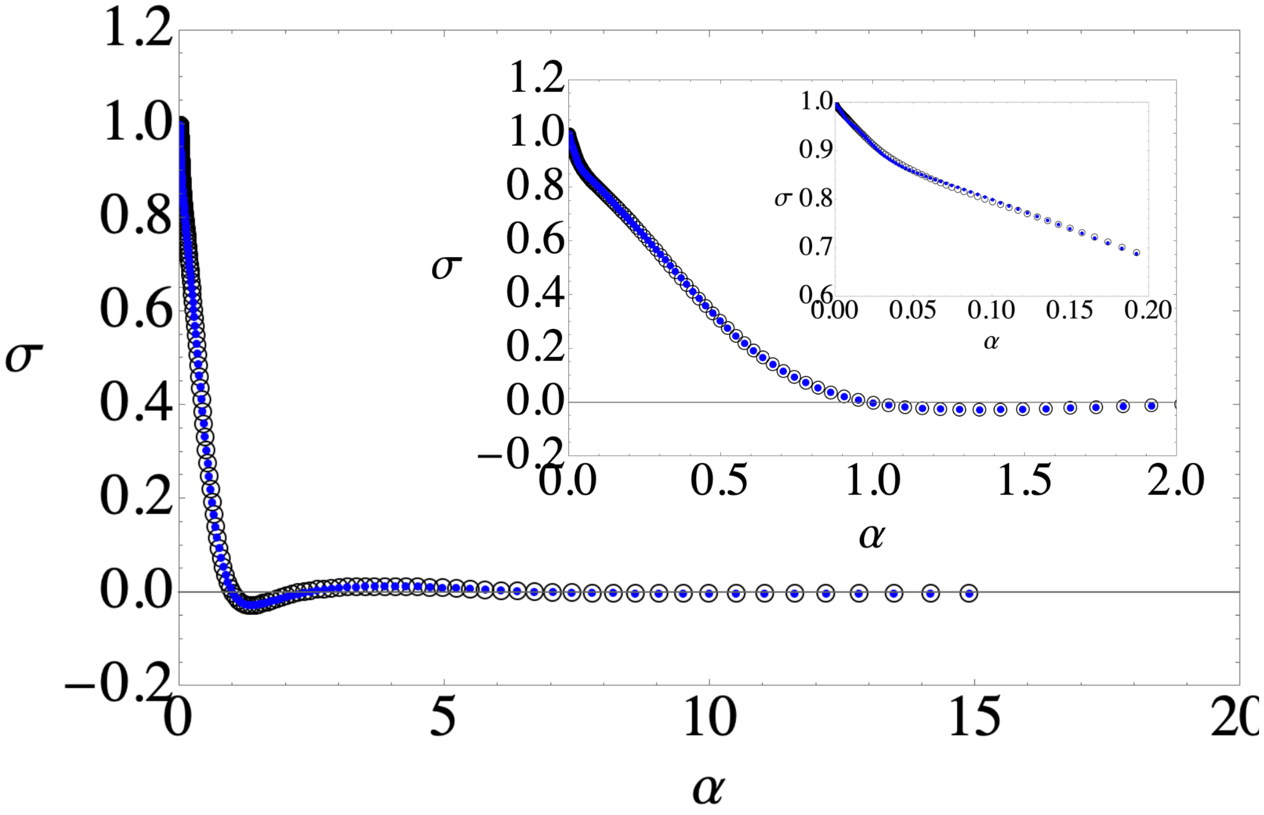
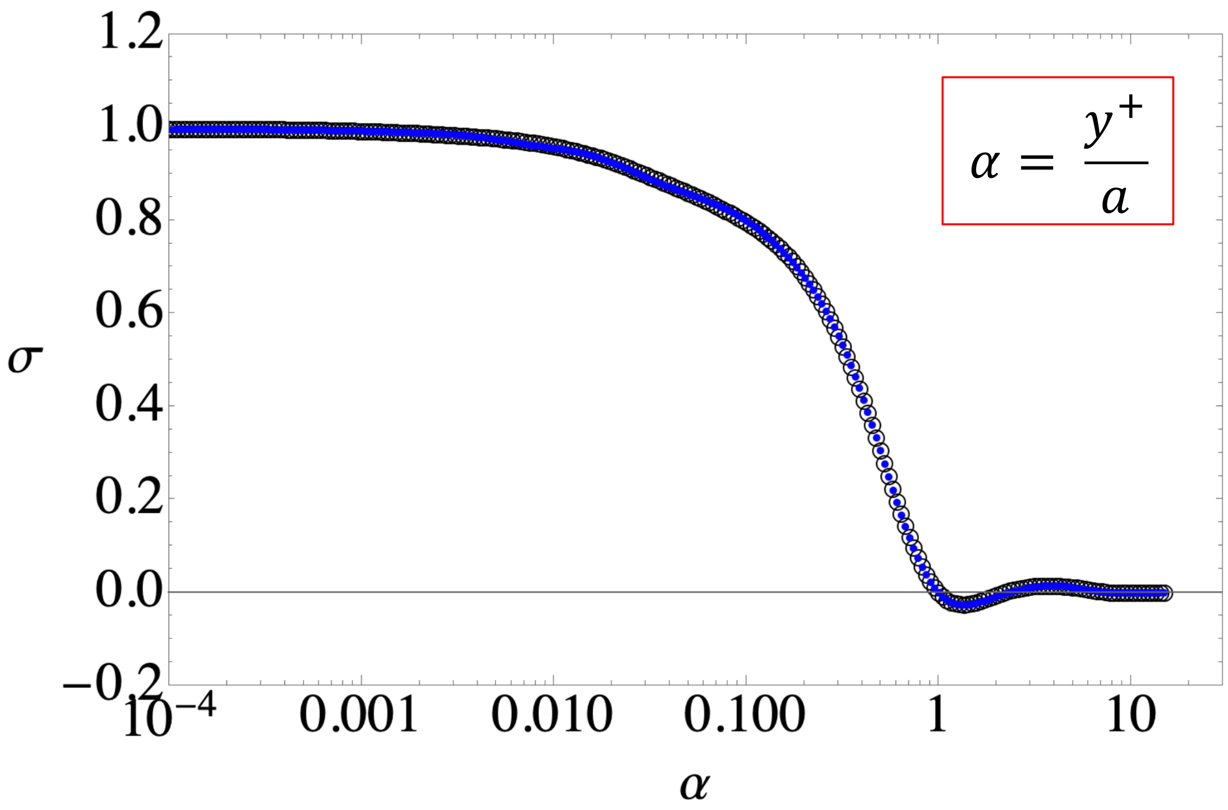
New wall damping function $\sigma(\alpha)$





$$\alpha = \frac{y^+}{a}$$

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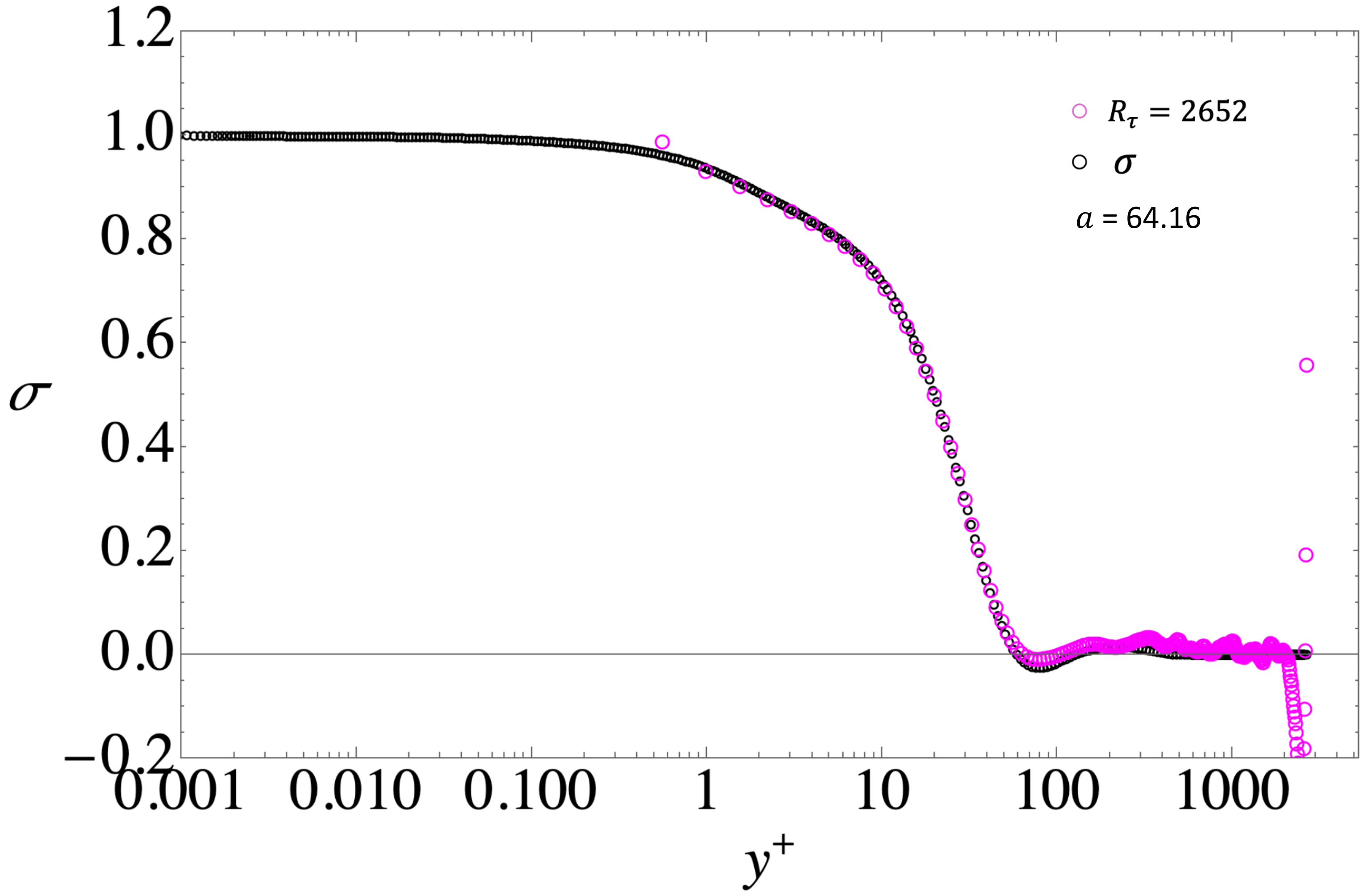
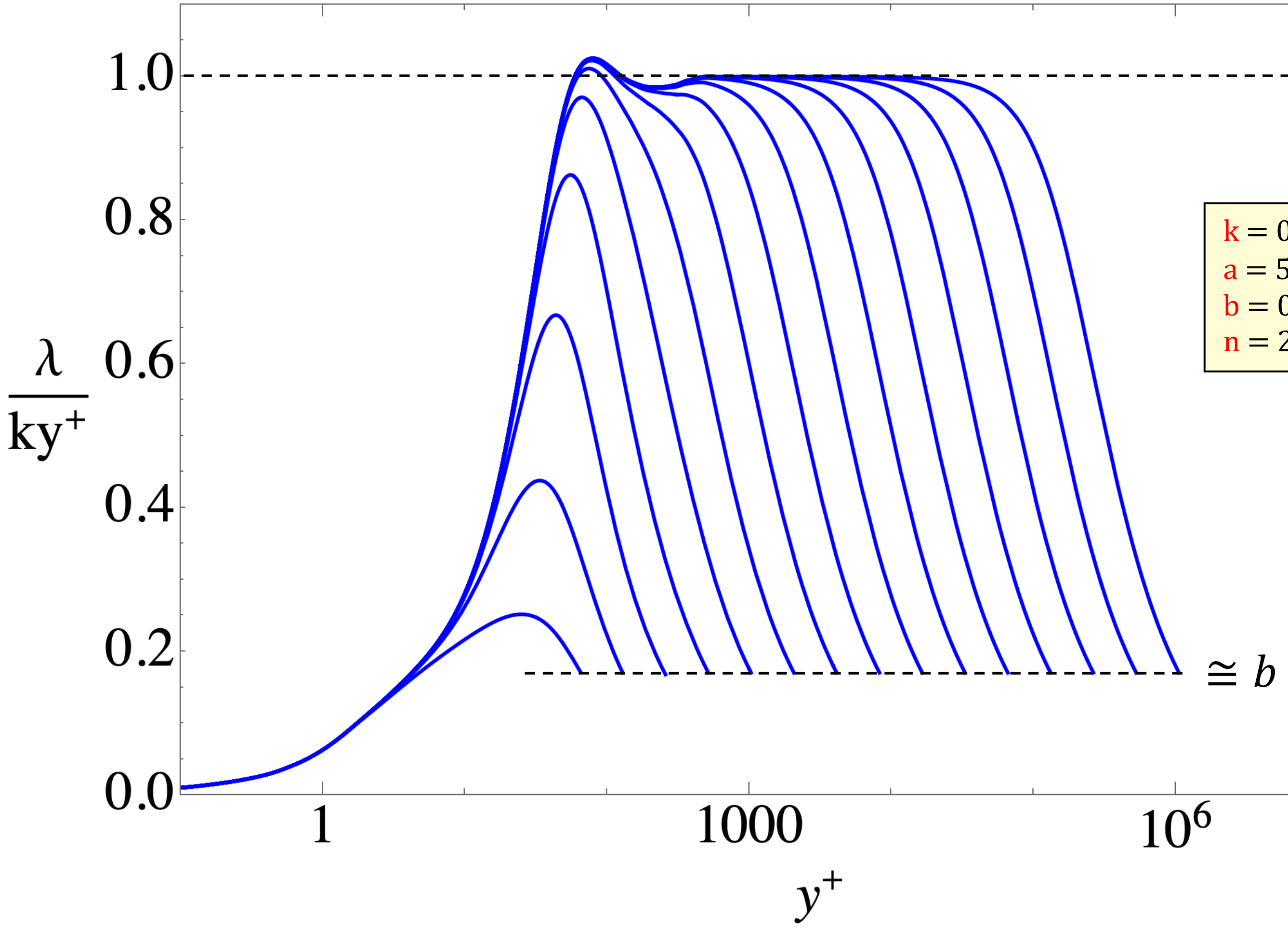
Comparison between the normalized and averaged wall damping function and the $R_\tau = 2652$ case



