Log-Linearizing the Model (Campbell, 1994)

The general model has $\delta < 1$, variable labor supply, $G_t \neq 0$, etc $\Rightarrow$ richer dynamics.
- Typically, numerical solutions
- We show a common way of proceeding as far as possible before resorting to numerical solutions – more details in Cooley (1995)

• Basic Model

1. Production $\quad Y_t = K_t^{\alpha}(A_t L_t)^{1-\alpha}$
2. Capital $\quad K_{t+1} = Y_t - C_t + (1-\delta)K_t - G_t$ (govt financed w/ lump sum taxes)
3. FOC (labor) $\quad \dfrac{C_t}{\theta(1-L_t)^{\gamma}} = W_t = (1-\alpha)\dfrac{Y_t}{L_t}$
4. FOC (EE) $\quad \dfrac{1}{C_t} = \beta E_t\left(\dfrac{R_{t+1}}{C_{t+1}}\right) \qquad R_{t+1} = 1-\delta+\alpha\dfrac{Y_{t+1}}{K_{t+1}}$
5. Technology $\quad \log A_t = \log \bar{A} + gt + \tilde{A}_t \qquad \tilde{A}_t = \rho \tilde{A}_{t-1} + \varepsilon_t$
6. Govt Spending $\quad \log G_t = \log \bar{G} + gt + \tilde{G}_t \qquad \tilde{G}_t = \rho_G \tilde{G}_{t-1} + v_t$

• For the log-linearized version, let $\tilde{X}$ denote (log) deviation from Balanced Growth Path, as $\tilde{A}, \tilde{G}$

① Production $\quad \tilde{Y}_t = \alpha \tilde{K}_t + (1-\alpha)\tilde{A}_t + (1-\alpha)\tilde{L}_t$ (Easy!)

② Capital Accumulation – Hard b/c of the linearity. (Not obvious how to proceed - Campbell tells us)

$$\frac{K_{t+1}}{K_t} = \frac{Y_t}{K_t} - \frac{C_t}{K_t} - \frac{G_t}{K_t} + (1-\delta)$$

Rewrite as

$$\underbrace{\log\left[e^{\Delta \log K_{t+1}} - (1-\delta)\right]}_{Ⓐ} = \log\left[\frac{Y}{K}\left(1-\frac{C}{Y}-\frac{G}{Y}\right)\right]$$

$$= \log Y - \log K + \underbrace{\log\left[1 - e^{\log C - \log Y} - e^{\log G - \log Y}\right]}_{Ⓑ}$$

Ⓐ: $f(\Delta\log K) = \log\left[e^{\Delta\log K} - (1-\delta)\right] \approx Const + f'(\cdot)\,\Delta\log K$

$$f'(\cdot) = \frac{1}{e^{\Delta\log K} - (1-\delta)}\, e^{\Delta\log K} = \frac{K_{t+1}/K_t}{K_{t+1}/K_t - (1-\delta)} \approx \frac{1+g}{g+\delta} \quad \text{in BGP.}$$

Ⓑ: $f(\log C - \log Y, \log G - \log Y) = \log\left[1 - e^{\log C - \log Y} - e^{\log G - \log Y}\right] \approx Const + f_1'(\cdot)(\log C - \log Y) + f_2'(\cdot)(\log G - \log$

$$\left.\begin{aligned} f_1'(\cdot) &= \frac{-1}{1 - C/Y - G/Y} \times C/Y \\ f_2'(\cdot) &= \frac{-1}{1 - C/Y - G/Y} \times \frac{G}{Y} \end{aligned}\right\} \quad \begin{aligned} &G/Y \equiv S_G \text{ constant along BGP} \\ &C/Y = 1 - S_K - S_G \qquad S_K \equiv I/Y \end{aligned}$$

