Every problem is worth 5 points. You are encouraged to work in groups, but everyone should turn in their own solutions. This problem set is due in class on Wednesday. The ideal homework is neatly and carefully written, stapled(!!!), and problems are clearly marked (i.e. by starting a new page).

Problem 1. Show that every \( n \)-connected CW-complex of dimension \( n \) is contractible.

Problem 2. Let \( Q \) be a set with two different binary operation \( +, \oplus : Q \times Q \to Q \) which both have a unit. Further assume that
\[
(a \oplus b) + (c \oplus d) = (a + c) \oplus (b + d) \quad \text{for all } a, b, c, d \in Q.
\]
Show that \( + \) and \( \oplus \) define in fact the same operation and that this operation is commutative and associative. Use this to find an alternate proof that \( \pi_n(X) \) is abelian for all spaces \( X \) and \( n \geq 2 \).

Problem 3. Let \( f : X \to Y \) be a map between CW-complexes. Show that for any given \( n \), \( f \) can be factored as
\[
X \xrightarrow{g} Z \xrightarrow{h} Y
\]
for some CW-complex \( Z \) such that \( g \) induces an isomorphism \( \pi_k(X) \to \pi_k(Z) \) for \( k \leq n \) and \( h \) induces an isomorphism \( \pi_k(Z) \to \pi_k(Y) \) for any \( k > n \).

Problem 4. Calculate the homotopy groups \( \pi_k(\mathbb{R}P^n) \) for \( k \leq n \) and determine the action of the fundamental group \( \pi_1(\mathbb{R}P^n) \) on those groups.

Problem 5. Construct a CW-complex \( X \) with given homotopy groups \( \pi_i(X) \) and a given action of \( \pi_1(X) \) on the \( \pi_i(X) \). Use this to give an example of a non-abelian space with abelian fundamental group.

Problem 6 (Challenge). In this problem we will construct the Poincare homology sphere.

1. Interpret \( S^3 \) as the unit quaternions and \( \mathbb{R}^3 \) as the pure quaternions (no real part). Then \( S^3 \) acts on \( \mathbb{R}^3 \) by conjugation \( (P_x(y) = xyx^{-1}, \text{where } x \in S^3, \text{ and } y \in \mathbb{R}^3. \) Show that this defines a double-covering \( S^3 \to SO(3) \).

2. Let \( I \subset SO(3) \) be the symmetry group of the regular icosahedron in \( \mathbb{R}^3 \) and let \( \tilde{I} \subset S^3 \) be its cover. Show that the projection \( S^3 \to S^3/\tilde{I} \) induces an isomorphism on homology.

3. Calculate the fundamental group of \( S^3/\tilde{I} \) and show that \( S^3/\tilde{I} \) is not homotopy equivalent to \( S^3 \). (They both are, however, CW-complexes.)