

Power Generation Management Under Time-Varying Power and Demand Conditions

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Abstract—A multi-period optimal power dispatching problem is considered for a network of energy utilities connected via multiple transmission lines, where the goal is to find the lowest operational-cost dispatching of diverse generation sources to satisfy demand over a time horizon comprised of multiple periods, and consisting of varying power and demand conditions. Our model captures various interactions among the time-varying periods including which generators should be allocated, when they should be brought into use, and the operational costs associated with each. An efficient algorithm is derived that exploits the structure inherent in this multi-period economic dispatch problem. The control options of our optimization model consist of the dispatching order and dispatching amount of available power generators. Our solutions are shown to be globally optimal under conditions that often arise in practice. Numerical experiments based on these solutions and analysis are presented to illustrate our findings.

I. INTRODUCTION

With the advent of the Smart Grid, the infrastructure for energy supply generation and transmission is experiencing a fundamental transition from the current centralized system to one that is decentralized. The ability to access real-time information on supply availability offers unique opportunities to improve the overall efficiency of the energy grid, in terms of both long-term supply-demand management and near-term dispatching of diverse generation facilities to meet current demand. The responsiveness and flexibility envisioned for the Smart Grid provide additional advantages in facing the significant new challenges of integrating distributed and intermittent power generation capability, such as small-scale generators and renewable energy sources (e.g., wind and solar), at a scale that current grid technology simply cannot achieve; refer to, e.g., [1], [2]. This is becoming more critical as renewable energy technologies are playing an increasingly important role in the portfolio mix of electricity generation.

The main objective of this paper is to provide grid control mechanisms that respond to unforeseen supply shortfalls at fine timescales, which include time-varying power and demand conditions. We consider a multi-period optimal dispatching problem where energy generation and transmission decisions are made in order to balance consumption with supply across a network of multiple buses interconnected

via transmission lines. Our optimization model considers the generation capability of the grid to consist of large thermal and hydro generators that have been scheduled by a longer time horizon (e.g., day-ahead) unit-commitment decision process, augmented by additional smaller capacity “peaker” thermal generators connected to a subset of buses. The goal is to find the lowest operational-cost dispatching of the aggregate generation capacity to satisfy demand over a horizon on the order of a few hours within a given day, where the operational cost structure includes the costs of both dispatching capacity and making adjustments to such capacity allocations over time. The time horizon is broken into multiple smaller intervals and generation decisions are made within each interval subject to power balance constraints at each bus and transmission constraints between all pairs of connected buses. Such dispatching of peaker generators at finer time scales is also likely due to wind and other sources of unforeseen supply shortfalls.

Our multi-period optimal dispatching problem represents a generalization of the familiar and well-studied Optimal Power Flow (OPF) problem. Previous work on the *single-period* OPF problem has focused on different optimization approaches (e.g., interior point methods, Lagrange relaxation, nonlinear programming and quadratic programming) to solve this strictly nonlinear and NP-hard problem in general, often leading to locally optimal solutions; see, e.g., [3], [4], [5], [6], [7], [8], [9], [10], [11] and the references cited therein. The recent work of Lavaei and Low [10] is the closest related to our present study from among this vast literature of previous work. In contrast, we establish mathematical results for the general multi-period optimization problem and illustrate the benefits of our approach through a large collection of numerical experiments. To the best of our knowledge, the present study is the first to consider a general multi-period OPF problem and provide conditions under which the multi-period OPF problem is convex with an efficiently computable global optimum.

The multi-period formulation of the OPF problem arises naturally in the Smart Grid as the result of several important factors. On the one hand, power output from thermal generators can change only within a specified range over short time intervals, and there are costs associated with such adjustments. In addition, the demand of users varies over time, which can in turn create transmission shortfalls. It also is increasingly likely that unforeseen transmission shortfalls will result from large-scale introduction of renewable energy sources to the

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grid, with generation from wind being a prime example. Since renewables have negligible operational costs (at fine time-scales), they should be the first power generation dispatched even though the intermittent nature of output (e.g., weather conditions affecting wind turbines) is a potential obstacle to dispatching wind power in the classical sense. On the other hand, the responsiveness and flexibility envisioned for the Smart Grid makes it possible to adjust the dispatching amounts of thermal generators at finer time scales, in order to address these time-varying aspects of both user demand and renewable energy sources. These factors motivate the need to consider multi-period generalizations of the OPF problem.

The complete set of control options modeled in our multi-period OPF problem consists of the dispatching amount of peaker thermal generators and the adjustments of these dispatching amounts from one period to the next. After formulating the general multi-period OPF problem, we derive a Lagrangian dual of the original optimization problem and we show that this dual is a convex semidefinite program whose solution can be efficiently computed. The global optimality of these identified optimal solutions are established under certain conditions that are often found to hold in practice. A large number of numerical experiments have been performed, using both semidefinite program solvers and recent nonlinear optimization methods. A representative set of these results are presented, which clearly demonstrate the importance of our multi-period OPF problem. By adjusting the dispatching amounts of thermal generators at finer time scales, our numerical experiments illustrate and quantify the potentially significant benefits of such capacity allocation adjustments to address time-varying aspects of both user demand and renewable energy sources.

The remainder of the paper is organized as follows. Section II introduces both our model notation and problem formulation. Our mathematical analysis of the problem is summarized in Section III. A representative collection of numerical experiments are presented in Section IV. Concluding remarks and related ongoing work are briefly described in Section V.

