3 Parabolic induction and supercuspidal

Let us briefly recall some representation theory. Our representations will always be on vector spaces over \( \mathbb{C} \), despite that we remark all theory actually stays the same if we work on vector spaces over \( \bar{\mathbb{Q}}_\ell \). Recall that a vector \( v \) in a representation of \( G(F) \) is called smooth if there exists an open subgroup \( K \subset G(F) \) (by shrinking we may and always assume \( K \) compact) such that \( v \) is fixed by \( K \). A representation \( \pi \) is called smooth if all vectors are smooth. Write \( \pi^K \) those vectors fixed by \( K \). Then \( \pi \) is smooth iff \( \pi = \sum \pi^K \) where \( K \) runs over compact open subgroups of \( G(F) \). We say \( \pi \) is admissible if for any open compact subgroup, \( \dim_\mathbb{C} \pi^K \) is finite.

Now we are ready to come back to the question: Let \( G \) be a connected reductive group over \( F \), our fixed non-archimedean local field. We assume the residual characteristic \( p \) does not divided the Weyl group of \( G \otimes F^{sep} \). Given \( \rho : W_F \to L^G \) which factors as \( l: j : L^S \to L^G \) and \( \rho_S : W_F \to L^S \). By local Langlands for tori we have from \( \rho_S \) a character \( \chi : S(F) \to \mathbb{C}^\times \), and we will like to “induce” \( \chi \) to an irreducible admissible representation of \( G(F) \), or rather a finite number (could be zero) of representations of \( G(F) \).

Recall that by the construction of \( L^G = G^\vee \rtimes \text{Gal}(E/F) \), we have \( T^\vee \subset B^\vee \subset G^\vee \) and \( \text{Gal}(E/F) \) acts on \( G^\vee \) by stabilizing a pinning associated to \( (G^\vee, B^\vee, T^\vee) \). A simpler case of our scenario will be if \( L^S \subset L^T := T^\vee \rtimes \text{Gal}(E/F) \). Note that from our definition of \( L^T \) we have a torus \( T \) defined over \( F \). This torus can be identified with a maximal torus contained a Borel of \( G^* \). In fact, if \( G^* \) is split, then \( T \) is split and having an embedding \( T \hookrightarrow G \) will imply \( G = G^* \) and \( T \) is a maximal split torus in \( G \).

**Lemma 3.1.** Suppose there is an embedding \( T \subset G \). Then \( G \) has a Borel \( B \supset T \) that is defined over \( F \). In particular, \( G \) is quasi-split.

We now suppose \( G = G^* \), and \( T \subset B \subset G^* \) are defined over \( F \). We are at the question that given a character \( \chi : T(F) \to \mathbb{C}^* \), how should we construct a representation of \( G(F) \)? We may inflate \( \chi \) to a character of \( B(F) \) and define

\[
\text{Ind}_B^G \chi := \{ f : G(F) \to \mathbb{C} \mid f(bg) = \chi(b)f(g), \forall t \in T(F), g \in G(F) \}
\]

While this looks very nice at the first glance, examples in \( GL_2 \) (if you have been to Zev’s talk on 1/17) tells us that we need a shift. Let \( \Delta_B \) be the modulus character of \( B(F) \), i.e. \( \mu(bEb^{-1}) = \Delta_B(b)\mu(E) \) for any either left or right Haar measure \( \mu \) on \( B \), any \( b \in B(F) \), and any measurable subset \( E \subset B(F) \). Let \( U \) be the unipotent radical of \( B \) and identify \( T = B/U \). We also have \( T(F) = B(F)/U(F) \) as e.g. \( H^1(F, U) = 0 \). A Haar measure on \( T(F) \) can be realized as the quotient measure of a (left or right) Haar measure on \( B(F) \) by another on \( U(F) \). Since \( T \) is unimodular, we have \( \mu(bE_Eb^{-1}) = \Delta_B(b)\mu(E_U) \) also for any Haar measure on \( U \) and measurable subset \( E_U \subset U(F) \). One has a \( B \)-stable filtration of \( U = U_n \supset U_{n-1} \supset ... \supset U_0 = 1 \) such that each \( U_n/U_{n-1} \) is a product of \( \mathbb{G}_a \), and \( B \) acts on
each $U_i/U_{i-1}$ through $T$ via some positive roots, so that each positive root appears in some $U_i/U_{i-1}$ exactly once.