Recall: $\dfrac{K_{t+1}}{K_t} = S_K \dfrac{Y_t}{K_t} - \delta - 1$

$$\alpha \frac{Y_t}{K_t} = r + \delta \Rightarrow \frac{Y_t}{K_t} = \frac{1}{\alpha}(r+\delta)$$

$$\Rightarrow \text{Along BGP, } S_K^* = \frac{(g+\delta)\alpha}{r+\delta} = \alpha\,\frac{g+\delta}{r+\delta}$$

$$f_1'(\cdot) = \frac{S_K^* + S_G - 1}{S_K^*} = 1 - \left(\frac{1-S_G}{S_K^*}\right) < 0$$

$$f_2'(\cdot) = -\frac{S_G}{S_K^*} < 0$$

$\Rightarrow$ Linearized Accumulation Equation:

$$\frac{1+g}{g+\delta}\,\Delta\log K_{t+1} = Const + \log Y_t - \log K_t + \left(1 - \frac{1-S_G}{S_K^*}\right)(\log C_t - \log Y_t) - \frac{S_G}{S_K^*}(\log G_t - \log Y_t)$$

$$= Const + \left(\frac{1-S_G}{S_K} + \frac{S_G}{S_K}\right)\log Y_t - \log K_t - \left(\frac{1-S_G}{S_K} - 1\right)\log C_t - \frac{S_G}{S_K}\log G_t$$

$$\frac{1+g}{g+\delta}\log K_{t+1} = Const + \frac{1}{S_K^*}\log Y_t + \left(\frac{1+g}{g+\delta} - 1\right)\log K_t - \left(\frac{1-S_G}{S_K^*} - 1\right)\log C_t - \frac{S_G}{S_K^*}\log G_t$$

$$\Rightarrow \boxed{\tilde{K}_{t+1} = \frac{g+\delta}{1+g}\left[\frac{1}{S_K^*}\tilde{Y}_t + \frac{1-\delta}{g+\delta}\tilde{K}_t - \left(\frac{1-S_G}{S_K^*} - 1\right)\tilde{C}_t - \frac{S_G}{S_K^*}\tilde{G}_t\right]}$$

Dynamic evolution of capital stock around steady state as function of last period's situation.

Continue by substituting from production:

$$\tilde{Y}_t = \alpha \tilde{K}_t + (1-\alpha)(\tilde{A}_t + \tilde{L}_t)$$

$$\Rightarrow \tilde{K}_{t+1} = \frac{g+\delta}{1+g}\left[\frac{1}{S_K^*}\right]\left(\tilde{Y}_t + S_K^*\left(\frac{1-\delta}{g+\delta}\right)\tilde{K}_t - (1-S_G-S_K)\tilde{C}_t - S_G\tilde{G}_t\right)$$

$$= \frac{\cancel{g+\delta}}{1+g}\left(\frac{r+\delta}{\cancel{g+\delta}}\right)\frac{1}{\alpha}\left(\tilde{Y}_t + \alpha\frac{1-\delta}{r+\delta}\tilde{K}_t - (1-S_G-S_K)\tilde{C}_t - S_G\tilde{G}_t\right)$$

$$= \frac{1}{\alpha}\,\frac{r+\delta}{1+g}\left[\alpha\left(1+\frac{1-\delta}{r+\delta}\right)\tilde{K}_t + (1-\alpha)(\tilde{A}_t+\tilde{L}_t) - (1-S_G-S_K)\tilde{C}_t - S_G\tilde{G}_t\right]$$

$$= \frac{\cancel{r+\delta}}{1+g}\left(\frac{1+r}{\cancel{r+\delta}}\right)\tilde{K}_t + \frac{1-\alpha}{\alpha}\left(\frac{r+\delta}{1+g}\right)(\tilde{A}_t+\tilde{L}_t) - \frac{1}{\alpha}\left(\frac{r+\delta}{1+g}\right)\left[(1-S_G-S_K)\tilde{C}_t + S_G\tilde{G}_t\right]$$

$$= \lambda_1\tilde{K}_t + \lambda_2(\tilde{A}_t+\tilde{L}_t) + \beta\tilde{C}_t + \lambda_4\tilde{G}_t$$

$$\boxed{\begin{aligned}
\lambda_1 &= \frac{1+r}{1+g} = \frac{1+r}{1+g}\\
\lambda_2 &= \frac{1-\alpha}{\alpha}\left(\frac{r+\delta}{1+g}\right)\\
\lambda_4 &= -\frac{(r+\delta)S_G}{\alpha(1+g)} = -\frac{(r+\delta)S_G}{\alpha(1+g)}
\end{aligned}}$$