II. MULTI-PERIOD DISPATCHING PROBLEM

Consider the OPF problem over a time-horizon comprised of a set of discrete time periods \mathcal{T} . The model generation and transmission variables are defined over the sets \mathcal{B} of buses and \mathcal{G} of generators. To elucidate the exposition, we shall use ℓ to index the set \mathcal{B} , j to index \mathcal{G} , and t to index \mathcal{T} . Let $\tilde{P}_{j,t}^d, \tilde{Q}_{j,t}^d$ denote the active and reactive loads of node j at time t . Renewable energy sources, connected to the grid in an always-on state, impact the demands in each time period through the model variables $\zeta_{j,t}^p$ and $\zeta_{j,t}^q$. Hence, the time-varying (effective) demands that need to be supplied by thermal power generators are given by

$$\begin{aligned} P_{j,t}^d &= \tilde{P}_{j,t}^d - \zeta_{j,t}^p \\ Q_{j,t}^d &= \tilde{Q}_{j,t}^d - \zeta_{j,t}^q. \end{aligned}$$

Let $P_{j,t}^g$ and $Q_{j,t}^g$ denote the active and reactive parts of the power extracted from generator $j \in \mathcal{G}$ at time t , respectively.

The production cost of energy associated with generator j at time t is then given by

$$\mathfrak{C}_{j,t}(P_{j,t}^g) = c_{j,t}(P_{j,t}^g)^2 + \tilde{c}_{j,t}P_{j,t}^g + \hat{c}_{j,t},$$

where $c_{j,t}$, $\tilde{c}_{j,t}$ and $\hat{c}_{j,t}$ are cost coefficients. Transmission is controlled at time t by the voltages and phase angles, V_t and δ_t , respectively. Limits V^{\min}, V^{\max} apply to voltages, whereas P^{\min}, P^{\max} and Q^{\min}, Q^{\max} similarly denote active and reactive power generator limits. To ease the notation, we assume that $P_\ell^{\min} = P_\ell^{\max} = Q_\ell^{\min} = Q_\ell^{\max} = 0$ if $\ell \in \mathcal{B} \setminus \mathcal{G}$, and we take $\tilde{P}_{\ell,t}^d = \tilde{Q}_{\ell,t}^d = 0$ and $\xi_{\ell,t}^p = \xi_{\ell,t}^q = 0$ for any node ℓ that has zero demand or zero renewable generation, respectively.

In addition to the production costs, there are costs associated with adjustments to the dispatching capacity allocations from one period to the next. Specifically, the allocation adjustment cost for generator j in time period t is given by

$$\tilde{\mathfrak{C}}_{j,t}(P_{j,t}^g) = \max\left(s_{j,1}|P_{j,t}^g - P_{j,t-1}^g|, s_{j,2}|P_{j,t}^g - P_{j,t-1}^g| - \Delta_j\right),$$

where $0 \leq s_{j,1} \leq s_{j,2}$ are the slopes of the cost function, and $\Delta_j > 0$ is used to determine the exact cutoff point.¹

The objective of the multi-period optimal power flow problem is to minimize, over the variables $P_{j,t}^g, Q_{j,t}^g, V_t$, the total aggregation of costs:

$$\sum_{t \in \mathcal{T}} \sum_{j \in \mathcal{G}} \left(\mathfrak{C}_{j,t}(P_{j,t}^g) + \tilde{\mathfrak{C}}_{j,t}(P_{j,t}^g) \right), \quad (1)$$

subject to the constraints:

$$\begin{aligned} P_{\ell,t}^d &= \tilde{P}_{\ell,t}^d - \zeta_{\ell,t}^p \\ Q_{\ell,t}^d &= \tilde{Q}_{\ell,t}^d - \zeta_{\ell,t}^q \\ P_{\ell,t}^g - P_{\ell,t}^d &= F_p^\ell(V_t) \\ Q_{\ell,t}^g - Q_{\ell,t}^d &= F_q^\ell(V_t) \\ V_\ell^{\min} &\leq |V_{\ell,t}| \leq V_\ell^{\max} \\ P_\ell^{\min} &\leq P_{\ell,t}^g \leq P_\ell^{\max} \\ Q_\ell^{\min} &\leq Q_{\ell,t}^g \leq Q_\ell^{\max} \end{aligned}$$

where $j \in \mathcal{G}$, $\ell \in \mathcal{B}$, $t \in \mathcal{T}$, and $F_p^\ell(V_t)$ and $F_q^\ell(V_t)$ are the AC active and reactive power flow equations for bus ℓ at time t , respectively. Throughout the rest of the paper, we make the tacit assumption that $V \equiv 0$ is *not* a feasible solution of the problem (this condition is overwhelmingly satisfied in practice, since any realistic energy network will have at least one V_ℓ^{\min} be nonzero).

Note that because of the relation between currents and voltages, the AC power flow equations can be written entirely in terms of the voltages V_t , without reference to the phase angles δ_t . This multi-period OPF formulation is a continuous, nonlinear, and nonconvex optimization problem that is difficult to solve in general. It is well-known that the dual of any optimization problem is convex, and can be efficiently solved by convex optimization methods, for several classes of interesting

¹These cost coefficients can certainly be made time-varying, at the expense of more notation; such details are omitted in the interest of space.

problems. For the standard single-period OPF (i.e., $|\mathcal{T}| \equiv 1$ and $\tilde{\mathcal{C}}_{j,t}(\cdot) \equiv 0$), it has been recently shown that the duals obtained through semidefinite programming [10] or trust-region based Lagrangian duality [11] identify a zero duality gap under certain properties of the operational parameters of the AC power flow equations, thus guaranteeing a globally optimal solution. In the next section, we shall establish an analogous set of results under the same operational properties for the AC power flow equations, which in turn show that our identified multi-period solution is globally optimal.

