2 $\ell$-adic representations and Weil-Deligne representations

Recall $F$ is a non-archimedean local field. Let $p$ be the residual characteristic of $F$. Fix a prime $\ell \neq p$, and let $G = G_{\bar{\mathbb{Q}_\ell}}$ where $G$ is a possibly disconnected reductive group over $\bar{\mathbb{Q}_\ell}$. Let $G_F \supset I_F \supset P_F$ be the absolute Galois group, the inertia, and the wild inertia, respectively. For an $\ell$-adic Galois representation $\rho : G_F \to G$, we first look at how $\rho|_{I_F}$ looks like. Recall that we have $I_F/P_F \cong (\mathbb{Z}_p^{\times}) = \prod_{\ell' \neq \ell} \mathbb{Z}_{\ell'}$. (Sketch of proof: By Hensel lemma style argument, any degree $m$ extension of $F^{ur}$ with $p \nmid m$ is of the form $F^{ur}(\sqrt[p]{\ell})$ for a fixed uniformizer $\varpi_F$.) Write $\ell^t : I_F \to \mathbb{Z}_\ell$ the composition $I_F \to I_F/P_F \to \mathbb{Z}_\ell$.

**Definition 2.1.** Say a compact topological group $H$ is prime-to-$\ell$ if it is an inverse limit of finite discrete prime-to-$\ell$ groups.

We have $\ker(\ell)$ is prime to $\ell$. Note that $x \mapsto x^\ell$ is a homeomorphism for prime-to-$\ell$ groups. One proves the following

**Lemma 2.2.** If $H$ is a prime-to-$\ell$ compact topological group and $\rho : H \to G$ is a continuous homomorphism, then $\ker(\rho)$ is open and thus with finite index.

**Proof.** We may embed $G \hookrightarrow GL_N$ and $G \hookrightarrow GL_N(\mathbb{Q}_\ell)$. Let $U_n \subset G$, $n \geq 0$ be those who are congruent to the identity modulo $\ell^{n+2}$ in $GL_N(\mathbb{Q}_\ell)$. Let $V = \rho^{-1}(U_0)$. Being an open subgroup of a prime-to-$\ell$ group $V$ is itself prime-to-$\ell$, thus $\rho(V) \subset \{g^\ell \mid g \in U_0\}$ which is contained in $U_1$. In particular $V = \rho^{-1}(U_1)$. The same argument implies $V = \rho^{-1}(U_n)$ for any $n$. Thus $V = \rho^{-1}(\bigcap_n U_n) = \rho^{-1}(\{e\}) = \ker(\rho)$.

**Corollary 2.3.** (Grothendieck’s $\ell$-adic monodromy theorem) Let $\rho : G_F \to G$ be a continuous homomorphism. Then there exists a relatively open subgroup $U \subset I_F$ and a necessarily unique nilpotent element $N \in G$ such that

$$\rho(\tau) = \exp(\ell^t(\tau)N), \forall \tau \in U.$$

**Proof.** By Lemma 2.2, $\ker(\rho)$ contains an open subgroup of $\ker(\ell)$. By shrinking $U$ we may assume $\rho|_{U \cap \ker(\ell)}$ is trivial. By again shrinking $U$, we may assume that $\rho(U)$ lies in an open neighborhood of the identity on which log converges, i.e. it makes sense to talk about $\log \rho(\tau)$ for $\tau \in U$. We have $\ell^t(U) \cong U/(U \cap \ker(\ell)) \overset{\log \circ \rho}{\longrightarrow} \text{Lie } G$. Since $\ell^t(U) \cong \ell^m \mathbb{Z}_\ell$ for some $m$, the above composition has to be given by $x \mapsto xN$ for some $N \in M_n(E)$. This is the $N$ we seek.