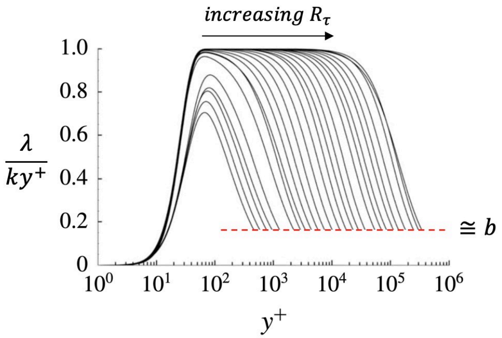
Continue to denote the new wall length scale a. Using $\sigma(\alpha)$, the optimum value of a for a given case will generally not coincide with the original first root of that case.

New mixing length function versus y^+ using parameters from $R_\tau = 2652$ data

The new λ/ky^+ has a peak of 1.027 at $y^+ = 77.54$



With exponential damping

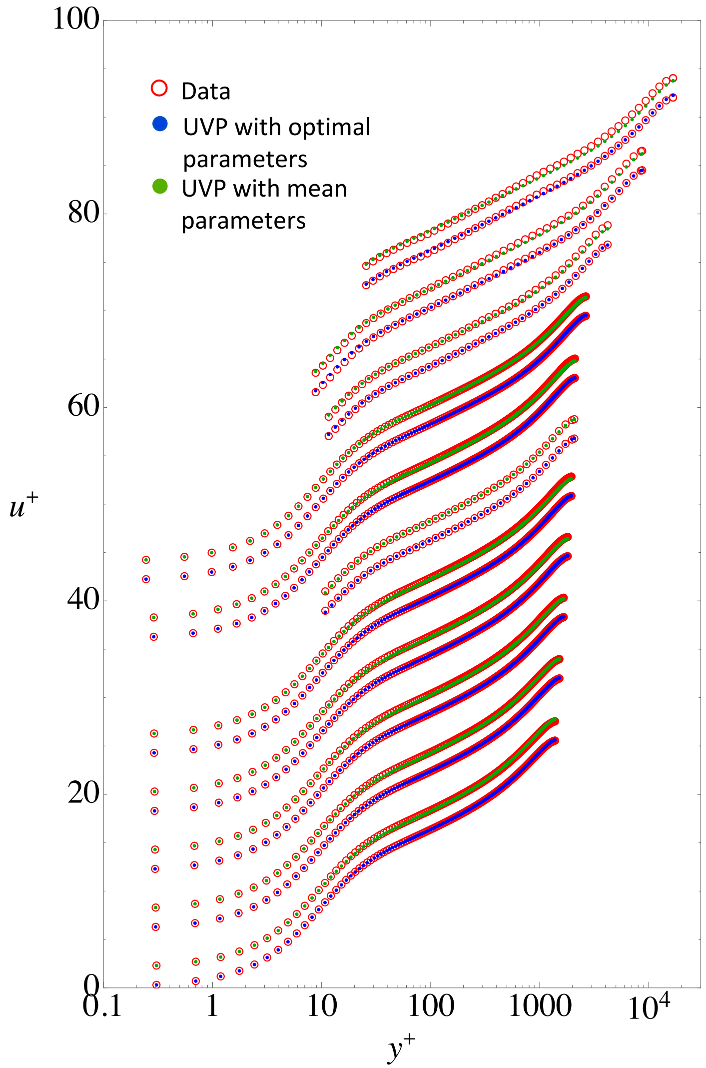


$R_\tau =$ 64, 128, 256, 512, 1024, 2048, 4096, 8192, 16384, 32768, 65536, 131072, 262144, 524288, 1,048576

Improved accuracy of the UVP using $\sigma\left(\frac{y^+}{a}\right)$

Turbulent boundary layers with $\sigma\left(\frac{y^+}{a}\right)$. All use $\delta_{h_{995}}$

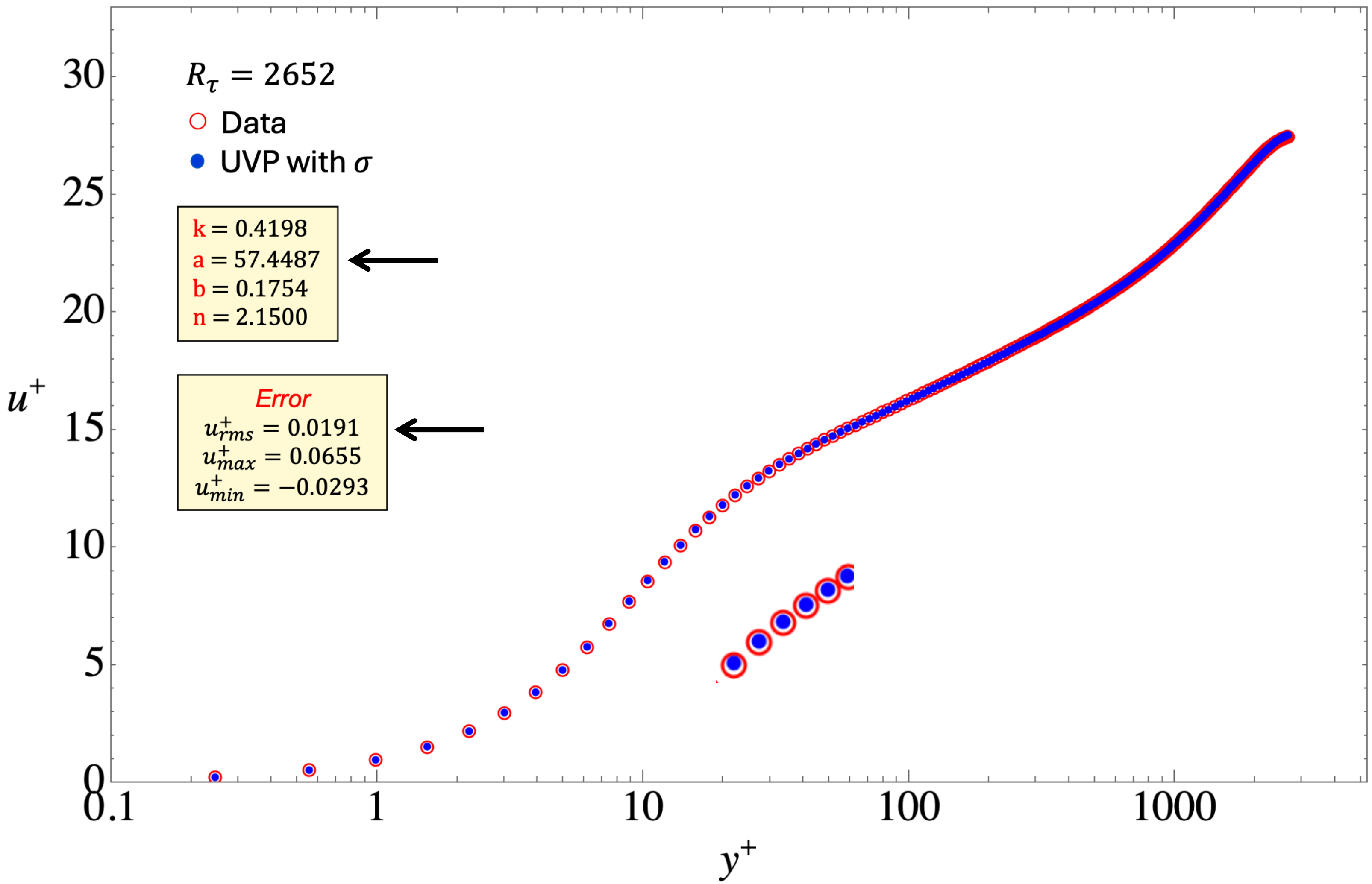
	R_τ	$\frac{u_e}{u_\tau}$ data	$\frac{u_e}{u_\tau}$ uvp	k	a	b	n	$u^+{}_{rms}$	$u^+{}_{max}$	$u^+{}_{min}$
	1383.36	25.5602	25.6235	0.4184	58.3896	0.1848	2.2718	0.0224	0.0740	-0.0355
	1516.99	25.9827	26.0290	0.4175	58.4200	0.1797	2.3300	0.0178	0.0592	-0.0231
	1660.91	26.3251	26.3616	0.4184	58.6000	0.1774	2.2750	0.0145	0.0481	-0.0226
	1806.16	26.6193	26.6531	0.4187	58.6144	0.1757	2.2392	0.0128	0.0464	-0.0198
	1955.32	26.8496	26.8735	0.4194	58.7980	0.1747	2.2373	0.0109	0.0364	-0.0226
Exp.	2077.23	26.7834	26.8127	0.4211	57.7682	0.1757	2.2343	0.0412	0.0425	-0.1539
	2104.15	27.0464	27.0696	0.4201	59.4125	0.1737	2.2843	0.0115	0.0332	-0.0182
	2651.82	27.4730	27.5296	0.4198	57.4487	0.1754	2.1500	0.0191	0.0655	-0.0293
Exp.	4199.96	28.8297	28.7843	0.4167	56.5855	0.1825	1.8796	0.0594	0.1823	-0.0751
Exp.	8708.43	30.5176	30.5136	0.4160	56.8617	0.1885	1.8066	0.1193	0.2890	-0.1789
Exp.	16711.52	32.0348	32.2555	0.4130	55.3048	0.1986	1.5777	0.1281	0.2210	-0.2507