III. MATHEMATICAL ANALYSIS

In this section, we introduce a convex relaxation of the original multi-period OPF problem, which is generally a nonconvex problem. Upon deriving the dual of this convex relaxation, we then discuss a set of sufficient conditions under which the relaxation is tight. Namely, the derived primal and dual deliver the exact same objective as the original problem, allowing us to reconstruct a (globally) optimal solution for the former. Note that these conditions are consistent with those considered in [10], [11] for the standard single-period OPF problem.

For convenience, we can use the AC power flow equations to express the active and reactive powers at all the generators, $P_{\ell,t}^g$ and $Q_{\ell,t}^g$, in terms of the corresponding loads and the nodal voltages and currents, i.e., $P_{\ell,t}^g = \text{Re}\{V_{\ell,t} I_{\ell,t}^*\} + P_{\ell,t}^d$ and $Q_{\ell,t}^g = \text{Im}\{V_{\ell,t} I_{\ell,t}^*\} + Q_{\ell,t}^d$. With this substitution, we rewrite the original multi-period OPF problem as follows:

$$\begin{aligned}
& \min_{\alpha, V, I, \beta} \sum_{t \in \mathcal{T}} \sum_{j \in \mathcal{G}} (\alpha_{j,t} + \beta_{j,t}) \\
& \text{s.t. } P_{\ell,t}^{\min} - P_{\ell,t}^d \leq \text{Re}\{V_{\ell,t} I_{\ell,t}^*\} \leq P_{\ell,t}^{\max} - P_{\ell,t}^d \\
& \quad Q_{\ell,t}^{\min} - Q_{\ell,t}^d \leq \text{Im}\{V_{\ell,t} I_{\ell,t}^*\} \leq Q_{\ell,t}^{\max} - Q_{\ell,t}^d \\
& \quad (V_{\ell,t}^{\min})^2 \leq |V_{\ell,t}|^2 \leq (V_{\ell,t}^{\max})^2 \\
& \quad c_{j,t} \text{Re}\{V_{j,t} I_{j,t}^*\}^2 + \tilde{c}_{j,t} \text{Re}\{V_{j,t} I_{j,t}^*\} + \hat{c}_{j,t} \leq \alpha_{j,t} \\
\text{(MOPF)} \quad & s_{j,1} (\text{Re}\{V_{j,t} I_{j,t}^*\} - \text{Re}\{V_{j,t} I_{j,t}^*\}) + \\
& \quad P_{j,t}^d - P_{j,t-1}^d \leq \beta_{j,t} \\
& \quad -s_{j,1} (\text{Re}\{V_{j,t} I_{j,t}^*\} - \text{Re}\{V_{j,t} I_{j,t}^*\}) + \\
& \quad P_{j,t}^d - P_{j,t-1}^d \leq \beta_{j,t} \\
& \quad s_{j,2} (\text{Re}\{V_{j,t} I_{j,t}^*\} - \text{Re}\{V_{j,t} I_{j,t}^*\}) + \\
& \quad P_{j,t}^d - P_{j,t-1}^d - \Delta_j \leq \beta_{j,t} \\
& \quad -s_{j,2} (\text{Re}\{V_{j,t} I_{j,t}^*\} - \text{Re}\{V_{j,t} I_{j,t}^*\}) + \\
& \quad P_{j,t}^d - P_{j,t-1}^d - \Delta_j \leq \beta_{j,t},
\end{aligned}$$

where, just as in the previous section, $j \in \mathcal{G}$, $\ell \in \mathcal{B}$, $t \in \mathcal{T}$.

Observe that this formulation continues to be nonconvex due to the (potentially indefinite) quadratic inequalities. Following [10], we note that the terms $\text{Re}\{V_{\ell,t} I_{\ell,t}^*\}$ and $\text{Im}\{V_{\ell,t} I_{\ell,t}^*\}$

can be rewritten as

$$\begin{aligned}
\text{Re}\{V_{\ell,t} I_{\ell,t}^*\} &= \text{Re}\{V_{\ell,t}^* e_{\ell,t} e_{\ell,t}^* I\} = \text{Re}\{V_{\ell,t}^* Y_{\ell} V_{\ell,t}\} \\
&= U_{\ell,t}^T \begin{bmatrix} \text{Re}\{Y_{\ell}\} & -\text{Im}\{Y_{\ell}\} \\ \text{Im}\{Y_{\ell}\} & \text{Re}\{Y_{\ell}\} \end{bmatrix} U_{\ell,t} \\
&= \frac{1}{2} U_{\ell,t}^T \begin{bmatrix} \text{Re}\{Y_{\ell} + Y_{\ell}^T\} & -\text{Im}\{Y_{\ell}^T - Y_{\ell}\} \\ \text{Im}\{Y_{\ell} - Y_{\ell}^T\} & \text{Re}\{Y_{\ell} + Y_{\ell}^T\} \end{bmatrix} U_{\ell,t} \\
&= U_{\ell,t}^T \mathbf{Y}_{\ell} U_{\ell,t} = \text{trace}\{\mathbf{Y}_{\ell} U_{\ell,t} U_{\ell,t}^T\},
\end{aligned}$$