It remains to prove that $N$ is nilpotent. Let $\Phi \in G_F$ be any (lift of) arithmetic Frobenius. Then for any $\tau \in I$, we have $\ell^t(\Phi \tau \Phi^{-1}) = q \cdot \ell^t(\tau)$ where $q$ is the order of the residue field of $F$. Applying this to the definition of $N$ we see $\rho(\Phi) \exp(N)\rho(\Phi)^{-1} = \exp(qN)$, or $\text{Ad}(\rho(\Phi))N = qN$. In particular $N$ lies in the same orbit as $qN$ and is thus nilpotent. \qed
Example 4. We have the following \( \ell \)-adic Galois representation \( \rho : G_F \to GL_2(\mathbb{Q}_\ell) \) given by

\[
\rho = \begin{pmatrix}
\chi_{\text{cyc}} & t_{\ell} \\
0 & 1
\end{pmatrix}.
\]

In fact, this is the Tate module of an elliptic curve that has semistable reduction.

Let us denote by \( | \cdot | : W_F \to \mathbb{C}^* \) the normalized norm, so that \( |\Phi| = q \). Fix a continuous homomorphism \( \rho : W_F \to \mathcal{G} \). From the proof we see that Corollary 2.3 also holds for \( \rho \). Fix again \( \Phi \) any (lift of) Frobenius. We have

**Theorem 2.4.** (Deligne) Let \( t_{\ell} \) be as above and \( N \) as in Corollary 2.3. The formula

\[
\rho_{\ell}(\Phi \rho^{m}) := \rho(\Phi)^{m} \rho(\tau) \exp(-t_{\ell}(\tau)N), \quad \forall m \in \mathbb{Z}, \quad \tau \in I_{F}
\]

defines a new homomorphism \( \rho_{\ell} : W_{F} \to \mathcal{G} \). We have \( \ker(\rho_{\ell}) \) is open in \( W_{F} \). Moreover, changing the choice of \( \Phi \) only changes \( \rho_{\ell} \) up to conjugation by an element in \( \mathcal{G}^{0} := \mathcal{G}^{0}(\mathbb{Q}_{\ell}) \).

**Proof.** Recall that we had \( t_{\ell}(\Phi \tau^{-1}) = q \cdot t_{\ell}(\tau) \Rightarrow \rho(\Phi) \rho(\Phi)^{-1} = q \cdot N \). A similar argument gives \( t_{\ell}(\sigma \tau^{-1}) = |\sigma| \cdot t_{\ell}(\tau) \) and \( \rho(\sigma) \rho(\sigma)^{-1} = |\sigma| \cdot N \) for any \( \sigma \in W_{F} \). This implies \( \rho(\sigma) \exp(xN) \rho(\sigma)^{-1} = \exp(|\sigma|xN) \) for any \( x \in \mathbb{Q}_{\ell} \).

Now for the first statement of the theorem, suppose we have \( \Phi^{m_{1}\tau_{1}} \Phi^{m_{2}\tau_{2}} = \Phi^{m_{1}+m_{2}\tau}, \) i.e. \( \tau = \Phi^{-m_{2}\tau_{1}} \Phi^{m_{2}\tau_{2}} \). Then

\[
\rho_{\ell}(\Phi^{m_{1}+m_{2}\tau}) = \rho(\Phi^{m_{1}+m_{2}\tau}) \exp(-t_{\ell}(\tau)N)
\]

\[
= \rho(\Phi^{m_{1}\tau_{1}}) \rho(\Phi^{m_{2}\tau_{2}}) \exp(-t_{\ell}(\Phi^{-m_{2}\tau_{1}} \Phi^{m_{2}\tau_{2}})N) \exp(-t_{\ell}(\tau_{2})N)
\]

\[
= \rho(\Phi^{m_{1}\tau_{1}}) \rho(\Phi^{m_{2}\tau_{2}}) \exp(-q^{-m_{2}t_{\ell}(\tau_{1})}N) \exp(-t_{\ell}(\tau_{2})N)
\]

\[
= \rho(\Phi^{m_{1}\tau_{1}}) \exp(-t_{\ell}(\tau_{1})N) \rho(\Phi^{m_{2}\tau_{2}}) \exp(-t_{\ell}(\tau_{2})N) = \rho_{\ell}(\Phi^{m_{1}\tau_{1}}) \rho_{\ell}(\Phi^{m_{2}\tau_{2}}).
\]