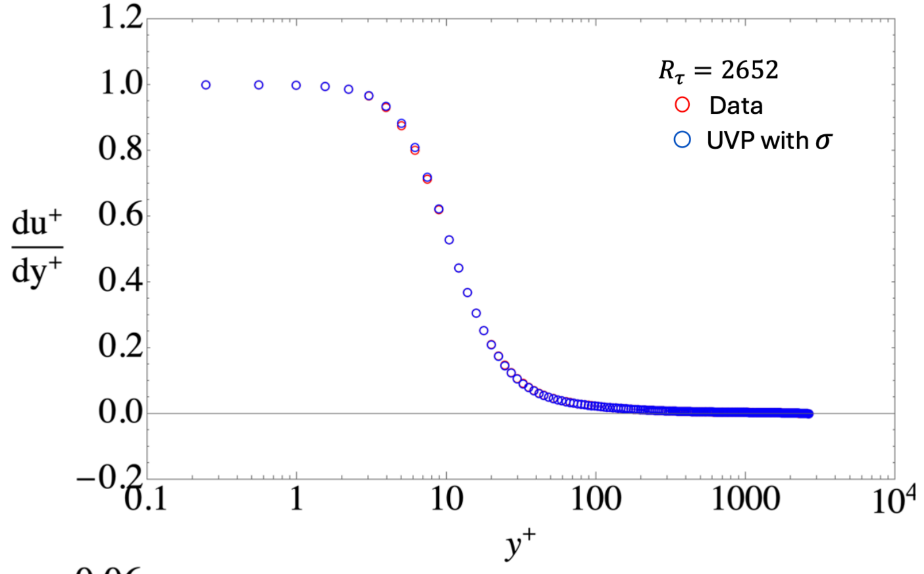
Mean parameter values

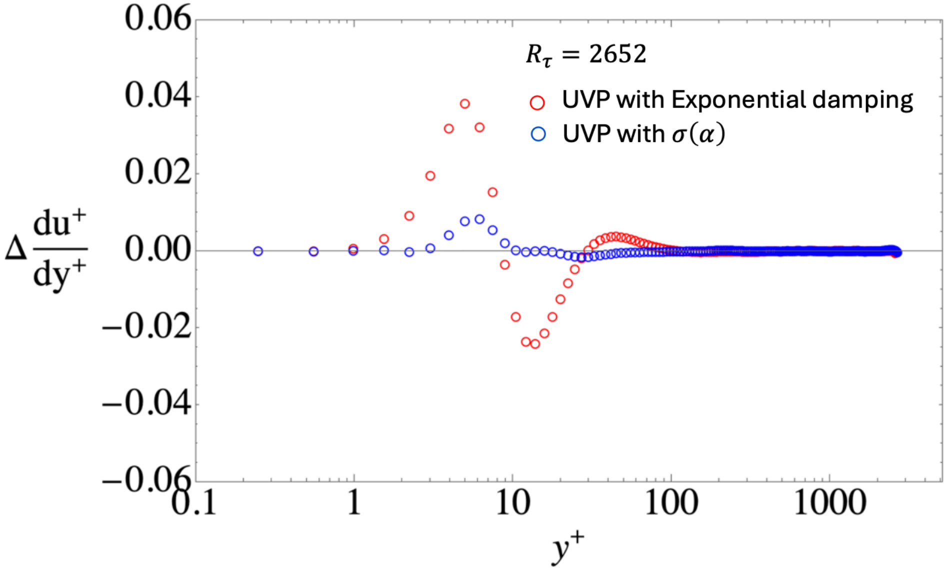
Flow	$\bar{k}$	k_{rms}	$\bar{a}$	a_{rms}	$\bar{b}$	b_{rms}	$\bar{n}$	n_{rms}
Channel (7 profiles)	0.3986	0.0057	53.7693	2.1647	0.4771	0.0429	1.4497	0.0843
Pipe (26 profiles)	0.4098	0.0061	55.5509	2.4906	0.3205	0.0361	1.6340	0.1768
ZPG Boundary Layer (11 profiles)	0.4180	0.0022	57.7730	1.1208	0.1806	0.0076	2.1169	0.2470
All Boundary Layers (41 profiles)	0.4201	0.0032	57.4137	1.0742	$f_b(0) = 0.1806$	0.0076	$f_n(0) = 2.1169$	0.2470

The case $R_\tau = 2652$ using $\sigma\left(\frac{y^+}{a}\right)$

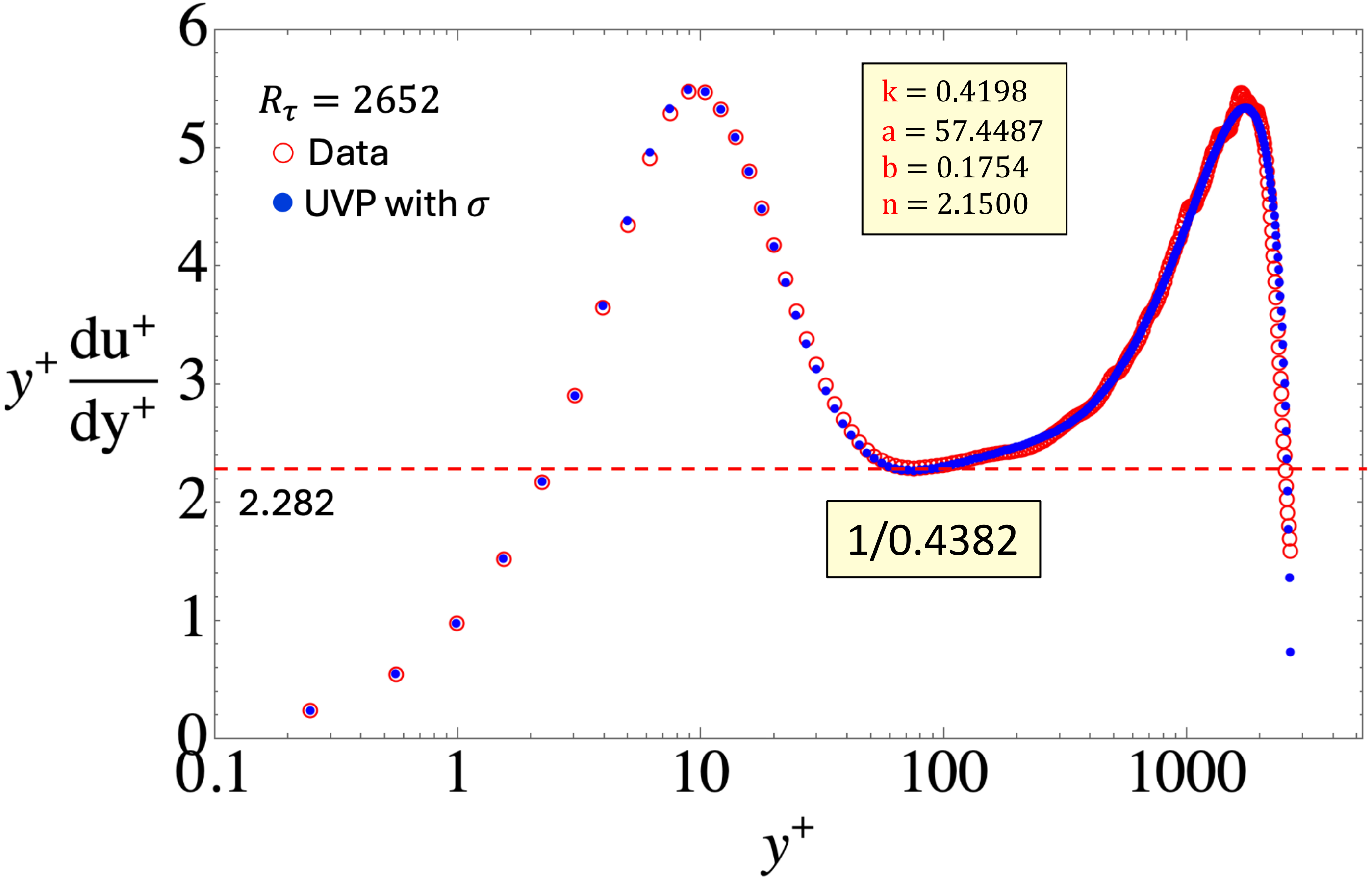


Velocity derivative and log indicator function are more accurate

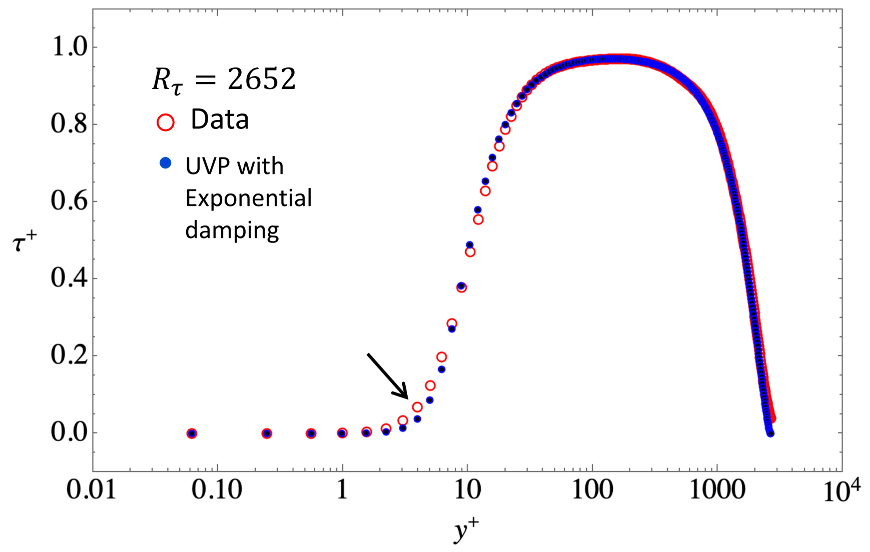


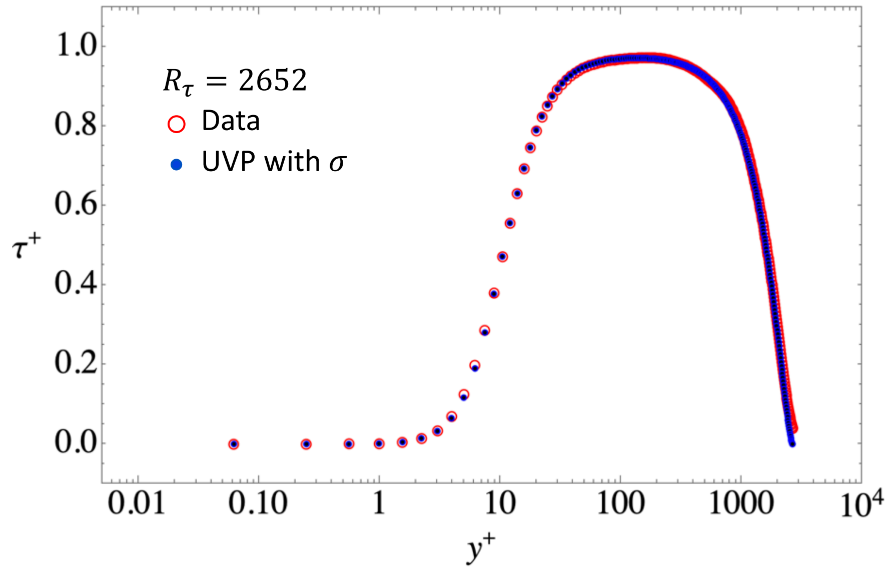


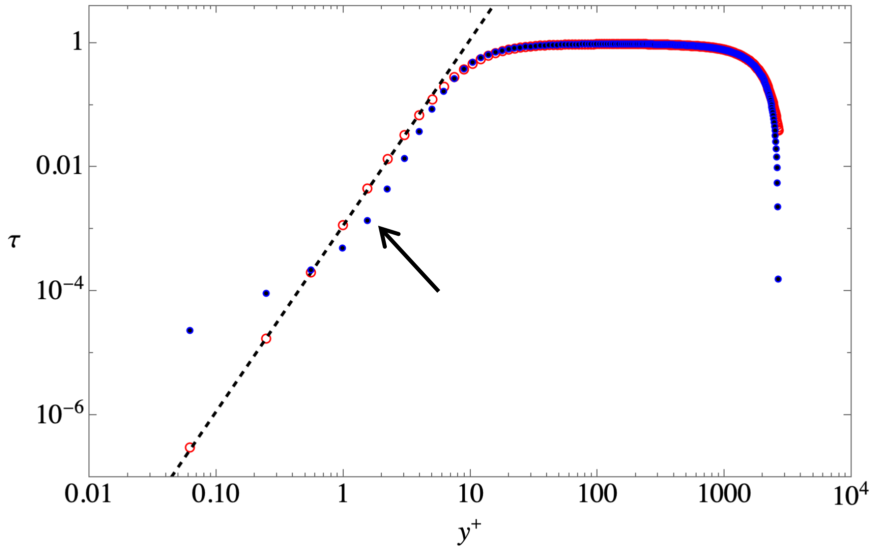
$R_\tau = 2652$

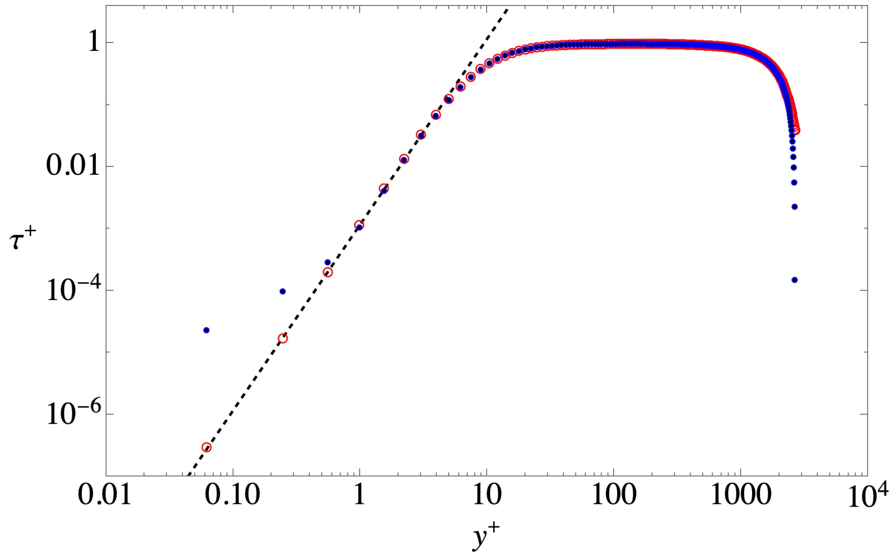


UVP Reynolds shear stresses near the wall are more accurate









Visualizing optimal parameters

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The number of parameters is reduced from 5 to 4

Only 4 parameters!