where $e_{\ell,t}$ denotes the ℓ -th unit vector in $\mathbb{R}^{|\mathcal{B}|}$, while the vector $U_{\ell,t} \in \mathbb{R}^{2|\mathcal{B}|}$ is given by

$$U_{\ell,t} = [\text{Re}\{V_{\ell,t}\}^T \text{Im}\{V_{\ell,t}\}^T]^T.$$

Similarly, we have

$$\begin{aligned}
\text{Im}\{V_{\ell,t} I_{\ell,t}^*\} &= U_{\ell,t}^T \bar{\mathbf{Y}}_{\ell} U_{\ell,t} = \text{trace}\{\bar{\mathbf{Y}}_{\ell} U_{\ell,t} U_{\ell,t}^T\}, \\
|V_{\ell,t}|^2 &= U_{\ell,t}^T M_{\ell} U_{\ell,t} = \text{trace}\{M_{\ell} U_{\ell,t} U_{\ell,t}^T\},
\end{aligned}$$

where M_{ℓ} is a zero square matrix of size $2|\mathcal{B}|$, having a 1 in the ℓ -th and $(\ell + |\mathcal{B}|)$ -th diagonal entries.

A natural relaxation of the above nonconvex formulation can be obtained by replacing $U_{\ell,t} U_{\ell,t}^T$ with a matrix $W_{\ell,t}$ constrained to be positive semidefinite, i.e., $W_{\ell,t} \succeq 0$. This yields the following semidefinite programming (SDP) relaxation:²

$$\min_{\alpha, W, \beta} \sum_{t \in \mathcal{T}} \sum_{j \in \mathcal{G}} (\alpha_{j,t} + \beta_{j,t}) \quad (2)$$

$$\text{s.t. } P_{\ell,t}^{\min} - P_{\ell,t}^d \leq \text{trace}\{\mathbf{Y}_{\ell} W_{\ell,t}\} \leq P_{\ell,t}^{\max} - P_{\ell,t}^d \quad (3)$$

$$Q_{\ell,t}^{\min} - Q_{\ell,t}^d \leq \text{trace}\{\bar{\mathbf{Y}}_{\ell} W_{\ell,t}\} \leq Q_{\ell,t}^{\max} - Q_{\ell,t}^d \quad (4)$$

$$(V_{\ell,t}^{\min})^2 \leq \text{trace}\{M_{\ell} W_{\ell,t}\} \leq (V_{\ell,t}^{\max})^2 \quad (5)$$

$$\text{(PSDP)} \quad \begin{bmatrix} x_{j,t} & z_{j,t} \\ z_{j,t} & -1 \end{bmatrix} \succeq 0 \quad (6)$$

$$W_{\ell,t} \succeq 0 \quad (7)$$

$$-\beta_{j,t} + s_{j,1} (\text{trace}\{\mathbf{Y}_j W_{\ell,t}\} - \text{trace}\{\mathbf{Y}_j W_{\ell,t-1}\} + P_{j,t}^d - P_{j,t-1}^d) \leq 0 \quad (8)$$

$$-\beta_{j,t} - s_{j,1} (\text{trace}\{\mathbf{Y}_j W_{\ell,t}\} - \text{trace}\{\mathbf{Y}_j W_{\ell,t-1}\} + P_{j,t}^d - P_{j,t-1}^d) \leq 0 \quad (9)$$

$$-\beta_{j,t} + s_{j,2} (\text{trace}\{\mathbf{Y}_j W_{\ell,t}\} - \text{trace}\{\mathbf{Y}_j W_{\ell,t-1}\} + P_{j,t}^d - P_{j,t-1}^d) - \Delta_j \leq 0 \quad (10)$$

$$-\beta_{j,t} - s_{j,2} (\text{trace}\{\mathbf{Y}_j W_{\ell,t}\} - \text{trace}\{\mathbf{Y}_j W_{\ell,t-1}\} + P_{j,t}^d - P_{j,t-1}^d) - \Delta_j \leq 0, \quad (11)$$

where $x_{j,t} = \tilde{c}_j \text{trace}\{\mathbf{Y}_j W_{\ell,t}\} - \alpha_{j,t} + \hat{c}_j + \tilde{c}_j P_{j,t}^d$, and $z_{j,t} = \sqrt{\tilde{c}_j} \text{trace}\{\mathbf{Y}_j W_{\ell,t}\} + \sqrt{\tilde{c}_j} P_{j,t}^d$. The above replacement typically results in a genuine relaxation, i.e., the optimal value of (PSDP) is smaller than the optimal value of (MOPF). However, if all the matrices $W_{\ell,t}$ constituting an optimal solution to the former have rank 1, then it is well known that the two formulations are equivalent [12]. In particular, an optimal solution for (MOPF) can be extracted from the eigen-decompositions of the matrices $W_{\ell,t}$.

²For ease of notation, we use the convention that any variable/quantity indexed by $t = 0$ or $t = |\mathcal{T}| + 1$ is void (i.e., it does not exist in the formulation).

