A Journey From Elementary to Advanced Mathematics: The Unknotting Problem

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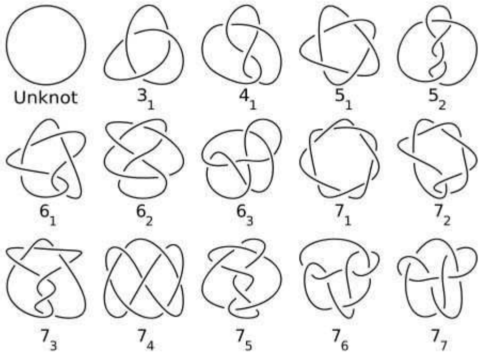
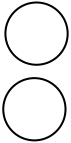
A *knot* is a piece of string tied together in 3-dimensional space. Here is a table of small knots:



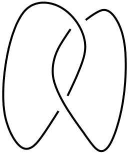
(from R. Lickorish, *An Introduction to Knot Theory*)

Links

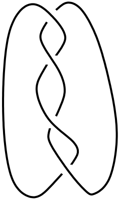
A *link* is a union of knots in three-space. The knots can be linked with each other. Examples:



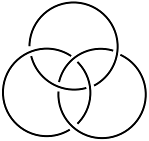
Unlink of 2 components



Hopf link



T(2,4) link



Borromean rings

Planar diagrams

Knots and links are represented by their projections to the plane, together with crossing information (which arc goes under and which arc goes above). Of course, the same link can have different diagrams:

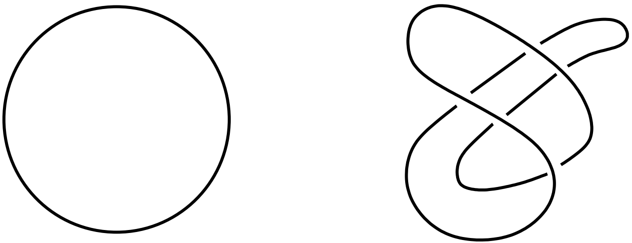
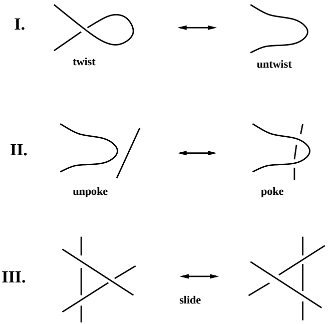


Figure: Two diagrams of the unknot.

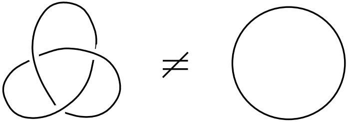
Reidemeister moves

Two diagrams represent the same link if they can be related by a deformation process. A deformation process consists of a sequence of moves, where each move is one of the three types shown here.



Distinguishing knots

We want to distinguish knots (and links). For example, we seek to show that the trefoil cannot be unknotted:



Applications of knot theory: biology (folding of DNA), cosmology (models for the universe), etc.

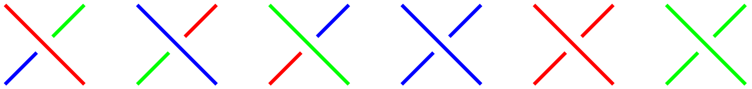
The idea is to find an invariant of knots: a quantity (or property) associated to diagrams that is unchanged by Reidemeister moves. Then, if the trefoil and the unknot have different invariants, we would know that they cannot be deformed into each other.

3-colorability

A diagram is made of several arcs (from one underpass to the next), drawn as uninterrupted curves. There are three arcs meeting at each crossing: two from the underpass and one from the overpass.

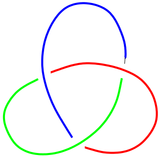
We say that a diagram is 3-**colorable** if we can color its arcs in red, blue and yellow such that:

- All three colors are used;
- At each crossing, either one color or all three colors appear:

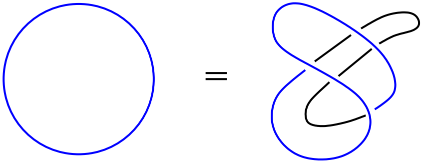


3-colorability

Examples:



3−colorable



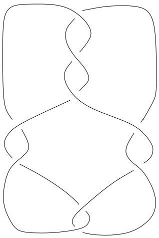
not 3−colorable

Elementary problem: Show that 3-colorability is an invariant property with respect to the Reidemeister moves.

Conclusion: the trefoil cannot be unknotted.