Wall: k, a

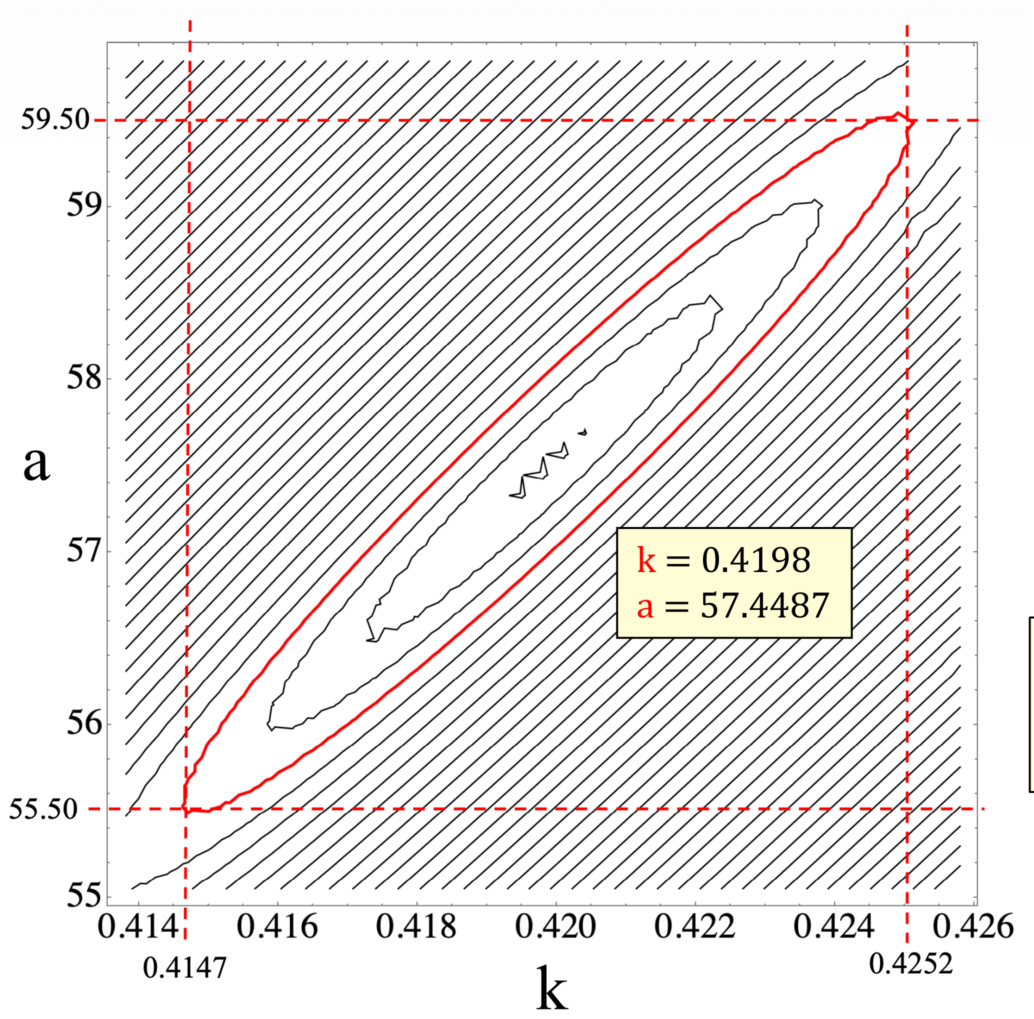
Wake: b, n

$$u^+(k, a, b, n, R_\tau, y^+) = \int_0^{y^+} \frac{2\left(1-\frac{s^+}{R_\tau}\right)}{\left[1+\left(1+4\left(\frac{k s^+\left(1-\sigma\left(\frac{s^+}{a}\right)\right)}{\left(1+\left(\frac{s^+}{b R_\tau}\right)^n\right)^{1/n}}\right)^2\left(1-\frac{s^+}{R_\tau}\right)\right)\right]^{1/2}} ds^+$$

Minimize with respect to (k, a), then (b, n), then (k, a) and so forth

$$u^+_{rms}(k,a) = \left(\frac{1}{N}\sum_{i=1}^{N}\left(u^+(k,a,b,n,R_\tau,y_i^+) - u_i^+(y_i^+)\right)^2\right)^{1/2}_{(b,n)constant}$$

$$u^+_{rms}(b,n) = \left(\frac{1}{N}\sum_{i=1}^{N}\left(u^+(k,a,b,n,R_\tau,y_i^+) - u_i^+(y_i^+)\right)^2\right)^{1/2}_{(k,a)constant}$$



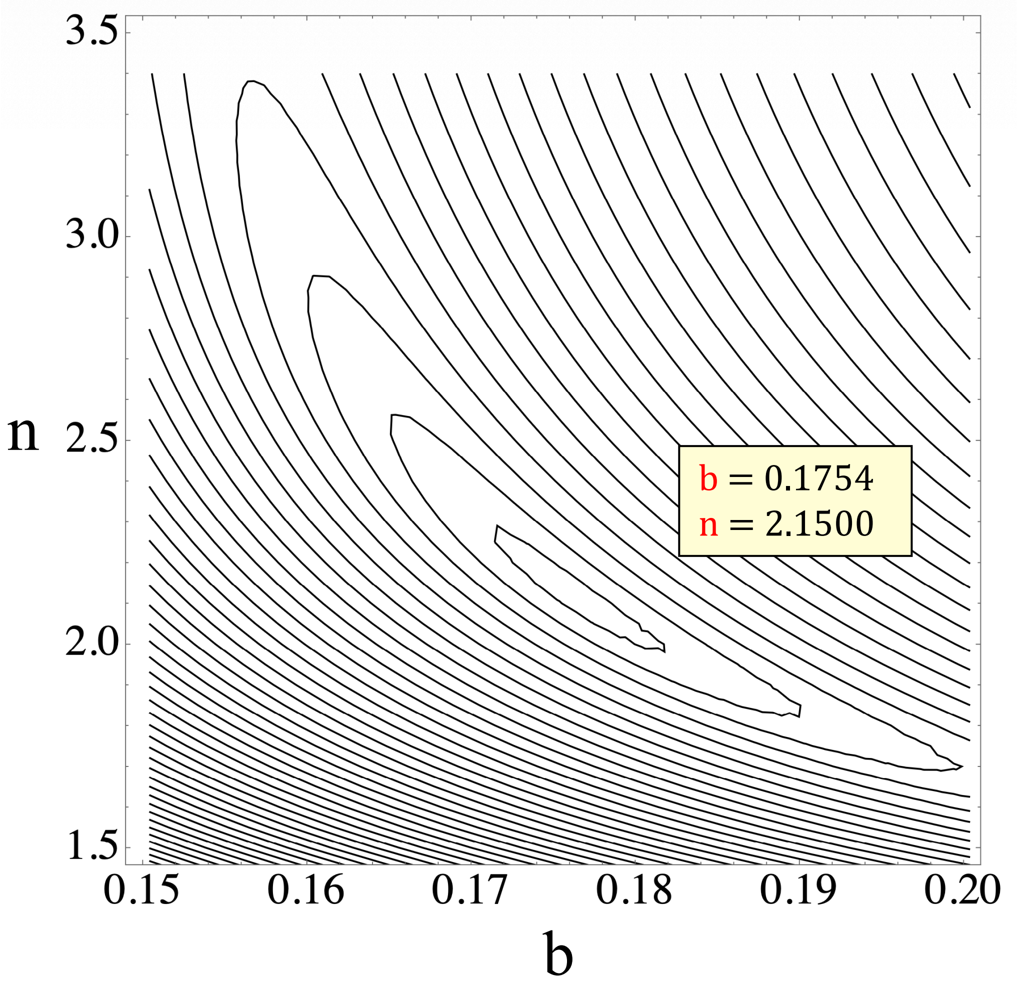
Error

$u^+_{rms} = 0.0191$

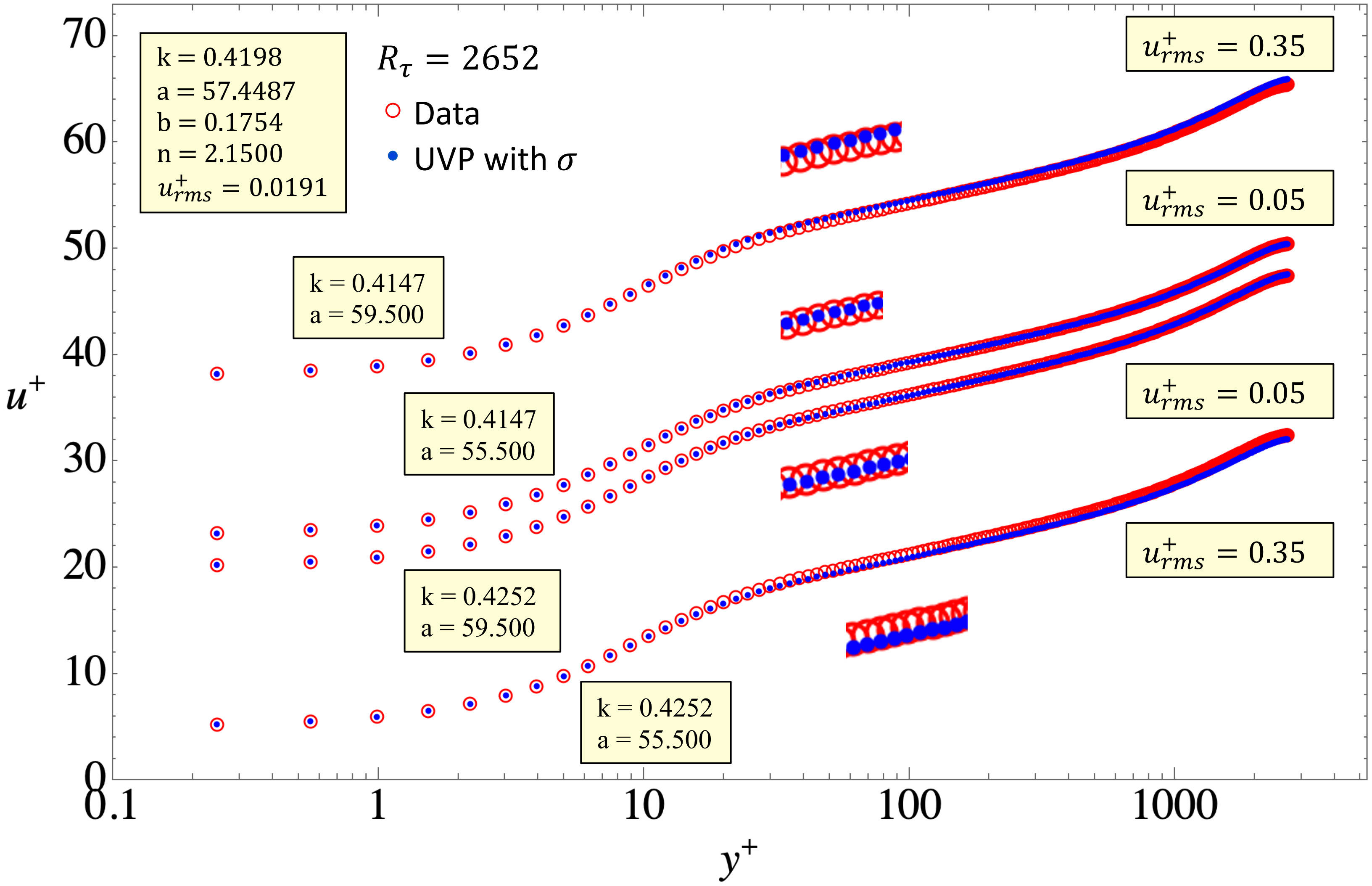
$u^+_{max} = 0.0655$

$u^+_{min} = -0.0293$

This procedure makes the UVP easy to optimize!