Following [10], we can introduce dual variables $\lambda_{\ell,t}^{\min}, \lambda_{\ell,t}^{\max}, \bar{\lambda}_{\ell,t}^{\min}, \bar{\lambda}_{\ell,t}^{\max}, \mu_{\ell,t}^{\min}, \mu_{\ell,t}^{\max}$ for constraints (3)-(5), respectively, matrix variables $\begin{bmatrix} r_{j,t,0} & r_{j,t,1} \\ r_{j,t,1} & r_{j,t,2} \end{bmatrix}$ for constraints (6), and variables $\nu_{j,t}^{\min}, \nu_{j,t}^{\max}, \eta_{j,t}^{\min}, \eta_{j,t}^{\max}$ for constraints (8)-(11), respectively. This results in the following Lagrangian function:

$$\begin{aligned} \mathcal{L}(\lambda, \bar{\lambda}, \mu, r, \nu, \eta, W, \alpha, \beta) = & \\ & g(\lambda, \bar{\lambda}, \mu, r, \nu, \eta) + \sum_{t \in \mathcal{T}} \sum_{j \in \mathcal{G}} (1 - r_{j,t,0}) \alpha_{j,t} + \\ & \sum_{t \in \mathcal{T}} \sum_{j \in \mathcal{G}} (1 - \nu_{j,t}^{\min} - \nu_{j,t}^{\max} - \eta_{j,t}^{\min} - \eta_{j,t}^{\max}) \beta_{j,t} + \\ & \sum_{t \in \mathcal{T}} \sum_{\ell \in \mathcal{B}} \text{trace} \left\{ \left[(-\lambda_{\ell,t}^{\min} + \lambda_{\ell,t}^{\max}) \mathbf{Y}_\ell + \right. \right. \\ & \quad \left. \left. (-\bar{\lambda}_{\ell,t}^{\min} + \bar{\lambda}_{\ell,t}^{\max}) \bar{\mathbf{Y}}_\ell + (-\mu_{\ell,t}^{\min} + \mu_{\ell,t}^{\max}) M_\ell + \right. \right. \\ & \quad \mathbf{1}_{\ell \in \mathcal{G}} [r_{\ell,t,0} \tilde{c}_\ell + 2r_{\ell,t,1} \sqrt{c_\ell} + (\nu_{\ell,t}^{\min} - \nu_{\ell,t}^{\max}) s_{\ell,1} + \\ & \quad \quad \left. (\eta_{\ell,t}^{\min} - \eta_{\ell,t}^{\max}) s_{\ell,2}] \mathbf{Y}_\ell + \right. \\ & \quad \left. \left. \mathbf{1}_{\ell \in \mathcal{G}} [-(\nu_{\ell,t+1}^{\min} - \nu_{\ell,t+1}^{\max}) s_{\ell,1} - \right. \right. \\ & \quad \quad \left. \left. (\eta_{\ell,t+1}^{\min} - \eta_{\ell,t+1}^{\max}) s_{\ell,2}] \mathbf{Y}_\ell \right] W_t \right\}, \end{aligned}$$

where

$$\begin{aligned} g(\lambda, \bar{\lambda}, \mu, r, \nu, \eta) = & \sum_{t \in \mathcal{T}} \sum_{\ell \in \mathcal{B}} \left\{ \lambda_{\ell,t}^{\min} P_\ell^{\min} - \lambda_{\ell,t}^{\max} P_\ell^{\max} + \right. \\ & \left[\lambda_{\ell,t}^{\max} - \lambda_{\ell,t}^{\min} + \mathbf{1}_{\ell \in \mathcal{G}} (\tilde{c}_{\ell,t} + 2\sqrt{c_{\ell,t}} r_{\ell,t,1}) \right] P_{\ell,t}^d + \\ & \bar{\lambda}_{\ell,t}^{\min} (Q_\ell^{\min} - Q_\ell^d) - \bar{\lambda}_{\ell,t}^{\max} (Q_\ell^{\max} - Q_\ell^d) + \\ & \mu_{\ell,t}^{\min} (V_\ell^{\min})^2 - \mu_{\ell,t}^{\max} (V_\ell^{\max})^2 \left. \right\} + \\ & \sum_{t \in \mathcal{T}} \sum_{j \in \mathcal{G}} \left\{ (\nu_{j,t}^{\min} - \nu_{j,t}^{\max}) s_{j,1} (P_{j,t}^d - P_{j,t-1}^d) + \right. \\ & (\eta_{j,t}^{\min} - \eta_{j,t}^{\max}) \left[s_{j,2} (P_{j,t}^d - P_{j,t-1}^d) \right] - \\ & \left. (\eta_{j,t}^{\min} + \eta_{j,t}^{\max}) \Delta_j + (\hat{c}_{j,t} - r_{j,t,2}) \right\}. \end{aligned}$$

Minimizing the Lagrangian subject to $W \succeq 0$ and no constraints on α, β yields the following semidefinite programming problem, which is the dual of (PSDP):

$$\max_{\lambda, \bar{\lambda}, \mu, r, \nu, \eta} g(\lambda, \bar{\lambda}, \mu, r, \nu, \eta) \quad (12)$$

$$\sum_{\ell \in \mathcal{B}} (\lambda_{\ell,t} \mathbf{Y}_\ell + \bar{\lambda}_{\ell,t} \bar{\mathbf{Y}}_\ell + \mu_{\ell,t} M_\ell) \succeq 0, \forall t \in \mathcal{T}, \quad (13)$$

$$\text{(DSDP)} \quad \begin{bmatrix} 1 & r_{j,t,1} \\ r_{j,t,1} & r_{j,t,2} \end{bmatrix} \succeq 0, \forall j \in \mathcal{G}, \forall t \in \mathcal{T}, \quad (14)$$

$$\nu_{j,t}^{\min} + \nu_{j,t}^{\max} + \eta_{j,t}^{\min} + \eta_{j,t}^{\max} = 1, \forall t \in \mathcal{T}, \forall j \in \mathcal{G}, \quad (15)$$