Question

Given a planar diagram for a knot, develop an algorithm to decide if it represents the unknot.



Is this the unknot?

Yes!

The Unknotting Problem

The first solution to the unknotting problem was found by Haken in 1961. His idea was to partition the Euclidean space into many small tetrahedra, such that the knot is composed of some of their edges. Then, one looks for a union of triangles that forms a disk with the knot as its boundary. Such a disk exists if and only if the knot is the unknot.

Other unknotting algorithms:

- Hass-Lagarias (2001): There is an upper bound (exponential in the number of crossings) on how many Reidemeister moves are needed to turn a diagram of the unknot into the one without crossings.
- Birman-Hirsch (1998);
- Dynnikov (2002), etc.

The Unknotting Problem

A more conceptual solution would be to have an invariant that detects the unknot.

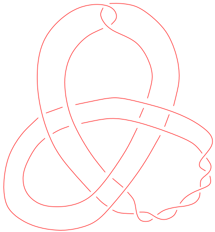
3-colorability distinguishes some knots. However, there are many non-trivial knots that are not 3-colorable.

Other invariants:

- The *Alexander polynomial* (1920's) distinguishes most small knots from the unknot, but does not do so for larger knots:

$\Delta($  $) = 1$

$\Delta($  $) = t - 1 + t^{-1}$

$\Delta\Big($  $\Big) = 1$

More invariants

- The *Jones polynomial* (1985) distinguishes more knots. It is unknown if it always detects the unknot.
- *Combinatorial knot Floer homology* was developed by M.-Ozsváth-Sarkar (2006), following work of Ozsváth-Szabó and Rasmussen (2002). It detects the unknot!
- *Khovanov homology* (1999). Kronheimer-Mrowka showed that it detects the unknot (2010).

Floer homology has its origin in *gauge theory*. Gauge theory is a part of physics, which started with the introduction of the *Yang-Mills equations* in 1954. These are partial differential equations that govern the weak and strong interactions of elementary particles.

Donaldson (1982) and Floer (1988) realized that the Yang-Mills equations can say something interesting about topology.

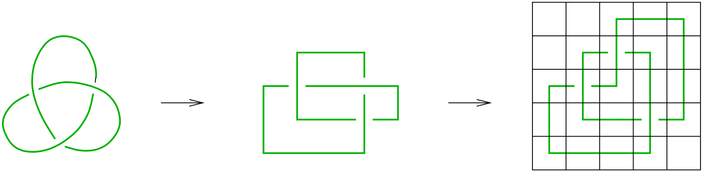
Combinatorial knot Floer homology

Strategy

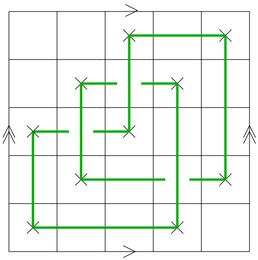
Diagram of a knot → Grid diagram → Graph → Homology → Rank

Then: Rank $= 1 \iff$ the knot is the unknot.

To go from a diagram to a grid diagram, make all segments vertical or horizontal, then arrange so that the vertical segments are on top:



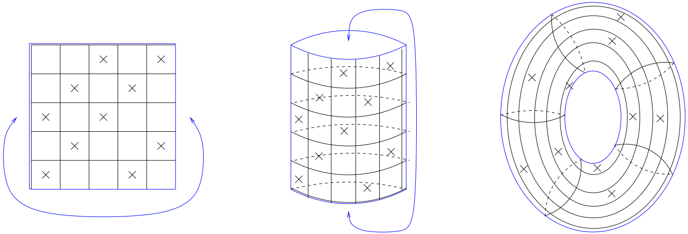
A grid diagram for the trefoil



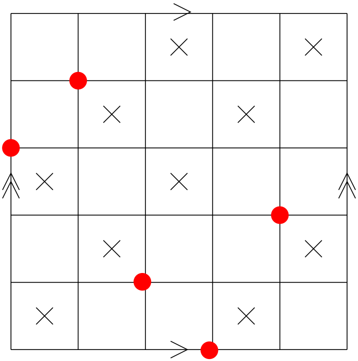
In a grid diagram, we have two X markings in each row and in each column. These markings determine the knot, so we might as well delete the green arcs.