Specifying a value of k is not meaningful without also specifying a wall damping length scale a.



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We can visualize the optimum parameter zone in 3D

$$u^+(k,a,b,n,R_\tau,y^+) = \int_0^{y^+} \frac{2\left(1-\frac{s^+}{R_\tau}\right)}{1+\left(1+4\left(\frac{ks^+\left(1-\sigma\left(\frac{s^+}{a}\right)\right)}{\left(1+\left(\frac{s^+}{bR_\tau}\right)^n\right)^{1/n}}\right)^2\left(1-\frac{s^+}{R_\tau}\right)\right)^{1/2}} ds^+$$

Map the UVP from 7 to 6 dimensions, k and a ⟶ ka

$$(u^+, \underline{k, a}, b, n, R_\tau, y^+) \rightarrow (ku^+, \underline{ka}, b, n, kR_\tau, ky^+)$$

$$ku^+(ka,b,n,kR_\tau,ky^+) = \int_0^{ky^+} \frac{2\left(1-\frac{ks^+}{kR_\tau}\right)}{1+\left(1+4\left(\frac{ks^+\left(1-\sigma\left(\frac{ks^+}{ka}\right)\right)}{\left(1+\left(\frac{ks^+}{bkR_\tau}\right)^n\right)^{1/n}}\right)^2\left(1-\frac{ks^+}{kR_\tau}\right)\right)^{1/2}} dks^+$$

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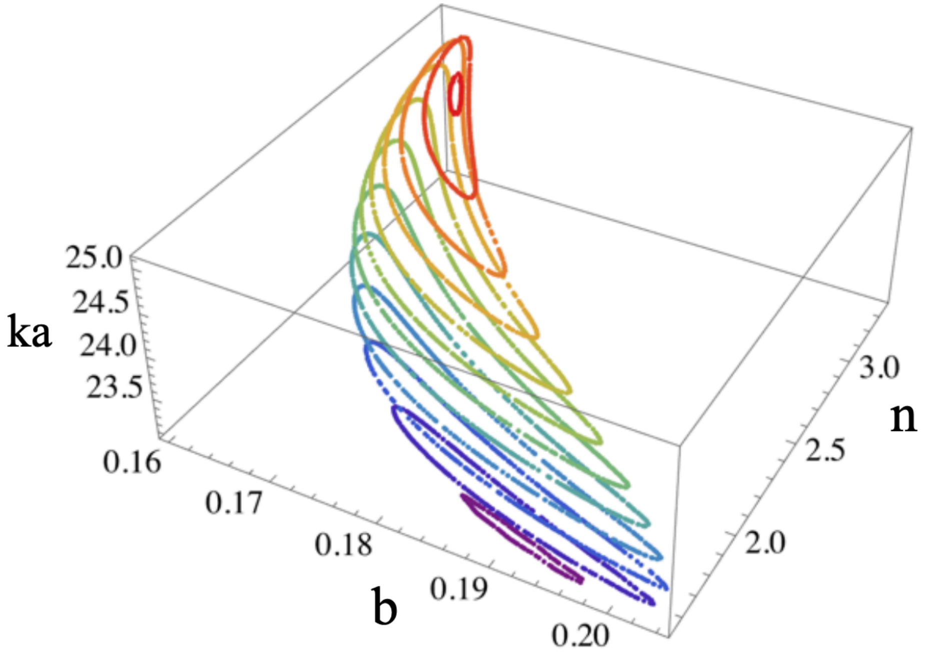
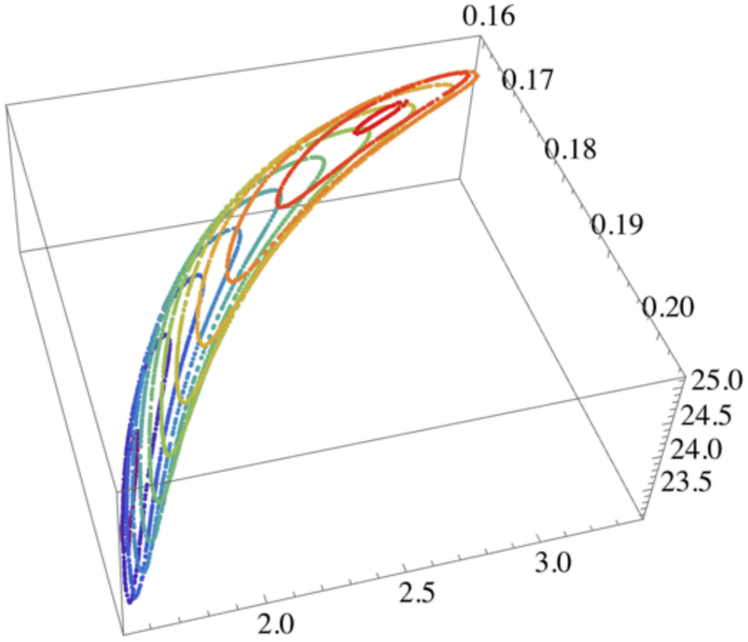
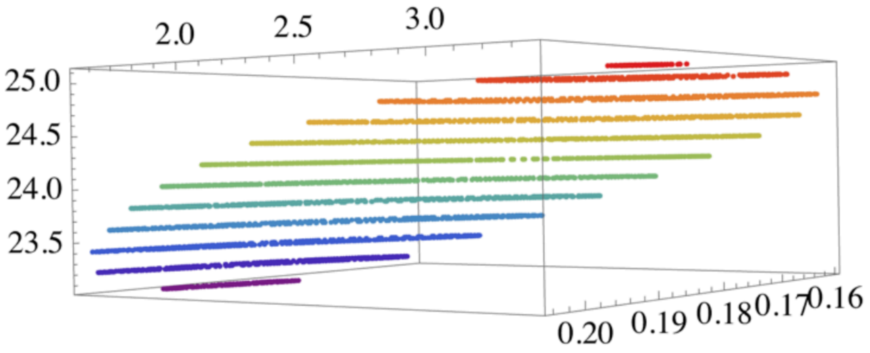
Minimum error zone in (ka, b, n) parameter space

The surface corresponds to $k\, u^{+}_{rms} < 0.04198$ or $u^{+}_{rms} < 0.1$



There are not multiple minima, but rather one smooth, slightly flattened, ellipsoidal volume.





Nonzero pressure gradient and a modified Clauser parameter

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The Kármán boundary layer integral equation

$$\frac{d\delta_2}{dx} + (2\delta_2 + \delta_1)\frac{1}{u_e}\frac{du_e}{dx} - \left(\frac{u_\tau}{u_e}\right)^2 = 0$$

$$U = \frac{u_e(x)}{U_\infty}$$

Re-express the Kármán equation in terms of R_{δ_1}, R_{δ_2} and u_e/u_τ.

$$R_x = \frac{xU_\infty}{\nu}$$

$$\frac{dR_\tau}{dR_x} = \frac{U}{dR_{\delta_2}/dR_\tau}\left(\frac{u_\tau}{u_e}\right)^2\left(1 - \frac{R_{\delta_1} + R_{\delta_2}}{\left(\frac{u_\tau}{u_e}\right)^2}\frac{1}{U^2}\frac{dU}{dR_x}\right)$$

$$R_\tau = \frac{\delta_h(x)U_\tau(x)}{\nu}$$

R_τ becomes the primary unknown with R_x as the independent variable. All the relevant functions depend on R_τ.

$$\frac{u_e}{u_\tau} = u^+(k,a,b,n,R_\tau,R_\tau) \equiv F_0(R_\tau)$$

$$R_{\delta_1} = \frac{\delta_1 u_e}{\nu} = \frac{u_e}{u_\tau}\int_0^{R_\tau}\left(1 - \frac{u_\tau}{u_e}u^+\right)dy^+ \equiv F_1(R_\tau)$$

$$R_{\delta_2} = \frac{\delta_2 u_e}{\nu} = \int_0^{R_\tau} u^+\left(1 - \frac{u_\tau}{u_e}u^+\right)dy^+ \equiv F_2(R_\tau)$$

$$\frac{dF_2(R_\tau)}{dR_\tau} \equiv F_3(R_\tau)$$

$$\frac{dR_\tau}{dR_x} = \frac{U}{{F_0}^2 F_3}(1 + \beta_c)$$

Modified Clauser parameter

$$\beta_c = -{F_0}^2(F_2 + F_1)\frac{1}{U^2}\frac{dU}{dR_x} = \rho\frac{\delta_2 + \delta_1}{\tau_w}\frac{dp_e}{dx}$$

The usual definition is

$$\beta = \rho\frac{\delta_1}{\tau_w}\frac{dp_e}{dx}$$

UVP fits APG and FPG Turbulent Boundary Layer data quite well

C_{px} $1/m$	x mm	u_{e995} m/sec	R_τ	R_{δ_h}	R_{δ_1}	R_{δ_2}	β	β_c	$\left(\frac{u_e}{u_\tau}\right)$ data	$\left(\frac{u_e}{u_\tau}\right)$ calc	k	a	b	n	u^+ rms	u^+ max	u^+ min
0.0	1200	10.330	840.79	19709	2962	2120	0.0	0.0	23.4405	23.4325	0.4225	57.8038	0.2082	2.312	0.0723	0.0949	-0.1496
0.288	1800	9.9457	1082.73	27390	4615	3248	0.664	1.131	25.2970	25.5736	0.4233	57.4112	0.1626	2.322	0.1108	0.2766	-0.2288
0.308	2240	9.2284	1128.48	31506	6058	4088	1.418	2.375	27.9189	27.9609	0.4270	58.4667	0.1215	2.655	0.0926	0.1771	-0.2638
0.284	2640	8.5621	1189.00	37310	8239	5255	2.820	4.618	31.3791	31.3475	0.4188	56.6600	0.0929	3.459	0.0833	0.2065	-0.2123
0.257	2880	8.1303	1233.35	42343	10167	6215	4.366	7.035	34.3316	34.2720	0.4188	56.5600	0.0770	4.350	0.1541	0.2654	-0.3363
0.251	3080	7.8722	1199.70	45604	11794	6849	6.725	10.630	38.0131	37.5840	0.4177	54.2600	0.0640	5.400	0.2735	0.5241	-0.4465
0.0	1200	30.611	2294.49	60244	8215	6131	0.0	0.0	26.2558	26.2082	0.4270	58.7234	0.2000	2.020	0.0249	0.0558	-0.0467
0.254	1800	28.967	2693.75	75666	11733	8554	0.689	1.192	28.0896	28.1956	0.4233	57.2152	0.1600	2.090	0.0931	0.0952	-0.1641
0.270	2240	26.954	2972.75	89072	15525	10993	1.370	2.341	29.9626	30.1104	0.4231	57.7102	0.1280	2.3840	0.0732	0.1596	-0.1272
0.268	2640	25.074	3204.70	104854	20681	14030	2.685	4.506	32.7188	32.7731	0.4239	57.8102	0.1000	2.994	0.0847	0.0733	-0.1894
0.242	2880	23.814	3289.93	115064	24380	16028	3.795	6.289	34.9745	34.8670	0.4170	56.7612	0.0874	3.987	0.1017	0.1811	-0.1497
0.233	3080	22.839	3370.72	127678	29347	18532	5.835	9.520	37.8787	37.7375	0.4137	55.7122	0.0744	4.982	0.1404	0.2590	-0.3114

APGTBL data

A. E. Perry and I. Marusic, "A wall-wake model for the turbulence structure of boundary layers. part 2. further experimental support," Journal of Fluid Mechanics 298, 389 – 407 (1995).