where the dual variables $\lambda_{\ell,t}, \bar{\lambda}_{\ell,t}$ and $\mu_{\ell,t}$ are defined as

follows:

$$\lambda_{\ell,t} \stackrel{\text{def}}{=} \begin{cases} -\lambda_{\ell,t}^{\min} + \lambda_{\ell,t}^{\max} + \tilde{c}_\ell + 2r_{\ell,t,1} \sqrt{c_\ell} + \\ (\nu_{\ell,t}^{\min} - \nu_{\ell,t}^{\max}) s_{\ell,1} + (\eta_{\ell,t}^{\min} - \eta_{\ell,t}^{\max}) s_{\ell,2} + \\ (\nu_{\ell,t+1}^{\max} - \nu_{\ell,t+1}^{\min}) s_{\ell,1} + (\eta_{\ell,t+1}^{\max} - \eta_{\ell,t+1}^{\min}) s_{\ell,2} \\ \text{if } \ell \in \mathcal{G} \\ -\lambda_{\ell,t}^{\min} + \lambda_{\ell,t}^{\max} \\ \text{otherwise} \end{cases}$$

$$\bar{\lambda}_{\ell,t} \stackrel{\text{def}}{=} -\bar{\lambda}_{\ell,t}^{\min} + \bar{\lambda}_{\ell,t}^{\max}$$

$$\mu_{\ell,t} \stackrel{\text{def}}{=} -\mu_{\ell,t}^{\min} + \mu_{\ell,t}^{\max}.$$

We note that this dual program resembles the **Dual OPF** in [10], with the main distinctions arising from the linear constraints, and the modification in the definition of $\lambda_{j,t}$.

In view of this analogy, by introducing the matrices

$$A_t^{\text{opt}} \stackrel{\text{def}}{=} \sum_{\ell \in \mathcal{B}} (\lambda_{\ell,t}^{\text{opt}} \mathbf{Y}_\ell + \bar{\lambda}_{\ell,t}^{\text{opt}} \bar{\mathbf{Y}}_\ell^{\text{opt}} + \mu_{\ell,t}^{\text{opt}} M_\ell), \quad \forall t \in \mathcal{T},$$

determined by the optimal solution of Problem (DSDP), we can immediately derive the following extension of the main Theorem 1 of [10]:

Theorem 1. *If there exists an optimal solution of (DSDP) such that every $(2|\mathcal{B}|) \times (2|\mathcal{B}|)$ matrix A_t^{opt} has a zero eigenvalue with multiplicity at most 2, then there is no duality gap between (MOPF), (PSDP) and (DSDP). Furthermore, given any vector $[U_{t,1}^T U_{t,2}^T]^T$ in the null-space of A_t^{opt} , an optimal solution to (MOPF) is given by*

$$V_t^{\text{opt}} = (\xi_{t,1} + \xi_{t,2}i) (U_{t,1} + U_{t,2}i), \quad (16)$$

where the real values $\xi_{t,1}, \xi_{t,2}$ can be computed from the KKT optimality conditions for (PSDP) and (DSDP).

Proof: The proof of the theorem resembles that of [10], for the single-period case. We only sketch the main differences here, and refer the interested reader to [10] for complete details.

It is easy to see that, in any optimal solutions of (PSDP) and (DSDP), the optimal matrices W_t^{opt} and A_t^{opt} must satisfy the complementary slackness conditions

$$\text{trace} \{ A_t^{\text{opt}} W_t^{\text{opt}} \} = 0, \quad \forall t \in \mathcal{T}.$$

In particular, since both W_t^{opt} and A_t^{opt} are positive semidefinite matrices, this implies that any eigenvector E_i of W_t^{opt} corresponding to a non-zero eigenvalue a_i must lie in the null-space of the matrix A_t^{opt} (otherwise, at least one term $a_i E_i^T A_t^{\text{opt}} E_i$ would be strictly positive, violating the above condition).

By the assumption in the theorem, the null-space of A_t^{opt} has dimension at most 2, hence the number of such orthogonal eigenvectors E_i is at most 2. We distinguish the following cases:

- If the number of such vectors were 0, then the matrix A_t^{opt} would be full-rank, implying that $W_t^{\text{opt}} \equiv 0$, a violation of the tacit assumption mentioned in Section II, that $V = 0$ cannot be a solution of (MOPF).

- If the number of such vectors were 1, then $\text{rank}(W_t^{\text{opt}}) = 1$, which would imply that an optimal voltage of the form (16) can be directly computed from the eigen-decomposition of W_t^{opt} .
- If the number of such vectors were 2, then, by letting $E_1 = [E_{11}^T E_{12}^T]^T$ denote one such eigenvector, it can be shown that $E_2 = [-E_{12}^T E_{11}^T]^T$ is also an eigenvector corresponding to the eigenvalue of zero, which is furthermore orthogonal to E_1 (the proof of the latter statement is trivial, while the former requires exploiting the structure of the matrix A_t^{opt} – the interested reader is referred to [10] for complete details). In this case, with $W_t^{\text{opt}} = a_1 E_1 E_1^T + a_2 E_2 E_2^T$, one can directly check that $\tilde{W}_t^{\text{opt}} = (a_1 + a_2) E_1 E_1^T$ also verifies all the constraints of the problem (PSDP) and delivers the exact same objective as W_t^{opt} . Hence, the rank-1 matrix \tilde{W}_t^{opt} is also an optimal solution to (PSDP), and an optimal voltage vector can be obtained as:

$$V_t^{\text{opt}} = (\xi_{t,1} + \xi_{t,2}i)(E_{11} + E_{12}i).$$

The coefficients $\xi_{t,1}, \xi_{t,2}$ can be computed directly from the KKT conditions for (PSDP) and (DSDP) (for instance, in our formulation, we derive one equation by equating the voltage-phase at a pre-specified reference bus to 0, while another equation is obtained from the complementary slackness condition written for the voltage amplitudes). ■

