THE KNIGHT MOVE CONJECTURE IS FALSE

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Abstract. The Knight Move Conjecture claims that the Khovanov homology of any knot decomposes as direct sums of some "knight move" pairs and a single "pawn move" pair. This is true for instance whenever the Lee spectral sequence from Khovanov homology to $\mathbb{Q}^2$ converges on the second page, as it does for all alternating knots and knots with unknotting number at most 2. We present a counterexample to the Knight Move Conjecture. For this knot, the Lee spectral sequence admits a nontrivial differential of bidegree $(1, 8)$.

1. Introduction

Almost 20 years ago, Khovanov [6] introduced a categorification of the Jones polynomial, now known by the name of *Khovanov homology*. This is an invariant of links in S^3 that is strictly more powerful than the Jones polynomial [3], and it detects the unknot [7]. Furthermore, using Khovanov homology, Rasmussen defined a concordance homomorphism $s\colon \mathcal{C} \to 2\mathbb{Z}$ from the smooth knot concordance group, and used it to give the first combinatorial proof of Milnor's conjecture [12].

Given a knot $K \subset S^3$, its Khovanov homology over $\mathbb{Q}$ is a bigraded vector space over $\mathbb{Q}$, endowed with a *homological grading* $i \in \mathbb{Z}$ and a *quantum grading* $j \in \mathbb{Z}$. We denote this bigrading by (i, j). We denote the Khovanov homology of a knot $K \subset S^3$ by

$$\mathrm{Kh}(K) = \bigoplus_{i,j\in\mathbb{Z}} \mathrm{Kh}^{i,j}(K),$$

and its Poincaré series by $\mathrm{Kh}(K)(t, q) = \sum_{i,j} \dim_{\mathbb{Q}} \left(\mathrm{Kh}^{i,j}(L)\right) t^i q^j$.

1.1. **The structure of Khovanov homology.**

An early conjecture about the structure of Khovanov homology [3, Conjecture 1], known as the *Knight Move Conjecture*, is due to Bar-Natan, Garoufalidis, and Khovanov. It says that it is always possible to decompose the Khovanov homology of a knot into the direct sum of elementary pieces.

Conjecture 1.1 (Knight Move Conjecture [3])**.** *Given a knot K, its Khovanov homology over $\mathbb{Q}$ is the direct sum of a single* pawn move *piece*

$$\mathbb{Q}\{0, s-1\} \oplus \mathbb{Q}\{0, s+1\},$$

where s is Rasmussen's invariant, and several knight move *pieces*

$$\mathbb{Q}\{i, j\} \oplus \mathbb{Q}\{i+1, j+4\},$$

for various $i, j \in \mathbb{Z}$.

In terms of Poincaré series, this conjecture can be rewritten as follows (see [5, Conjecture 5.2]):

Conjecture 1.2 (Reformulation of the Knight Move Conjecture)**.** *For any knot K, there is a Laurent polynomial $f_2 \in \mathbb{N}[t^{\pm 1}, q^{\pm 1}]$ so that*

$$\mathrm{Kh}(K)(t, q) = q^s(q + q^{-1}) + f_2(t, q)(1 + tq^4).$$

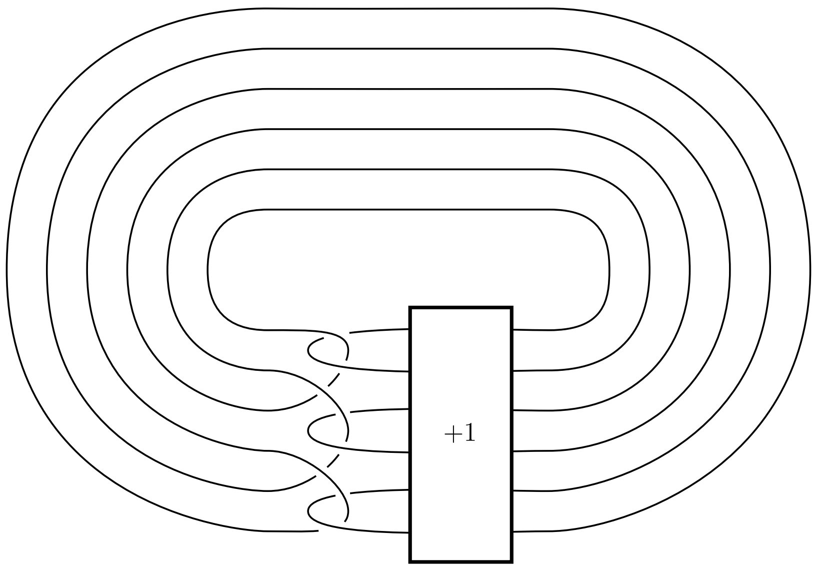

FIGURE 1. The knot K. The box labelled $+1$ denotes a full positive twist.

1.2. **Lee's deformation.** In [9], Lee introduced a deformation of the (co-)chain complex yielding Khovanov homology, which in fact is a filtered differential, where the filtration level is given by the quantum degree. The zeroth page of the resulting spectral sequence is the usual Khovanov complex, with the usual differential d_0. Thus, the first page E_1 is simply Khovanov homology.[1] The higher differentials d_n on E_n have degree $(1, 4n)$. Lastly, if K is a knot, the resulting spectral sequence converges to $\mathbb{Q}\{0, s-1\} \oplus \mathbb{Q}\{0, s+1\}$, where s is Rasmussen's invariant.