A grid diagram for the trefoil

In a grid diagram, the left and the right edge are identified, and so are the bottom and the top edge. We get a topological object called a *torus*:

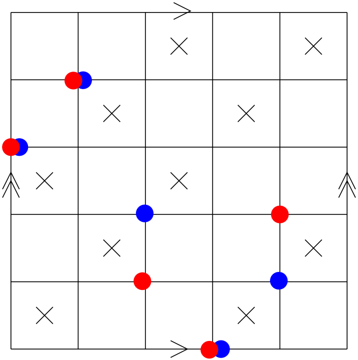


From a grid diagram to a graph



The vertices $\mathbf{x} = \{x_1, \ldots, x_n\}$ are n-tuples of points on the grid (one on each vertical and horizontal circle).

From a grid diagram to a graph



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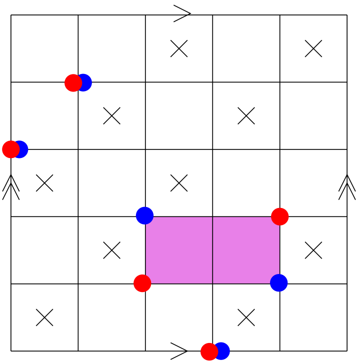
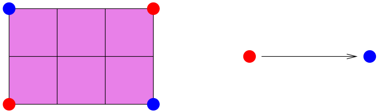


Let's look at the empty rectangles on the grid torus (that is, having no red or blue dots or X markings inside).

From a grid diagram to a graph

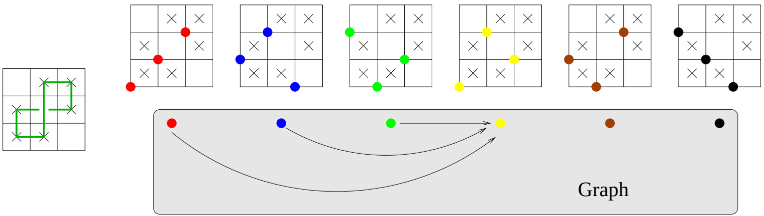
Whenever we see an empty rectangle, we draw an oriented edge from the red to the blue vertex in our graph:



In our example (the 5×5 grid diagram for the trefoil) we get an oriented graph with $5! = 120$ vertices and 90 edges.

Let's look at some simpler examples!

Here is a 3×3 grid diagram for the unknot. We show the 6 vertices and the rectangles corresponding to the 3 edges in the graph:



From a grid diagram to a graph

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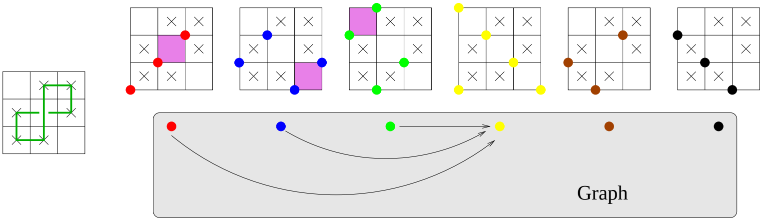
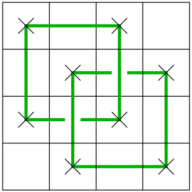
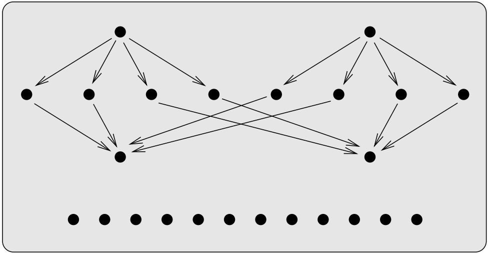


From a grid diagram to a graph

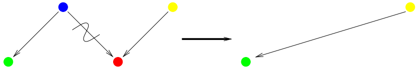
Here is a 4×4 grid diagram for the Hopf link, and the corresponding graph:



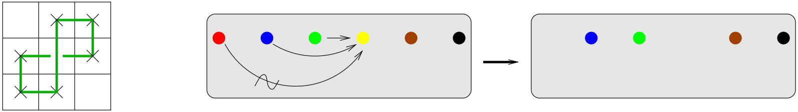


From a graph (of a chain complex) to homology

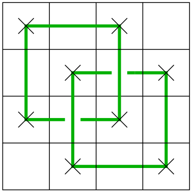
Now we have a graph! To get the Floer homology, we delete an edge and replace all zigzags with new edges, as follows:

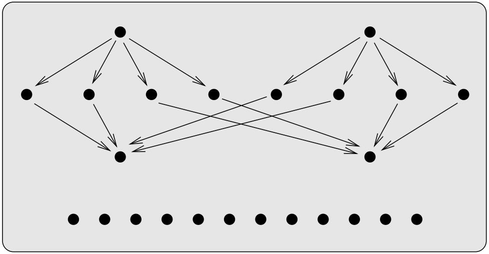


We repeat this process until there are no edges left. For example, for the 3×3 unknot, we get:



