3 Parabolic induction and supercuspidal

Let us briefly recall some representation theory. Our representations will always be on vector spaces over $\mathbb{C}$, despite that we remark all theory actually stays the same if we work on vector spaces over $\mathbb{Q}_p$. Recall that a vector $v$ in a representation of $G(F)$ is called smooth if there exists an open subgroup $K \subset G(F)$ (by shrinking we may and always assume $K$ compact) such that $v$ is fixed by $K$. A representation $\pi$ is called smooth if all vectors are smooth. Write $\pi^K$ those vectors fixed by $K$. Then $\pi$ is smooth iff $\pi = \sum \pi^K$ where $K$ runs over compact open subgroups of $G(F)$. We say $\pi$ is admissible if for any open compact subgroup, $\dim \pi^K$ is finite.

Now we are ready to come back to the question: Let $G$ be a connected reductive group over $F$, our fixed non-archimedean local field. We assume the residual characteristic $p$ does not divide the Weyl group of $G \otimes F^{sep}$. Given $\rho : W_F \to \mathcal{L}G$ which factors as $\rho : L_S \to \mathcal{L}G$ and $\rho_S : W_F \to \mathcal{L}S$. By local Langlands for tori we have from $\rho_S$ a character $\chi : S(F) \to \mathbb{C}^\times$, and we will like to “induce” $\chi$ to an irreducible admissible representation of $G(F)$, or rather a finite number (could be zero) of representations of $G(F)$.

Recall that by the construction of $\mathcal{L}G = G^\vee \rtimes \text{Gal}(E/F)$, we have $T^\vee \subset B^\vee \subset G^\vee$ and $\text{Gal}(E/F)$ acts on $G^\vee$ by stabilizing a pinning associated to $(G^\vee, B^\vee, T^\vee)$. A simpler case of our scenario will be if $\mathcal{L}S \subset \mathcal{L}T := T^\vee \rtimes \text{Gal}(E/F)$. Note that from our definition of $\mathcal{L}T$ we have a torus $T$ defined over $F$. This torus can be identified with a maximal torus contained a Borel of $G^\ast$. In fact, if $G^\ast$ is split, then $T$ is split and having an embedding $T \hookrightarrow G$ will imply $G = G^\ast$ and $T$ is a maximal split torus in $G$.

**Lemma 3.1.** Suppose there is an embedding $T \subset G$. Then $G$ has a Borel $B \supset T$ that is defined over $F$. In particular, $G$ is quasi-split.

We now suppose $G = G^\ast$, and $T \subset B \subset G^\ast$ are defined over $F$. We are at the question that given a character $\chi : T(F) \to \mathbb{C}^\ast$, how should we construct a representation of $G(F)$? We may inflate $\chi$ to a character of $B(F)$ and define

$$\text{Ind}_B^G \chi := \{ f : G(F) \to \mathbb{C} \mid f(bg) = \chi(b)f(g), \forall b \in B(F), g \in G(F) \}^\infty$$

This is a representation by right translation, and the superscript $\infty$ means to take the smooth vectors in the whole vector space. While this looks very nice at the first glance, examples in $GL_2$ (if you have been to Zev’s talk on 1/17) tells us that we need a shift. Let $\Delta_B$ be the modulus character of $B(F)$, i.e. $\mu(bE_b^{-1}) = \Delta_B(b)\mu(E)$ for any either left or right Haar measure $\mu$ on $B$, any $b \in B(F)$, and any measurable subset $E \subset B(F)$. Let $U$ be the unipotent radical of $B$ and identify $T = B/U$. We also have $T(F) = B(F)/U(F)$ as e.g. $H^1(F, U) = 0$. A Haar measure on $T(F)$ can be realized as the quotient measure of a (left or right) Haar measure on $B(F)$ by another on $U(F)$. Since $T$ is unimodular, we have $\mu(bE_Ub^{-1}) = \Delta_B(b)\mu(E_U)$ also for any Haar measure on $U$ and measurable subset.
Let $W \subseteq U(F)$. One has a $B$-stable filtration of $U = U_n \supset U_{n-1} \supset \ldots \supset U_0 = 1$ such that each $U_n/U_{n-1}$ is a product of $\mathbb{G}_a$, and $B$ acts on each $U_i/U_{i-1}$ through $T$ via some positive roots, so that each positive root appears in some $U_i/U_{i-1}$ exactly once.