K $\times 10^7$	x mm	u_{entry} m/sec	R_τ	R_{δ_h}	R_{δ_1}	R_{δ_2}	$-\beta$	$-\beta_c$	$\left(\frac{u_e}{u_\tau}\right)$ data	$\left(\frac{u_e}{u_\tau}\right)$ calc	k	a	b	n	u^+ rms	u^+ max	u^+ min
5.39	800	5.0	419.057	8439	1108	773	0.242	0.411	20.1395	20.1285	0.4209	58.2413	0.4020	1.5770	0.0760	0.1107	-0.1581
5.39	1600	5.0	666.197	13809	1616	1178	0.377	0.652	20.7287	20.8032	0.4194	56.4921	0.4850	1.5220	0.0621	0.1628	-0.0918
5.39	2200	5.0	696.784	14545	1692	1242	0.396	0.687	20.8750	20.8352	0.4270	58.3761	0.4110	1.8110	0.0438	0.1055	-0.0849
5.39	2800	5.0	837.259	17478	1912	1421	0.462	0.805	21.1776	21.1973	0.4222	57.6765	0.5850	1.4260	0.0464	0.1949	-0.0633
5.39	3280	5.0	937.175	20033	2073	1553	0.511	0.893	21.3760	21.3809	0.4201	57.3912	0.7250	1.2930	0.0340	0.0870	-0.0495
5.39	3580	5.0	1018.42	21786	2139	1614	0.529	0.928	21.3920	21.4182	0.4201	57.9203	0.9440	1.2180	0.0439	0.0757	-0.0912
3.59	800	7.5	546.032	11547	1495	1064	0.238	0.407	21.1474	21.0387	0.4234	58.4850	0.3700	1.5365	0.0652	0.1048	-0.1258
3.59	1600	7.5	783.150	16980	2059	1513	0.346	0.600	21.6813	21.6371	0.4249	58.5415	0.3640	1.7120	0.0534	0.1565	-0.0688
3.59	2200	7.5	940.331	20534	2331	1737	0.398	0.695	21.8366	21.8132	0.4220	57.2913	0.4440	1.5350	0.0315	0.0947	-0.0568
3.59	2800	7.5	1132.57	24965	2580	1951	0.451	0.791	22.0424	22.0581	0.4183	57.3330	0.6800	1.2700	0.0363	0.1060	-0.0735
3.59	3280	7.5	1273.59	28303	2819	2151	0.500	0.882	22.2230	22.2353	0.4214	58.5261	0.7119	1.2820	0.0345	0.0454	-0.0953
3.59	3580	7.5	1407.51	31178	2919	2247	0.517	0.914	22.1511	22.2007	0.4198	58.3628	0.9117	1.2665	0.0420	0.0739	-0.0715
2.70	800	10.0	676.967	14704	1916	1381	0.244	0.420	21.7204	21.7248	0.4194	56.8055	0.3445	1.5590	0.0568	0.1807	-0.1054
2.70	1600	10.0	966.537	21499	2567	1905	0.343	0.597	22.2435	22.2313	0.4216	57.0424	0.3790	1.5680	0.0473	0.1278	-0.0905
2.70	2200	10.0	1144.22	25655	2916	2191	0.395	0.691	22.4216	22.3922	0.4180	55.5805	0.4225	1.5360	0.0407	0.1486	-0.0817
2.70	2800	10.0	1411.65	32007	3338	2547	0.465	0.820	22.6737	22.7159	0.4190	56.8243	0.5743	1.3200	0.0367	0.0678	-0.0722
2.70	3280	10.0	1533.96	35167	3634	2780	0.518	0.915	22.9257	22.9858	0.4170	56.2243	0.6343	1.2200	0.0662	0.1517	-0.0887
2.70	3580	10.0	1702.99	38968	3812	2943	0.542	0.961	22.8822	22.9555	0.4166	56.2497	0.7439	1.2260	0.0708	0.1353	-0.1055

FPGTBL data

M. Jones, I. Marusic, and A. E. Perry, "Evolution and structure of sink-flow turbulent boundary layers," Journal of Fluid Mechanics 428, 1 – 27 (2001).

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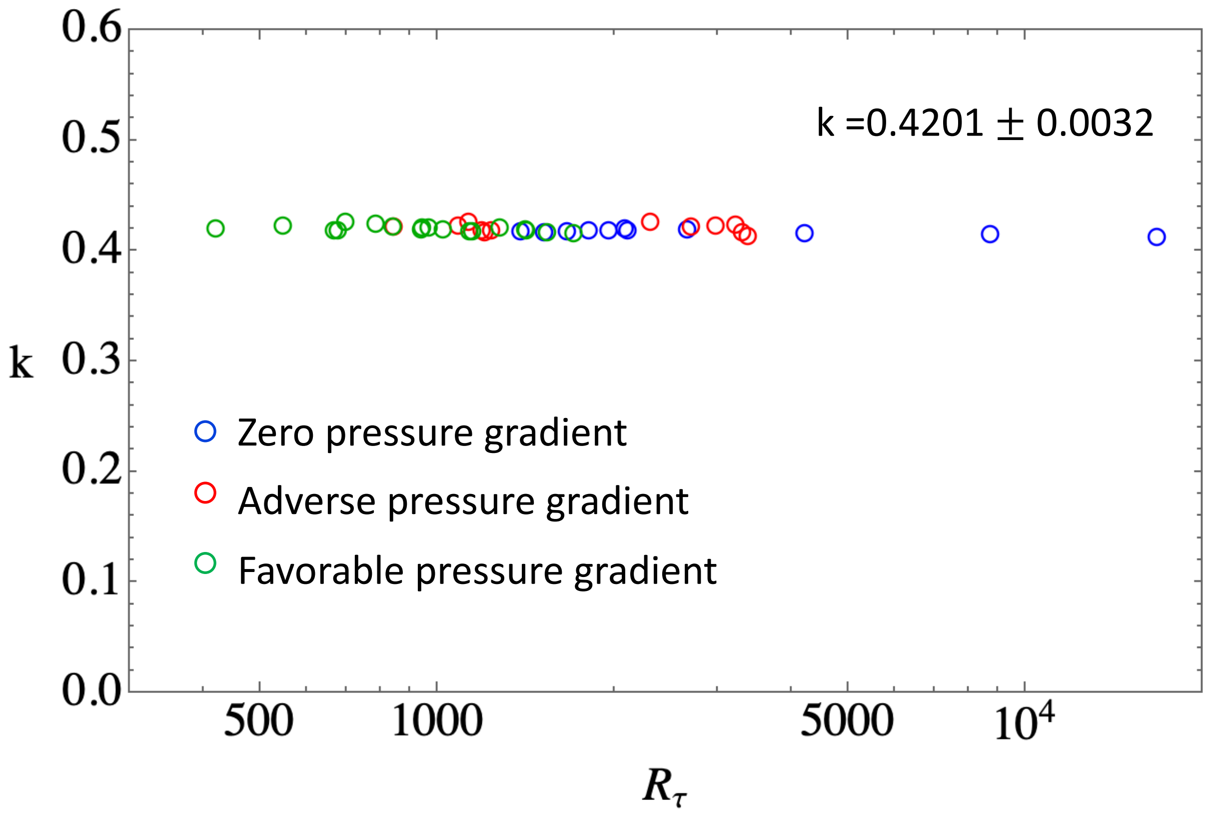
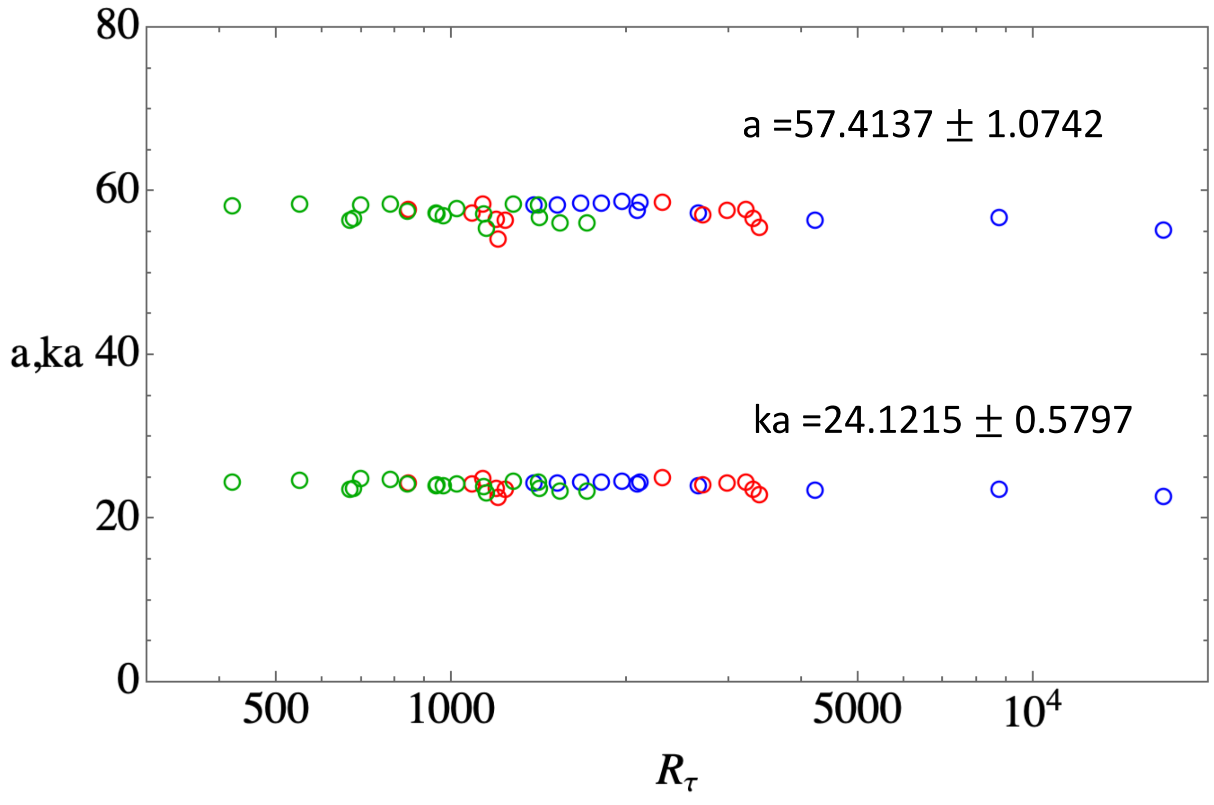
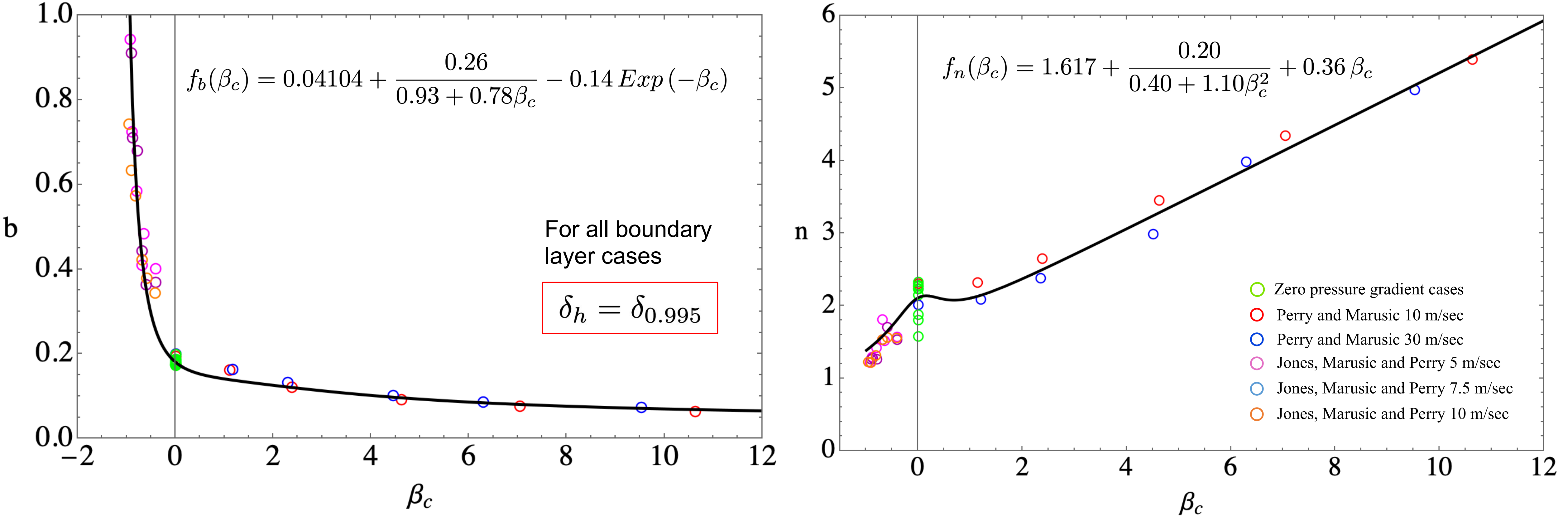
Optimal parameters k and a over all boundary layer cases