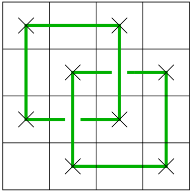
For the 4×4 Hopf link, we get:

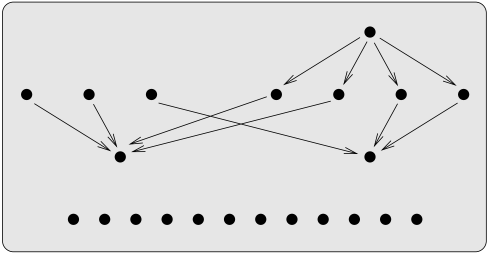




From a graph to homology

For the 4×4 Hopf link, we get:





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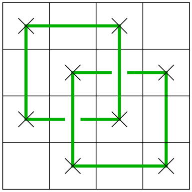
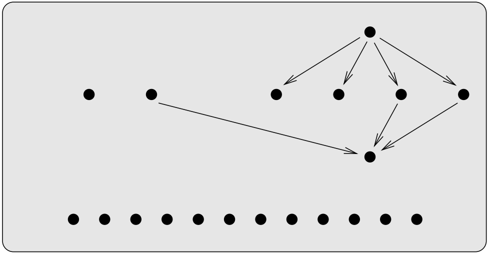
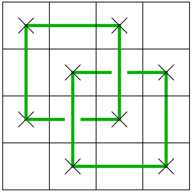
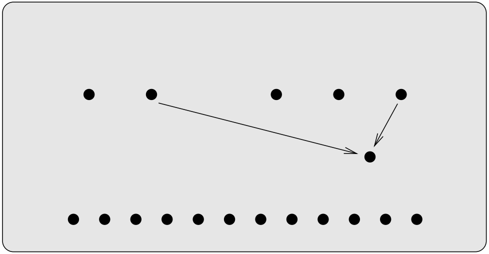




From a graph to homology

For the 4×4 Hopf link, we get:





From a graph to homology

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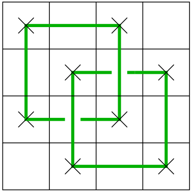
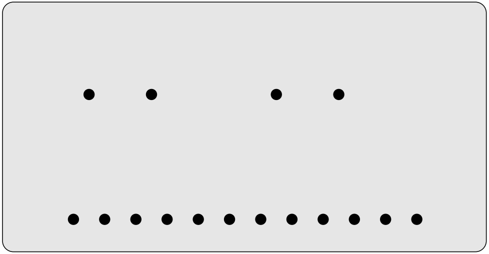




From homology to the rank

Once we get the homology, we just count the remaining vertices, and divide by $2^{n-\ell}$, where

$$\begin{aligned} n &= \text{size of the grid} \\ \ell &= \text{number of components of the link.} \end{aligned}$$

For example, for the 3×3 unknot, we get

$$\text{Rank} = \frac{4}{2^{3-1}} = 1.$$

For the 4×4 Hopf link, we get

$$\text{Rank} = \frac{16}{2^{4-2}} = 4.$$

Detecting the unknot

Theorem

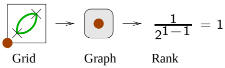
(a) If two grid diagrams represent the same knot, then their Floer homology has the same rank.
(b) A grid diagram has Floer homology of rank 1 $\iff$ it represents the unknot.

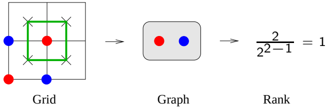
Part (a) has a combinatorial proof (M.-Ozsváth-Szabó-Thurston).

Part (b) is much more difficult. The proof (due to Ozsváth-Szabó) involves various techniques from three-dimensional topology, symplectic geometry, and contact geometry.

Examples

We saw that the 3×3 grid for the unknot gives rank 1. Part (a) of the theorem says that any other grid diagram for the unknot gives rank 1. Let's test this for two very simple grids:





On the other hand, if we start with the 5×5 grid for the trefoil, we get a graph with 120 vertices and 90 edges. After deleting edges, we are left with 48 vertices in the homology, so the rank is $\frac{48}{2^{5-1}} = 3$.

In fact, for any grid that represents a non-trivial knot, we will get rank at least 3.

This is how you can detect the unknot!