The importance of Theorem 1 is that it provides a sufficient condition which can be directly checked from the optimal solution of the dual problem (DSDP), a convex optimization problem (a semidefinite program) which can be efficiently solved. Therefore, one can immediately obtain a certificate that there is no gap between (MOPF) and (DSDP),³ together with an optimal solution to the original multiperiod problem (MOPF), in the form of V_t^{opt} . The condition in the body of the theorem is satisfied by many practical networks, and a very elegant discussion of the algebraic and geometrical properties underpinning the mathematical analysis is presented in [10].

IV. NUMERICAL EXPERIMENTS

To test the performance of our approach, we perform a sequence of numerical experiments, which we report in the current section. As a backbone of the tests, we use several classical power networks appearing in the literature, whose structure and characteristics are summarized in Table I (CH9 is the 9 bus example from [13, p.70], while IEEE14, IEEE30, IEEE57, IEEE118 and IEEE300 can all be found at [14]).

In order to test the multi-period formulation, we proceed with several classes of experiments, which we describe next.

³And hence also (PSDP), since the latter is the dual of the former convex problem.

TABLE I: Test systems characteristics

| Test system | Buses | Generators | Lines |
|-------------|-------|------------|-------|
| CH9 | 9 | 3 | 9 |
| IEEE14 | 14 | 5 | 20 |
| IEEE30 | 30 | 6 | 41 |
| IEEE57 | 57 | 7 | 80 |
| IEEE118 | 118 | 54 | 186 |
| IEEE300 | 300 | 69 | 411 |

A. Experiment 1

In the first experiment, we first replicate each of the systems in Table I for a total of $|\mathcal{T}| = 10$ time-periods. We then vary the demand in periods $t \in \{3, 4, \dots, 8\}$ by increasing it relative to the base demand (of periods $t \in \{1, 2, 9, 10\}$). By varying the change, we therefore obtain test-cases where the demand is higher by 10, 20, 30, 40 or 50%. These tests emulate a situation where the effective demand is highly variable in mid-periods, either due to high loads or poor availability of renewables.

For every such test-case, we then proceed to solve two different problems:

- First, we solve the programs (PSDP) and (DSDP), corresponding to the (primal and dual) convex relaxations discussed in Section III.
- Second, we solve $|\mathcal{T}|$ single-period OPF problems, which correspond to a model that completely ignores the switching cost between different load configurations. We then compute (using this de-coupled solution) the ensuing switching costs that would be incurred if this solution were implemented in a myopic fashion.

The reason why we resort to this comparison is to be able to pinpoint the benefits that could be obtained by solving a genuine multi-period formulation as opposed to several decoupled single-period problems.

In the course of the simulations, we observe the following two results:

- 1) The condition stated in Theorem 1 is always satisfied, i.e., the programs (PSDP) and (DSDP) both recover the true optimal value of the original (nonconvex) (MOPF).
- 2) The improvements from utilizing a multi-period formulation that accounts for switching costs can be substantial (even in the order of 2-3%). A typical result for the simulations is reported in Figure 1, which includes a breakdown of the total costs into switching and non-switching parts.

B. Experiment 2

In the second set of experiments, we again start by replicating a total of $|\mathcal{T}| = 10$ time-periods of the systems in Table I. We then proceed to select a (randomly generated) subset of the load nodes, \mathcal{S} , for which we increase the demand during periods $t \in \{1, \dots, 5\}$ by a particular percentage (between 10 and 20%), while for the other load nodes (i.e.,

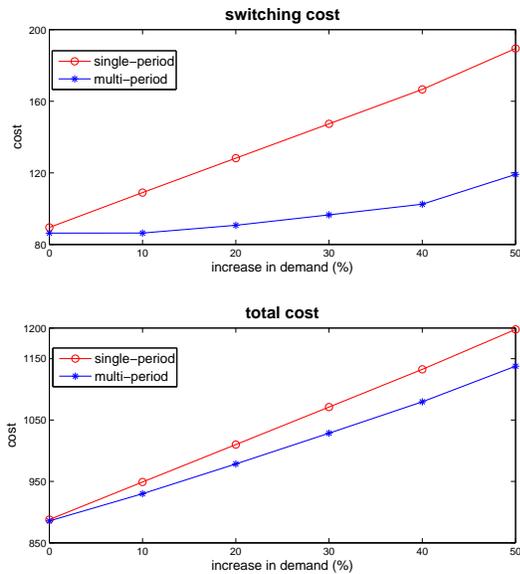


Fig. 1: Increase in demand.

the complement of the set \mathcal{S}), we increase the demands during periods $t \in \{6, \dots, 10\}$. This test-case is meant to emulate a situation where some nodes experience (proportionally) larger loads during earlier periods (e.g., factories), while other nodes experience larger demands in later periods (e.g., households).

The same observations as in Experiment 1 were noticed in this case, as well. In particular, there are relevant savings to be obtained by utilizing the multi-periods formulations (see Figure 2 for a typical example), and the effect becomes more pronounced for larger variations of the demand.