In any case, we now define

$$\Delta_B(b) = |\delta(b)|.$$ 

In any case, we now define

$$\iota_B^G(\chi) := \text{Ind}_B^G(\chi \otimes \Delta_B^{1/2})$$

Let $W := N_{G(F)}(T)/T(F)$ be the relative Weyl group. Apparently $W$ acts on the set of characters of $T(F)$. The big theorem is

**Theorem 3.2.** (Bernstein-Zelevinsky) The induction $\iota_B^G(\chi)$ always have finite length. For two characters $\chi_1, \chi_2 : T(F) \to \mathbb{C}^\times$, the composition series of $\iota_B^G(\chi_1)$ and that of $\iota_B^G(\chi_2)$ have a common factor if and only if $\chi_1$ and $\chi_2$ can be conjugate to each other by $W$. If this is the case, then they have the same Jordan-Hölder constituents with the same multiplicities, and $\text{Hom}(\iota_B^G(\chi_1), \iota_B^G(\chi_2)) \neq 0$.

We briefly describe the idea of Bernstein and Zelevinsky. Recall that Frobenius reciprocity says

$$\text{Hom}_{G(F)}(\pi, \text{Ind}_B^G \chi) = \text{Hom}_{B(F)}(\pi, \chi).$$

In particular

$$\text{Hom}_{G(F)}(\text{Ind}_B^G \chi_1, \text{Ind}_B^G \chi_2) = \text{Hom}_{B(F)}(\text{Res}_B^G \text{Ind}_B^G \chi_1, \chi_2). \tag{1}$$

By definition $\chi_2$ is inflated from a character from $T(F)$. On the RHS of (1), we can pass it to $\text{Hom}_{T(F)}$ by taking the $U(F)$-coinvariant of $\text{Res}_B^G \text{Ind}_B^G$; for $\lambda$ any representation of $B(F)$, let $J_U(\lambda)$ be the representation of $T(F)$ whose underlying vector space is the space of $U(F)$-coinvariants. Then we have

$$\text{Hom}_{B(F)}(\text{Res}_B^G \text{Ind}_B^G \chi_1, \chi_2) = \text{Hom}_{T(F)}(J_U(\text{Res}_B^G \text{Ind}_B^G \chi_1), \chi_2).$$

In fact, let $r_B^G \pi := J_U(\text{Res}_B^G(\pi) \otimes \Delta_B^{-1/2})$. Then we have $r_B^G$ is left adjoint to $\iota_B^G$. In particular

$$\text{Hom}_{G(F)}(\iota_B^G \chi_1, \iota_B^G \chi_2) = \text{Hom}_{T(F)}(r_B^G \chi_1, \chi_2).$$

When $G = GL_2$, one may do a bit of brute force and check that

$$\iota_B^G \chi = \chi \oplus \chi^w$$

where $w$ is the non-trivial element in $N_{G(F)}(T)/T(F)$, unless $\chi = \chi^w$. In the case $\chi = \chi^w$ things become trickier to check, and in fact $r_B^G \chi$ will turns out to be a non-trivial extension of $\chi$ by itself. In the case where $G$ is a general quasi-split group, we have

$$r_B^G \chi$$ has a composition series given by some permutation of $\{\chi^w \mid w \in W\}$.  

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Recall that we have a natural embedding \( L^T \hookrightarrow L^G \). Fix \( \chi : T(F) \to \mathbb{C}^\times \) a character and let \( \text{rec}(\chi) : W_F \to L^T \) be the corresponding homomorphism. Let \( L_j : L^T \to L^G \) be the natural embedding. We then have

**Conjecture 3.3.** A Jordan-Hölder constituents of \( \iota_T^G(\chi) \) should correspond under Local Langlands to a Langlands parameter with \( \rho_\sharp = L_j \circ \text{rec}(\chi) \) and some \( N \).

**Remark 3.4.** One can treat this conjecture either as a conjecture, or as a requirement of the Local Langlands. In either case, it is known for \( GL_n \) and split classical groups. I am not sure if it is known for unitary groups. The Local Langlands is not known for more general groups over a \( p \)-adic field.