Check: $1-\lambda_1-\lambda_2-\lambda_4 = 1 - \frac{1+r}{1+g} - \frac{1-\alpha}{\alpha}\left(\frac{r+\delta}{1+g}\right) + \frac{r+\delta}{1+g}\frac{S_G}{\alpha}$

$$= 1 - \frac{r+\delta}{1+g}\left[\frac{1+r}{r+\delta} + \frac{1-\alpha}{\alpha} - \frac{S_G}{\alpha}\right]$$

$$= 1 - \frac{1}{\alpha}\frac{r+\delta}{1+g}\left[\alpha\frac{1+r}{r+\delta} + 1-\alpha-S_G\right]$$

$$= -\frac{1}{\alpha}\frac{r+\delta}{1+g}\left[-\alpha\frac{1+g}{r+\delta} + \alpha\frac{1+r}{r+\delta} + 1-\alpha-S_G\right]$$

$$= \longleftrightarrow \left[1-S_G-\alpha\left(1+\frac{1+g}{r+\delta}-\frac{1+r}{r+\delta}\right)\right]$$

$$= \longleftrightarrow \left[1-S_G-\alpha\underbrace{\left(\frac{g+\delta}{r+\delta}\right)}_{S_K}\right]$$

$$\Rightarrow \boxed{\beta = 1-\lambda_1-\lambda_2-\lambda_4} \checkmark$$

$$\Rightarrow \boxed{\tilde{K}_{t+1} = \lambda_1\tilde{K}_t + \lambda_2(\tilde{A}_t+\tilde{L}_t) + (1-\lambda_1-\lambda_2-\lambda_4)\tilde{C}_t + \lambda_4\tilde{G}_t}$$

This is the difference equation for $\tilde{K}_{t+1}$.

(see Campbell, p. 492)

③ FOC: (Leisure)

$$\log C_t - \log\theta - \gamma\log(1-L_t) = \log(1-\alpha) + \log Y_t - \log L_t$$

• All terms are easy except

$$f(\log L) = \log(1-L) = \log(1-e^{\log L}) \approx f'(\cdot)\log L + \text{Const}$$

$$f'(\cdot) = \frac{-1}{1-e^{\log L}}e^{\log L} = -\frac{L^*}{1-L^*} \quad \text{(function of SS hours)}$$

– Can solve for L^* (implicitly unless $\gamma=1$) from FOC as function of S_K^* ⇒ done.

$$\Rightarrow \quad \tilde{C}_t + \gamma\frac{L^*}{1-L^*}\tilde{L}_t = \tilde{Y}_t - \tilde{L}_t$$

$$\tilde{C}_t = \alpha\tilde{K}_t + (1-\alpha)\tilde{A}_t + (1-\alpha)\tilde{L}_t - \left(1+\gamma\frac{L^*}{1-L^*}\right)\tilde{L}_t$$

$$\boxed{\tilde{C}_t = \alpha\tilde{K}_t + (1-\alpha)\tilde{A}_t - \left(\alpha + \gamma\frac{L^*}{1-L^*}\right)\tilde{L}_t}$$

Predetermined Exogenous ⇒ We just need to get one more eqn: EE!

④ FOC (Euler Eqn)

$$\frac{1}{C_t} = \beta E_t\left(\frac{R_{t+1}}{C_{t+1}}\right)$$

$E_t(x_{t+1}) = E(x_{t+1}|\Omega_t)$ $\Omega_t = \{$all information dated as of time $t\}$

The tricky thing here is how to deal with the Expectations operator.

Campbell Solution: Assume $\log R_{t+1}$ and $\log C_{t+1}$ have a joint normal distribution w/ constant variance-covariance

$$\Rightarrow \quad X_{t+1} \equiv R_{t+1}/C_{t+1} \text{ is log-normal.}$$

$\log E(e^{\log X}) = \mu + \frac{1}{2}\sigma^2.$

KEY PROPERTY: If X_{t+1} is log-normal

$$\ln E_t(X_{t+1}) = E_t \ln X_{t+1} + \frac{1}{2}\mathrm{Var}_t \ln X_{t+1}$$

Note that if ε_t and V_t are iid Normal, then this assumption will be valid here.