C. Experiment 3

In the third experiment, we seek to test the effect of the switching cost parameters on the performance. We again proceed by replicating $|\mathcal{T}| = 10$ time-periods of the systems in Table I, and then change the demands randomly for a subset of time-periods (and all the nodes). We then proceed to systematically increase the cost parameters $s_{j,1}$ and $s_{j,2}$, and record the ensuing total and switching costs in the multi-period model and the $|\mathcal{T}|$ -period (decoupled) model. The results (an example of which is reported in Figure 3) are consistent with our expectations, and confirm yet again that the multi-period models can potentially entail large savings, even of up to 1-2%.

We also proceed to test (and compare) the computational times involved in solving the two convex problems (PSDP) and (DSDP). In particular, Table II records the upper and lower bounds for the optimal objective function value, respectively, as well as the computational time (in seconds). As can be seen, the two problems deliver the same objective function (as

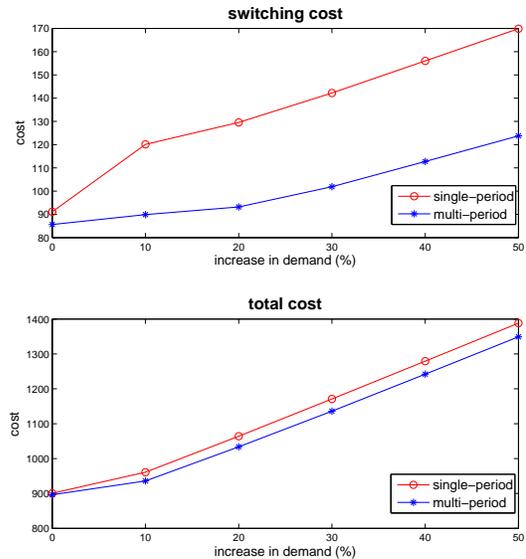


Fig. 2: Increase in demand.

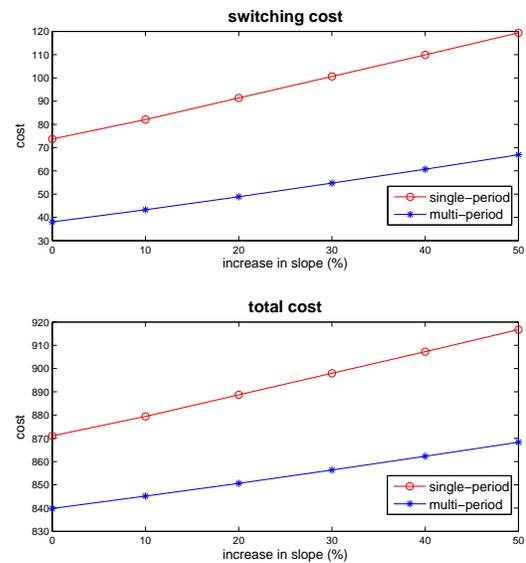


Fig. 3: Increase in slope.

expected), but the computational time is quite different, with a clear advantage for the dual program (DSDP). We remark that all the semidefinite programming problems were solved with SeDuMi 1.3, and all experiments were carried out on a PC using Matlab 7.10 with an Intel Xeon X5570 2.93 GHz under a Linux operating system.

V. CONCLUSIONS

In this paper we considered a multi-period optimal power dispatching problem within a network of energy utilities

TABLE II: The performance of algorithm. CPU time in seconds

| Test system | | | (DSDP) | (PSDP) |
|-------------|----------|----------|--------|--------|
| | UB | LB | Time | Time |
| CH9 | 279.88 | 279.88 | 0.43 | 0.79 |
| IEEE14 | 444.55 | 444.55 | 0.2 | 3.39 |
| IEEE30 | 508.92 | 508.92 | 2.07 | 157.84 |
| IEEE57 | 2432.32 | 2432.32 | 10.29 | > 600 |
| IEEE118 | 7416.79 | 7416.79 | 85.73 | >> 600 |
| IEEE300 | 42743.56 | 42743.56 | 504.62 | >> 600 |

connected via multiple transmission lines. Our objective was to find the lowest operational-cost dispatching of diverse generation sources to satisfy demand over a time horizon comprised of multiple periods consisting of varying power and demand conditions. Various interactions among the time-varying periods were captured in the optimization model, including the set of generators that should be allocated, when these generators should be brought into use, and the operational costs associated with each. We derived an efficient solution algorithm that exploits the structure inherent in this multi-period economic dispatch problem. The control options of our optimization model consisted of the dispatching order and dispatching amount of available power generators. We derived a Lagrangian dual of this original optimization problem and we established that this dual is a convex semidefinite program whose solution can be efficiently computed. The global optimality of these identified optimal solutions were established under certain conditions that are often found to hold in practice.

Numerical experiments based on our globally optimal solutions and mathematical analysis were presented to illustrate some of our findings. These results clearly demonstrate the importance of the multi-period generalizations of the Optimal Power Flow (OPF) problem. By exploiting the responsiveness and flexibility envisioned for the Smart Grid to adjust the dispatching amounts of thermal generators at finer time scales, our numerical experiments illustrate and quantify the potentially significant benefits of such capacity allocation adjustments to address time-varying aspects of both user demand and renewable energy sources.

Ongoing extensions of the present work include alternative approaches to compute the solution of the multi-period optimal power dispatching problem considered herein, as well as stochastic variants of this multi-period optimal power dispatching problem based on the AC power flow equations.

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