Using the above properties, it is immediate to check that if the Lee spectral sequence of a knot K degenerates after the first page, then there must be a Knight Move decomposition of $\mathrm{Kh}(K)$. This is true for example for all alternating knots [8], and more generally for all quasi-alternating knots [10], as well as for all knots with unknotting number not bigger than 2 [2].

For a general knot, a corollary of Lee's spectral sequence is that we have a decomposition of Khovanov homology into a pawn move and several, possibly "longer" knight moves, of the form

$$\mathbb{Q}\{i, j\} \oplus \mathbb{Q}\{i+1, j+4n\}.$$

In other words, for any knot K, there is a family of two variable Laurent polynomials $f_{2l} \in \mathbb{N}[t^{\pm 1}, q^{\pm 1}]$, for $l \geq 1$, so that

$$\mathrm{Kh}(K)(t, q) = q^s(q + q^{-1}) + \sum_{l \geq 1} f_{2l}(t, q)(1 + tq^{4l}).$$

The Knight Move Conjecture is equivalent to saying that f_{2l} can be set to 0 for all $l \geq 2$.

In this note, we present a counterexample to the Knight Move Conjecture. The example that we give has a non-trivial differential d_2.

2. The counterexample

Our counterexample is the knot K illustrated in Figure 1. It is obtained from an 8-crossing diagram of the unknot by doing a full positive twist along 6 strands. The resulting diagram has 38 crossings.

[1] In some of the literature, for example in [12], what we call the E_n page of the Lee spectral sequence is denoted E_{n+1}, and our differential d_n is their d_{n+1}.

	-18	-17	-16	-15	-14	-13	-12	-11	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4
13																							1
11																						1	
9																				1	2	1	
7																			3	4	2		
5																		5	4	1			
3																1	6	6	4	1			
1															3	9	10	4	1	1			
-1														3	9	8	3	1	1				
-3													3	10	12	6	1						
-5											1	5	10	10	2		1						
-7									1	2	4	6	7	3	1								
-9								2	1	3	7	8	2										
-11							2	2	4	5	3												
-13						3	4	2	2	2	1												
-15					3	3	1	2	1														
-17				2	3	2	1																
-19			2	3	1																		
-21		1	2																				
-23		2																					
-25	1																						

TABLE 1. The Khovanov homology of the knot K. The homological grading i is on the horizontal axis, and the quantum grading j on the vertical axis. The entry corresponding to column i and row j is the dimension of $\mathrm{Kh}^{i,j}(K)$. The red box marks an entry that cannot be canceled by a d_1 differential.

Theorem 2.1. *The knot K in Figure 1 does not satisfy the Knight Move Conjecture. Moreover, the second Lee differential d_2 of bidegree $(1, 8)$ is non-vanishing.*

Proof. The Khovanov homology of K is computed using the program "JavaKh-v2", an update by Scott Morrison of Jeremy Green's original program, both of which are available on the Knot Atlas [1]. The result is shown in Table 1. The entry tq (marked in red) is non-empty. If the Knight Move Conjecture were true, this should be matched by a non-zero entry in either q^{-3} or t^2q^5. However, these are both empty.

Regarding the Lee spectral sequence, the entry tq cannot be canceled by a d_1 differential, because both the entries q^{-3} and t^2q^5 are empty. It follows that it must be canceled by a higher differential, which is necessarily d_2, since there is no room for non-trivial maps of bidegree $(1, 4n)$ for $n \geq 3$, as one can easily check from Table 1. □

In fact, one can determine the whole structure of the Lee spectral sequence for K using the program "UniversalKh" of Scott Morrison [1, 11]. It turns out that the d_1 differential (the knight move) cancels most of the terms in Khovanov homology, leaving only four copies of $\mathbb{Q}$ on the E_2 page, in bidegrees $(0, -1)$, $(0, 1)$, $(1, 1)$ and $(2, 9)$. The last two are canceled by the d_2 differential, and the first two survive to the E_∞ page. Rasmussen's invariant for this knot is $s = 0$.

Remark 2.2. We came across the knot K while studying the *generalized crossing changes* introduced by Cochran and Tweedy in [4]. The full twist shown in Figure 1 is an example of a generalized negative crossing. The resulting knot K is slice in the blown-up ball $B^4\#\overline{\mathbb{CP}}^2$, but it is not slice in B^4, because its Alexander polynomial

$$\Delta_K(t) = -3t^{-1} + 7 - 3t$$

does not satisfy the Fox-Milnor criterion.

Remark 2.3. It is easy to see that the knot K can be unknotted by three crossing changes. Since the Knight Move Conjecture holds for knots of unknotting number at most two [2], it follows that K has unknotting number 3.

We end with an open problem.

Question 2.4. Given any $n \geq 3$, does there exist a knot for which the d_n differential in the Lee spectral sequence is nonzero?

In view of the work of Alishahi and Dowlin [2], if for a knot K we have $d_n \neq 0$, then K needs to have unknotting number at least $2n-1$.

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