- Applying this property here,

Note: For asset pricing, this will not work - essentially we've ignored the covariances that are the heart of asset pricing.

$$-\ln C_t = \ln\beta + \ln E_t\left(\frac{R_{t+1}}{C_{t+1}}\right)$$

$$= \ln\beta + E_t(\ln R_{t+1} - \ln C_{t+1}) + \underbrace{\tfrac{1}{2}\mathrm{Var}_t(\ln R_{t+1} - \ln C_{t+1})}_{\text{Constant}}$$

Constant $\Rightarrow$ drops out when we take deviations!

$$\Rightarrow \boxed{E_t(\Delta \ln C_{t+1}) = \text{Const} + E_t(\ln R_{t+1})}$$

- Now linearize $\ln R_{t+1}$

$$\ln R_{t+1} = \ln\left(1-\delta+\alpha\frac{Y_{t+1}}{K_{t+1}}\right) = \ln\left(1-\delta+\alpha K_{t+1}^{\alpha-1}A_{t+1}^{1-\alpha}L_{t+1}^{1-\alpha}\right)$$

$$= \ln\left(1-\delta+\alpha e^{(1-\alpha)[\ln A_{t+1}+\ln L_{t+1}-\ln K_{t+1}]}\right)$$

$$= f(\ln A_{t+1}+\ln L_{t+1}-\ln K_{t+1})$$

$$f'(\cdot) = \frac{1}{R_{t+1}}\alpha(1-\alpha)K_{t+1}^{\alpha-1}A_{t+1}^{1-\alpha}L_{t+1}^{1-\alpha}$$

$R \equiv 1+r$

$r = \alpha\frac{Y}{K}-\delta$

$$= \frac{1}{R_{t+1}}(1-\alpha)(r+\delta)$$

$$= (1-\alpha)\frac{r+\delta}{1+r}$$

$$\Rightarrow \boxed{E_t(\Delta\tilde{C}_{t+1}) = \overbrace{(1-\alpha)\frac{r+\delta}{1+r}}^{\lambda_3 \text{ acc to Campbell}} E_t(\tilde{A}_{t+1}+\tilde{L}_{t+1}-\tilde{K}_{t+1})}$$

Stop here.

Note: $E_t(\tilde{A}_{t+1}) = E_t(\rho\tilde{A}_t+\varepsilon_t) = \rho\tilde{A}_t$

$E_t(\tilde{K}_{t+1}) = K_{t+1}$ given above (predetermined!)

L_{t+1} - Obtain from FOC (Leisure) above.

L_{t+1}: From (3) and advancing 1 period:

$$\tilde{C}_{t+1} = \alpha\tilde{K}_{t+1} + (1-\alpha)\tilde{A}_{t+1} - \left(1+\alpha+\gamma\frac{L^*}{1-L^*}\right)\tilde{L}_{t+1}$$

$$\Rightarrow \tilde{L}_{t+1} = \frac{1}{1+\alpha+\gamma\frac{L}{1-L}}\left(\alpha\tilde{K}_{t+1}+(1-\alpha)\tilde{A}_{t+1}-\tilde{C}_{t+1}\right)$$

- At this point, we have 3 equations:

$$\tilde{K}_{t+1} = \lambda_1 \tilde{K}_t + \lambda_2 (\tilde{A}_t + \tilde{L}_t) + (1-\lambda_1-\lambda_2-\lambda_4)\tilde{C}_t + \lambda_4 \tilde{G}_t$$ K accumulation

$$E_t \Delta \tilde{C}_{t+1} = \lambda_3 E_t (\tilde{A}_{t+1} + \tilde{L}_{t+1} - \tilde{K}_{t+1})$$ Euler Eqn

$$\tilde{L}_t = \lambda_5 (\alpha \tilde{K}_t + (1-\alpha)\tilde{A}_t - \tilde{C}_t)$$ FOC static

(where $\lambda_5 = \frac{1}{\alpha + \gamma \frac{L^*}{1-L^*}}$)

- With these 3 equations, we can use the <u>method of undetermined coefficients</u> to solve further.

① Guess the form of the solution

$$\tilde{C}_t = \eta_{CK} \tilde{K}_t + \eta_{CA} \tilde{A}_t + \eta_{CG} \tilde{G}_t$$

$$\tilde{L}_t = \eta_{LK} \tilde{K}_t + \eta_{LA} \tilde{A}_t + \eta_{LG} \tilde{G}_t$$

$$\tilde{K}_{t+1} = \eta_{KK} \tilde{K}_t + \eta_{KA} \tilde{A}_t + \eta_{KG} \tilde{G}_t$$

η = Unknown Parameters

② Use the 3 eqns above together w/ the form of the solution guessed in ① to solve for the 9 η parameters and verify that the form in ① is correct

$$\Rightarrow \quad \eta = f(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)$$

and the λ's are in turn functions of the underlying structural parameters.