This implies the following: Let $\delta$ be the sum of all positive roots; in particular $\delta : T \to \mathbb{G}_m$ is a character, which we pull back to $B$ also. We have

$$\Delta_B(b) = |\delta(b)|.$$  

In any case, we now define

$$\iota^G_B(\chi) := \text{Ind}^G_B(\chi \otimes \Delta_B^{1/2})$$  

Let $W := N_{G(F)}(T)/T(F)$ be the relative Weyl group. Apparently $W$ acts on the set of characters of $T(F)$. The big theorem is

**Theorem 3.2.** (Bernstein-Zelevinsky) The induction $\iota^G_B(\chi)$ always have finite length. For two characters $\chi_1, \chi_2 : T(F) \to \mathbb{C}^\times$, the composition series of $\iota^G_B(\chi_1)$ and that of $\iota^G_B(\chi_2)$ have a common factor if and only if $\chi_1$ and $\chi_2$ can be conjugate to each other by $W$. If this is the case, then they have the same Jordan-Hölder constituents with the same multiplicities, and $\text{Hom}(\iota^G_B(\chi_1), \iota^G_B(\chi_2)) \neq 0$.

We briefly describe the idea of Bernstein and Zelevinsky. Recall that Frobenius reciprocity says

$$\text{Hom}_{G(F)}(\pi, \text{Ind}^G_B \chi) = \text{Hom}_{B(F)}(\pi, \chi).$$

In particular

$$\text{Hom}_{G(F)}(\text{Ind}^G_B \chi_1, \text{Ind}^G_B \chi_2) = \text{Hom}_{B(F)}(\text{Res}^G_B \text{Ind}^G_B \chi_1, \chi_2).$$  \hspace{1cm} (1)

By definition $\chi_2$ is inflated from a character from $T(F)$. On the RHS of (1), we can pass it to $\text{Hom}_{T(F)}$ by taking the $U(F)$-coinvariant of $\text{Res}^G_B \text{Ind}^G_B$; for $\lambda$ any representation of $B(F)$, let $J_U(\lambda)$ be the representation of $T(F)$ whose underlying vector space is the space of $U(F)$-coinvariants. Then we have

$$\text{Hom}_{B(F)}(\text{Res}^G_B \text{Ind}^G_B \chi_1, \chi_2) = \text{Hom}_{T(F)}(J_U(\text{Res}^G_B \text{Ind}^G_B \chi_1), \chi_2).$$

In fact, let $r^G_B \pi := J_U(\text{Res}^G_B(\pi) \otimes \Delta^{-1/2}_B)$. Then we have $r^G_B$ is left adjoint to $\iota^G_B$. In particular

$$\text{Hom}_{G(F)}(\iota^G_B \chi_1, \iota^G_B \chi_2) = \text{Hom}_{T(F)}(r^G_B \iota^G_B \chi_1, \chi_2).$$

When $G = \text{GL}_2$, one may do a bit of brute force and check that

$$\iota^G_B \iota^G_B \chi = \chi_1 \oplus \chi^w$$

where $w$ is the non-trivial element in $N_{G(F)}(T)/T(F)$, unless $\chi = \chi^w$. In the case $\chi = \chi^w$ things become trickier to check, and in fact $r^G_B \iota^G_B \chi$ will turns out to be a non-trivial extension of $\chi$ by itself. In the case where $G$ is a general quasi-split group, we have

$$r^G_B \iota^G_B \chi$$ has a composition series given by some permutation of $\{\chi^w \mid w \in W\}$.  

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