The converse is not true, in the sense that even if we have a \((\rho_\sharp, L_j \circ \text{rec}(\chi))\) which is of the form in the conjecture, it can be the image under the finite-to-1 Local Langlands correspondence from a representation not inside \( \iota_T^G(\chi) \). The easiest example is probably when \( G = GL_2 \), and \( \chi = \Delta^{1/2} \). In this case \( \rho_\sharp \) is unramified and \( \rho_\sharp(\Phi) = \begin{pmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{pmatrix} \).

There is a unique non-trivial choice of \( N \), and such \((\rho_\sharp, N)\) corresponds to both the Steinberg representation of \( GL_2(F) \) (which is in \( \iota_T^G(\chi) \)) as well as the trivial representation of \( D^\times(F) \), the group of non-zero quaternions. Note that \( D^\times \) is an inner form of \( GL_2 \).

The smallest counterexample I know that appears in the same group (unlike \( D^\times \) vs \( GL_2 \) above) happens in \( Sp_8 \), and is too complicated for me to understand.

Let us now drop the assumption that \( G = G^* \), so that \( G \) is a general connected reductive group over \( F \). Let \( P \subset G \) be a parabolic subgroup and we can decompose \( P = MU \) where \( M \) is some Levi subgroup and \( U = U_P \) is the unipotent radical of \( P \). For any smooth representation \( \lambda \) of \( M(F) = P(F)/U(F) \), we inflate it to a representation of \( P(F) \), denote also by \( \lambda \) the underlying vector space and define

\[
\text{Ind}^G_P \lambda := \{ f : G(F) \to \lambda \mid f(pg) = \chi(p) f(g), \forall p \in P(F), g \in G(F) \}^\infty.
\]

Suppose \( P^1 = M^1 U_{P^1} \subset M \) is another parabolic subgroup where \( M^1 \) is again the reductive quotient of \( P^1 \). Let \( P^m \) be the preimage of \( P^m \) in \( P \to M \). Then \( P^{(1)} \) is a parabolic subgroup of \( G \) and \( M^1 \) is naturally identified as its reductive quotient. One has

\[
\text{Ind}^G_{P^{(1)}} = \text{Ind}^G_P \text{Ind}^M_P \, \text{Ind}^M_{P^{(1)}}. \tag{2}
\]

On the other hand, for a smooth representation \( \pi \) of \( G(F) \) recall we denote by \( U \) the unipotent radical of \( P \). Denote by \( J_U \pi \) the \( U(F) \)-coinvariant of \( \text{Res}^G_P \pi \), as a smooth representation of \( P(F)/U(F) = M(F) \). We have seen that

\[
\text{Hom}_{G(F)}(J_U \pi, \lambda) = \text{Hom}_{M(F)}(\pi, \text{Ind}^G_P(\lambda))
\]
As in (2) we also have from definition

\[ J_{U_{p(1)}}(J_U \pi) = J_{U_{p1}} \pi. \]

**Lemma 3.5.** If \( \pi \) is finitely generated, then so is \( J_U(\pi) \).

**Proof.** Suppose \( \pi \) is generated by \( v_1, ..., v_n \). Then there exists some open subgroup \( K \subset G(F) \) such that \( v_i \in \pi^K \). The double quotient \( P(F) \backslash G(F)/K \) is finite, and thus there exists a finite set \( \Gamma \subset G(F) \) representing all double cosets. But then the image of \( \gamma v_i \) for all \( \gamma \in \Gamma, i = 1, ..., n \) generates \( \text{Res}_{P}^{G} \pi \) (as a \( P(F) \)-representation) and thus \( J_U(\pi) \).

**Definition 3.6.** Say an irreducible admissible representation \( \pi \) of \( G(F) \) is **supercuspidal** if \( J_U(\pi) = 0 \) for any proper parabolic \( P \) and its unipotent radical \( U \).

**Theorem 3.7.** Begin with any irreducible smooth representation \( \pi \) of \( G(F) \), there exists a parabolic \( P \subset G \) as above and a supercuspidal representation \( \lambda \) of \( M(F) \) such that \( \pi \) can be embedded into \( \text{Ind}_{P}^{G}(\pi) \).