By Corollary 2.3, \( \rho_{\ell} \) is trivial on some \( U \subset I_{F} \) open, yet \( I_{F} \) is open in \( W_{F} \) and thus \( \ker(\rho_{\ell}) \) is open. For the last statement, suppose \( \Phi' \) is a different choice of Frobenius, and \( \Phi = \Phi' \tau' \), with \( t_{\ell}(\tau') = s' \). Then the resulting \( \rho'_{\ell} \) has \( \rho'_{\ell}(\Phi) = \rho_{\ell}(\Phi' \tau') = \rho_{\ell}(\Phi) \exp(-sN) \). On the other hand, we have \( \rho(\Phi) \exp(xN) \rho(\Phi)^{-1} = \exp(xN) \Rightarrow \rho(\Phi) = \exp(xN)^{q} \rho(\Phi) \exp(xN)^{-1} \Rightarrow \rho(\Phi) \exp(xN)^{1-q} = \exp(xN)^{q} \rho(\Phi) \exp(xN)^{q-1} \). Taking \( x = \frac{s}{q-1} \), we get \( \rho_{\ell}(\Phi) = \rho_{\ell}(\Phi) \exp(-sN) = \rho'_{\ell}(\Phi) \exp(xN)^{q} \). Since for any \( \tau \in I \) we have \( \rho_{\ell}(\tau) = \rho_{\ell}(\tau') \) commutes with \( N \) and \( \exp(xN) \) by the formula, we have \( \rho'_{\ell} \) is conjugate to \( \rho_{\ell} \) by \( \exp(xN) \in \mathcal{G}^{0} \).

**Remark 2.5.** \( \rho_{\ell}|_{I_{F}} \) is trivial on the open subgroup \( U \subset I_{F} \) in Corollary 2.3, and thus has finite image. On the other hand, in Corollary 2.3 we have \( N = 0 \) iff \( \rho|_{I_{F}} \) has finite image, which is also equivalent to \( \rho = \rho_{\ell} \).
Now let us fix an isomorphism $\iota : \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ as abstract fields. Such an isomorphism exists because the transcendence degrees of both algebraically closed fields over $\mathbb{Q}$ are of the cardinality of $2^{\aleph_0}$.

**Definition 2.6.** A Weil-Deligne representation into $G(\mathbb{C})$ is a pair $(\rho_\sharp, N)$ where $\rho_\sharp : W_F \to G(\mathbb{C})$ is a continuous homomorphism, and $N \in \text{Lie} G(\mathbb{C})$ is a nilpotent element such that $\rho_\sharp(\sigma)N\rho_\sharp(\sigma)^{-1} = |\sigma| \cdot N$ for any $\sigma \in W_F$. Two Weil-Deligne representations $(\rho_\sharp, N)$ and $(\rho'_\sharp, N')$ are equivalent if there exists $g \in G(\mathbb{C})$ such that $\rho'_\sharp = g\rho_\sharp g^{-1}$ and $N' = \text{Ad}(g)N$.

When $N = 0$, it’s also common to call it a Weil representation.

**Corollary 2.7.** Upon the choice of $\iota$, Theorem 2.4 gives a bijection between the set of isomorphism classes of $\ell$-adic Weil representations and the set of isomorphism classes of Weil-Deligne representations.

**Lemma 2.8.** Let $(\rho_\sharp, N)$ be a Weil-Deligne representation. If $\rho_\sharp(\Phi)$ is semisimple, then $\rho_\sharp(\sigma)$ is semisimple for any $\sigma \in W_F$.

**Proof.** Let $U = \ker(\rho_\sharp)$ which is open and normal in $W_F$. For any $\sigma = \Phi^m \tau \in W_F$, we may write $\sigma^a = \Phi^a \tau_a$ for any $a \in \mathbb{Z}$. The image of $\tau_a$ in $I_F/U$ has to be periodic when $a$ runs over $\mathbb{Z}$, and thus $\tau_a \in U$ for some $a > 0$. This gives $\rho_\sharp(\sigma)^a = \rho_\sharp(\Phi)^a$ is semisimple. Since $\overline{\mathbb{Q}}_\ell$ has characteristic zero, $\rho_\sharp(\sigma)$ is also semisimple. (If $m = 0$, then $\sigma \in I_F$ and the semisimplicity of $\rho_\sharp(\sigma)$ is automatic as $\rho_\sharp | I_F$ has finite image.)