Optimal wake parameters b and n correlate with β_c



Mean parameter values

Flow	$\bar{k}$	k_{rms}	$\bar{a}$	a_{rms}	$\bar{b}$	b_{rms}	$\bar{n}$	n_{rms}
Channel (7 profiles)	0.3986	0.0057	53.7693	2.1647	0.4771	0.0429	1.4497	0.0843
Pipe (26 profiles)	0.4098	0.0061	55.5509	2.4906	0.3205	0.0361	1.6340	0.1768
ZPG Boundary Layer (11 profiles)	0.4180	0.0022	57.7730	1.1208	0.1806	0.0076	2.1169	0.2470
All Boundary Layers (41 profiles)	0.4201	0.0032	57.4137	1.0742	$f_b(0) = 0.1806$	0.0076	$f_n(0) = 2.1169$	0.2470

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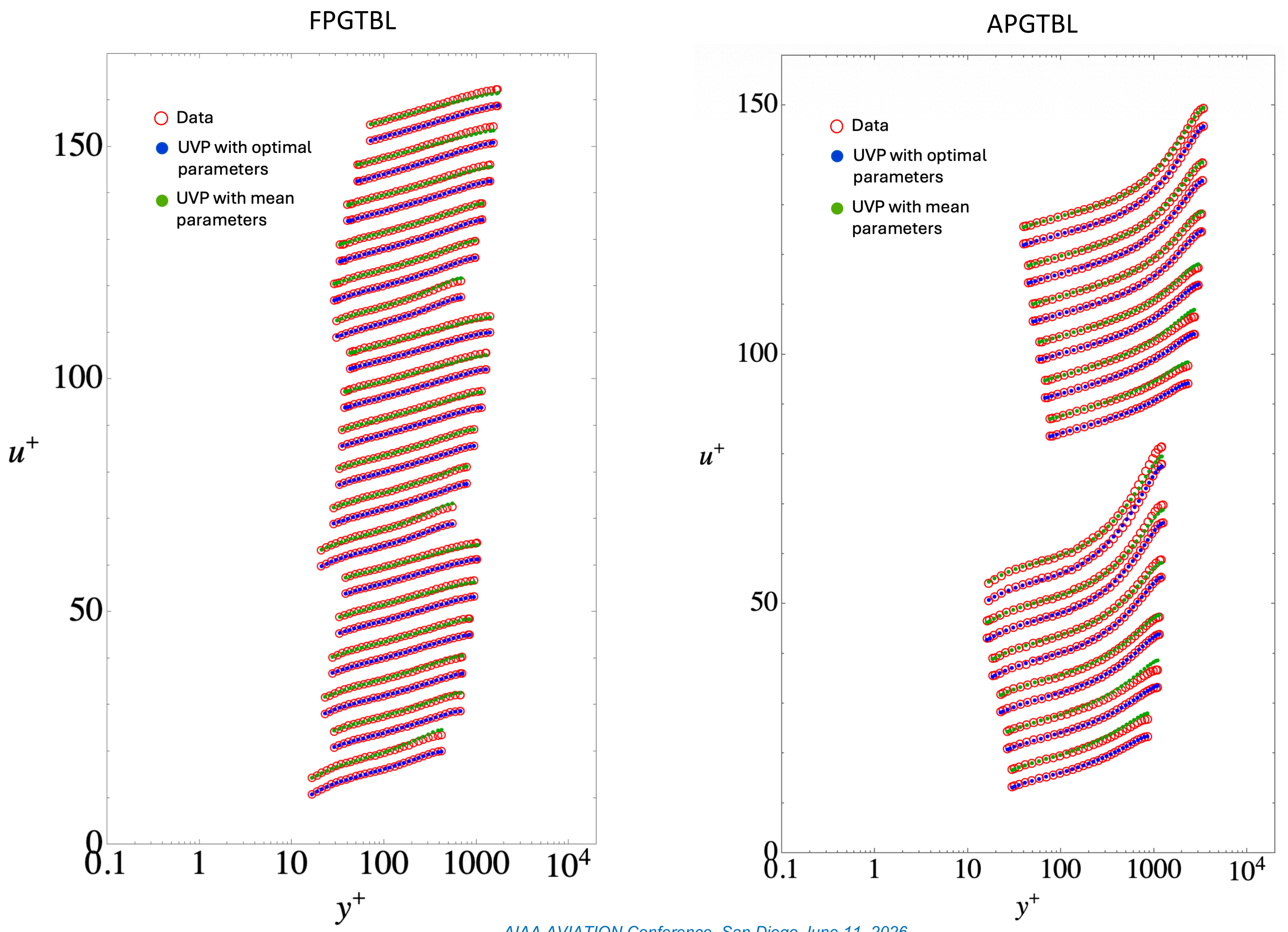
UVP versus APG and FPG data using optimal and mean parameters



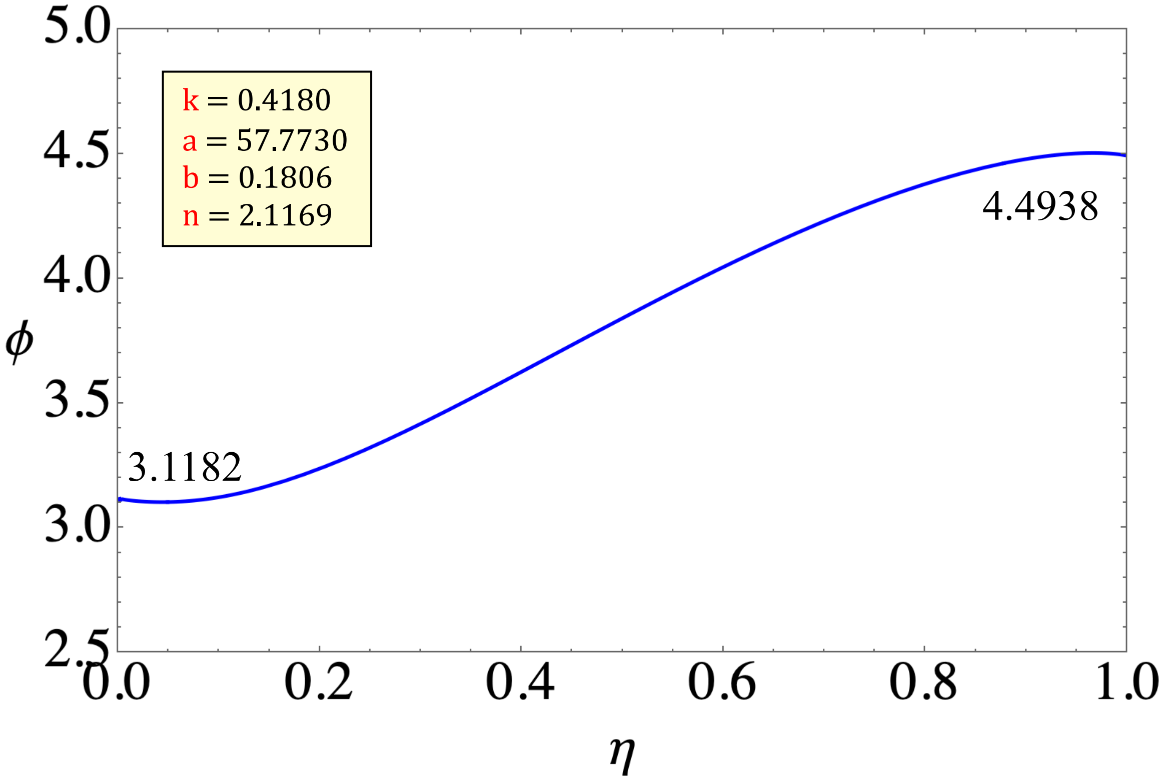
Explicit form of the UVP

The UVP simplifies to a form similar to the classical wall-wake formulation.

$R_\tau > 5000, \; \eta > 132/R_\tau$

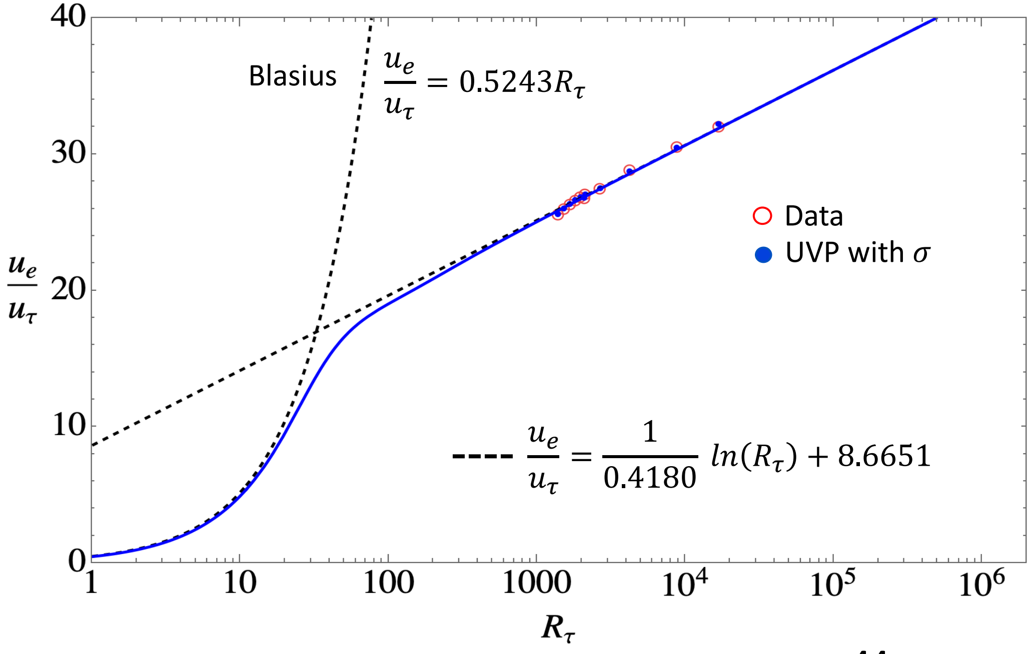
$$u^+ = \frac{1}{k} ln\left(ky^+\right) + \frac{1}{k}\phi\left(k,a,b,n,\eta\right)$$

$$\phi\left(k,a,b,n,\eta\right) = \underset{3.1182}{\phi_0(k,a)} + \int_0^\eta \frac{(1-\eta')^{1/2}\left(1+\left(\frac{\eta'}{b}\right)^n\right)^{1/n} - 1}{\eta'} d\eta'$$



Evaluate at the boundary layer edge to determine the friction law.

$$\frac{u_e}{u_\tau} = \frac{1}{k} ln\left(R_\tau\right) + \frac{1}{k} ln\left(k\right) + \frac{1}{k}\phi\left(k,a,b,n,1\right)$$