**Proof.** One can always find a minimal \( P \) such that \( J_U \pi \neq 0 \), yet \( J_{U_{p1}}(J_U \pi) = 0 \) for any proper parabolic \( P^1 \subset M \). By Lemma 3.5 \( J_U \pi \) is finitely generated. We now claim that \( J_U \pi \) has an irreducible quotient. Indeed, suppose a minimal generating set for \( J_U \pi \) is \( \{v_1, ..., v_k\} \). Then one may choose, via Zorn’s lemma, a maximal subrepresentation of \( J_U \pi \) containing \( v_1, ..., v_{k-1} \) but not \( v_k \). Then the quotient of \( J_U \pi \) by this subrepresentation is irreducible.

Let \( \lambda \) be this irreducible quotient. As a quotient we evidently have \( J_{U_{p1}} \lambda = 0 \) for any proper parabolic \( P^1 \subset M \). Thus \( \lambda \) is supercuspidal. Now we have \( \text{Hom}_{G(F)}(\pi, \text{Ind}_{P}^{G}(\pi)) = \text{Hom}_{M(F)}(J_U \pi, \lambda) \neq 0 \), which is what we want.

Now for the purpose of Langlands correspondence, we will like to say when does a Langlands parameter correspond to supercuspidal representations. The expected result is

**Conjecture 3.8.** Let \( (\rho, N) \) with \( \rho : W_F \to \mathbb{L}G \) be a Langlands parameter. Then all irreducible admissible representations that correspond to \( (\rho, N) \) (of \( G^* \) and all of its inner forms) are supercuspidal if and only if \( Z_{G^*}(\rho(W_F))^{0} \subset Z(G^{\vee}) \).

**Example 3.9.** When \( G = GL_n \), this says that \( \rho : W_F \to GL_n(\mathbb{C}) \) is irreducible when viewed as an \( n \)-dimensional representation, in which case we necessarily have \( N = 0 \), as \( \ker(N) \) will be a sub-representation. In fact we will see soon that the condition above implies \( N = 0 \).

In order to see how Conjecture 3.8 can be generalized in a way to incorporate with Theorem 3.7, we need to reformulate our Langlands parameter as
**Theorem 3.10.** Let $(\rho : W_F \to {}^LG, N)$ be a Langlands parameter. Then there exists a continuous homomorphism $\rho : W_F \times SL_2(\mathbb{C}) \to {}^LG$ such that

(i) $\rho|_{W_F}$ is Frobenius-semisimple.

(ii) $\rho|_{SL_2(\mathbb{C})} : SL_2(\mathbb{C}) \to G^\vee$ is algebraic.

(iii) $\exp(N) = \rho(id, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix})$.

(iv) There exists a lift of Frobenius $\Phi \in W_F$ such that $\rho(\Phi^m \tau, \begin{pmatrix} q^{m/2} & 0 \\ 0 & q^{-m/2} \end{pmatrix})$ for any $m \in \mathbb{Z}$ and $\tau \in I_F$. Moreover, such $\rho$ is unique up to conjugation by $G^\vee$.

**Example 3.11.** Suppose $G = GL_2$, and

$$\rho_2 = \begin{pmatrix} \chi_{cyc}^{1/2} & 0 \\ 0 & \chi_{cyc}^{-1/2} \end{pmatrix}, \quad N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$ 

Then we have $\rho$ is trivial on $W_F$ and is (conjugate to) the natural embedding $SL_2(\mathbb{C}) \hookrightarrow GL_2(\mathbb{C}) = {}^LG$.

**Example 3.12.** We note that it is particularly required that $\rho(SL_2(\mathbb{C}))$ commutes with $\rho(W_F)$. In particular, if $N$ is non-zero, then $\rho(\mathbb{C}^\times)$ must be non-trivial (where $\mathbb{C}^\times \subset SL_2(\mathbb{C})$ is the diagonal torus) and also non-central as $\rho(\mathbb{C}^\times) \subset \rho(SL_2(\mathbb{C}))$ has trivial image in the adjoint quotient of $G^\vee$. We have $\rho(\mathbb{C}^\times) \subset Z_{G^\vee}(\rho_2(W_F))^o$ and thus we must not be in the case of Conjecture 3.8.