**Definition 2.9.** A Weil-Deligne representation is called Frobenius-semisimple if it satisfies the property in Lemma 2.8.

A general conjecture is for an interesting (i.e. those that come from geometry) $\ell$-adic representations, Frobenius should act by semisimple elements. Thus it’s somewhat natural to focus on Frobenius-semisimple Weil-Deligne representation.

Let us now begin with $G$ a reductive group over $F$, without the assumption that $G$ is quasi-split. Let $G'$ be a $F$-split form of $G$. Let $\Phi'$ be the roots of $G'$ and $\Delta'$ some choice of positive roots. We have the exact sequence

$$1 \to G'_{\text{ad}} \to \text{Aut}(G') \to \text{Aut}(\Phi', \Delta') \to 1$$

which induces

$$H^1(G_F, G'_{\text{ad}}) \to H^1(G_F, \text{Aut}(G')) \to H^1(G_F, \text{Aut}(\Phi, \Delta))$$

Now $G$ corresponds to a class in $H^1(G_F, \text{Aut}(G'))$, which is further mapped to a class in $\alpha \in H^1(G_F, \text{Aut}(\Phi, \Delta))$. There is however a lift $\iota : \text{Aut}(\Phi', \Delta') \hookrightarrow \text{Aut}(G')$, and $H^1(\iota)(\alpha)$
corresponds to a quasi-split form $G^*$ of $G$. By the formalism of non-abelian $H^1$, we have $G$ is a twist of $G'$ by an element in $H^1(G_F, G_{ad}^*)$, i.e. an inner twist of $G^*$. In other words, we have shown that any form $G$ has a quasi-split inner form $G^*$.

This inner form is actually unique. Suppose $G$ and $G^*$ are both quasi-split and we have an inner twisting $\iota : G(k_{\text{sep}}) \rightarrow G^*(k_{\text{sep}})$. By the quasi-split assumption both $G$ and $G^*$ have Borels $B$ and $B^*$ that are stable under $G_F$. By twisting $\iota$ by a 1-coboundary we may assume $\iota(B) = B^*$ (since all Borels are (inner) conjugate). Since the stabilizer of $B$ in $G_{ad}$ is $B_{ad}$, we may assume the twisting $\iota$ is a class in $H^1(G_F, B_{ad})$, and it remains to prove $H^1(G_F, B_{ad}) = 0$. We have

$$H^1(G_F, U_{ad}) \rightarrow H^1(G_F, B_{ad}) \rightarrow H^1(G_F, T_{ad}).$$

The first term vanishes because $U_{ad}$ is unipotent. To prove that the third term vanishes, we claim that $T_{ad}$ is always an induced torus, i.e. a product or $\text{Res}_{F}^E(\mathbb{G}_m)$ for possible many different finite separable $E$. By the usual equivalence of category it is equivalent to prove that $X^*(T_{ad})$ is induced, i.e. a direct sum of $\mathbb{Z}[G_F/G_E]$ as a $\mathbb{Z}[G_F]$-module. But as $T_{ad}$ is the maximal torus of an adjoint group, the weight lattice is equal to the root lattice and thus $X^*(T_{ad}) = \mathbb{Z}^\Delta$ is induced. Thus we have proved that the quasi-split inner form $G^*$ of $G$ is unique.

We define $^L G := ^L(G^*)$ to be the same $L$-group as that of $G^*$. In fact, to make $\pi_0(^L G)$ finite, let us take the version $^L(G^*) = (G^*)^\vee \rtimes \text{Gal}(E/F)$ where $E/F$ is any finite Galois extension that splits $E$; it will be mostly tautological that extending $E$ does not affect everything we do later. This allows us to finally define

**Definition 2.10.** A **Langlands parameter** for $G$ is a Weil-Deligne representation $\rho : W_F \rightarrow ^L G$ that is Frobenius semisimple. Two Langlands parameters are equivalent if they are equivalent as Weil-Deligne representation.