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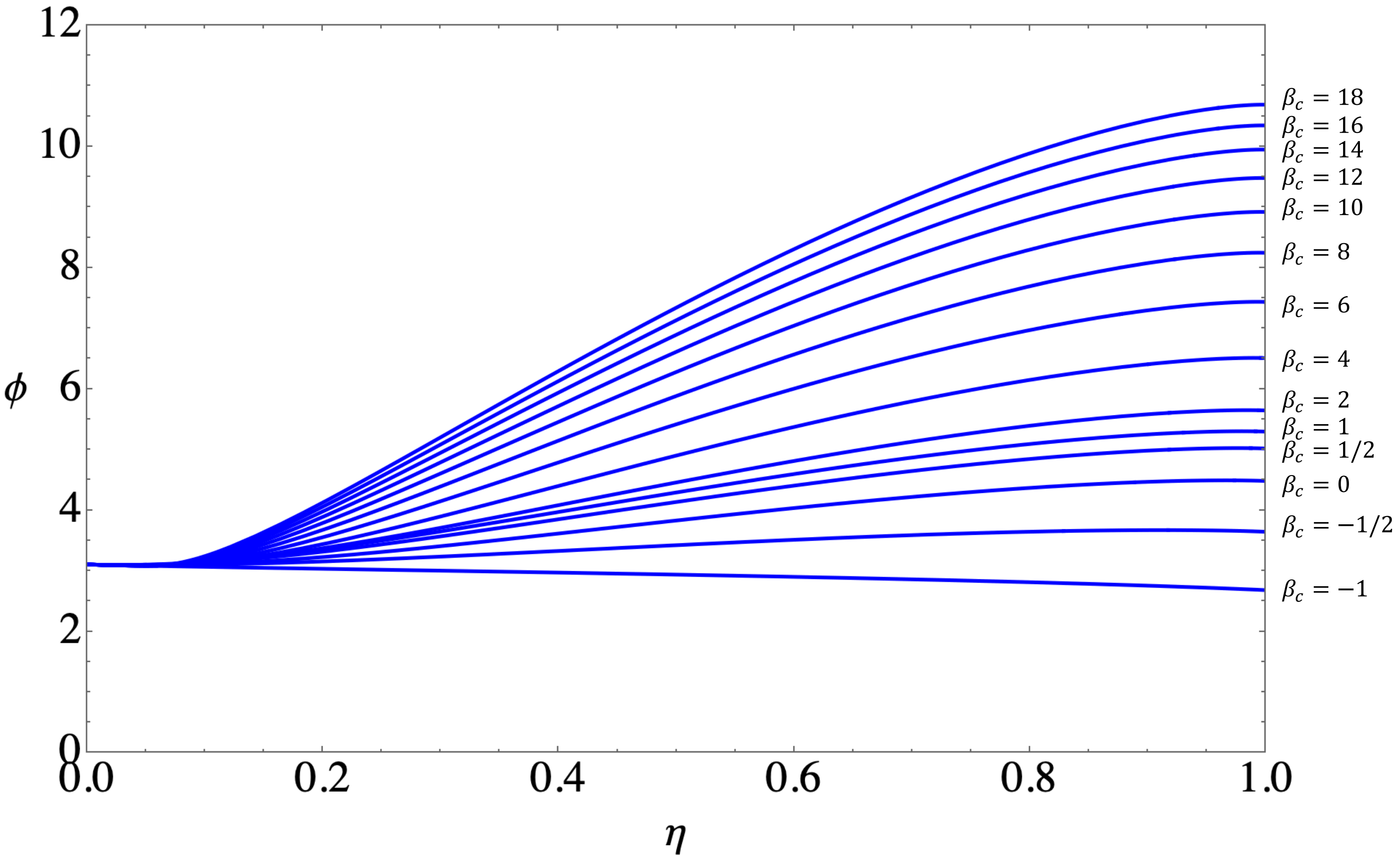
The parameters b and n depend on β_c.

$$u^+ = \frac{1}{k} ln\left(ky^+\right) + \frac{1}{k}\phi\left(k, a, f_b\left(\beta_c\right), f_n\left(\beta_c\right), \eta\right)$$

$\eta > 132/R_\tau$
$R_\tau > 5000$



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$\phi(k, a, f_b(\beta_c), f_n(\beta_c), \eta)$ can be used in a fast integral boundary layer method.

Approximate $\phi(k, a, fb(\beta_c), fn(\beta_c), \eta)$ by a bivariate polynomial in β_c and η.

The wall parameters (k, a) are kept constant at the mean boundary layer values

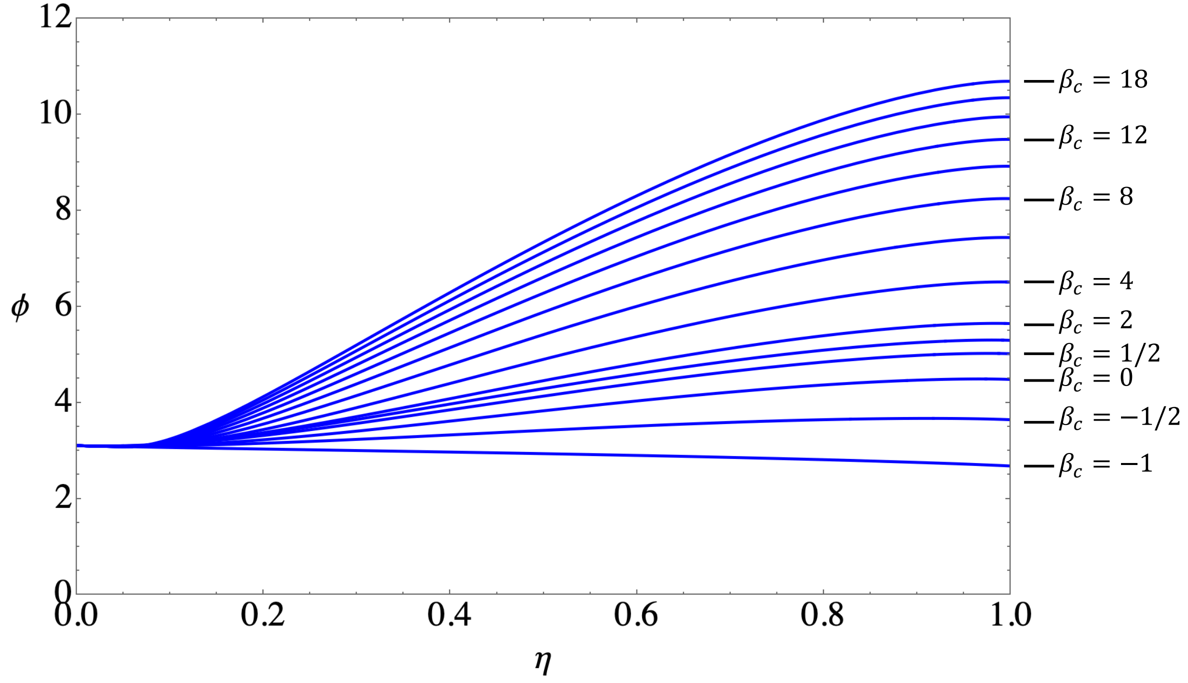
k = 0.4201 $\pm$ 0.0032

a = 57.4137 $\pm$ 1.0742

Eighth order polynomial provides a good approximation to $\phi(k, a, f_b(\beta_c), f_n(\beta_c), \eta)$.

$$\phi(k, a, f_b(\beta_c), f_n(\beta_c), \eta) = c_0(\beta_c) + c_1(\beta_c)\eta + c_2(\beta_c)\eta^2 + c_3(\beta_c)\eta^3 + c_4(\beta_c)\eta^4 + c_5(\beta_c)\eta^5 + c_6(\beta_c)\eta^6 + c_7(\beta_c)\eta^7 + c_8(\beta_c)\eta^8$$

$\phi(k, a, f_b(\beta_c), f_n(\beta_c), \eta)$ for various β_c



The displacement thickness and momentum thickness integrals can be carried out up to whatever R_τ is required. This leads to polynomial expressions for the friction and thickness functions $F_0(\beta_c, R_\tau)$, $F_1(\beta_c, R_\tau)$, $F_2(\beta_c, R_\tau)$ and $F_3(\beta_c, R_\tau)$.

Once these functions are known, the time to calculate a solution of the Kármán equation is independent of the Reynolds number.

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Viscous drag coefficient of J0012 and NACA0012 airfoils

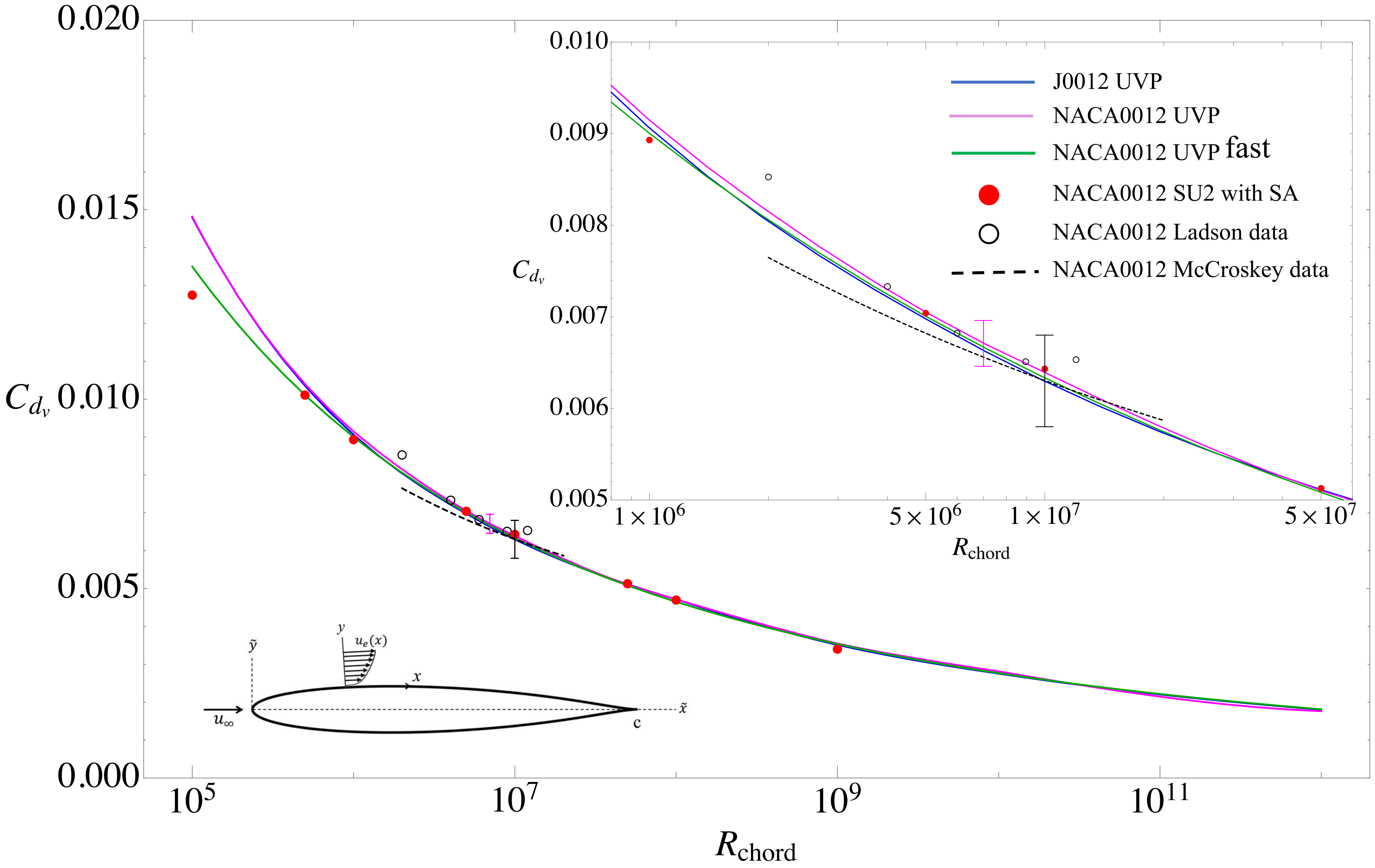

Figure source

B. J. Cantwell, E. Bilgin, and J. T. Needels, "A new boundary layer integral method based on the universal velocity profile," Physics of Fluids 34, 075130 (2022).

Concluding Remarks

1) $\sigma(y^+/a)$ produces distinctly better agreement between the UVP and the velocity as well as near wall data for the velocity gradient and Reynolds shear stress.

2) $\sigma(y^+/a)$ drops off more rapidly than the exponential model and oscillates several times as it decays to zero. The oscillations begin at about 1 and extend to 5-10 damping length scales above the wall.
The overall behavior resembles a damped linear system. How this stationary oscillatory behavior might relate to the physics of a spatially periodic momentum exchange process near the wall deserves to be studied and can probably only be determined from a detailed study of time-dependent data.

3) Along with improved accuracy, the most important result is the reduction in model parameters from 5 to 4 allowing the optimization process to be visualized.
This makes the UVP much easier to optimize.

4) No specification of the Kármán constant is complete without also specifying the damping length scale *a*.
Even then, two pairs with quite different values can give highly accurate approximations to the same data.

5) There is no R_τ limit. The UVP can be applied to very large scale aerodynamic, hydrodynamic and geophysical flows.

Thanks!

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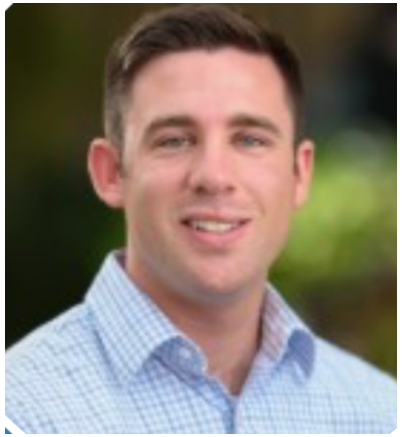

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