Online Supplement for "Loyalty Program Liabilities and Point Values"

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1 Comparative Statics For Microfounded Model

Our micro-founded model in §5.3 can be used to study how specific behavioral parameters or biases affect the firm's optimal decisions. Specifically, we examine how the customers' point value perception $\tilde{\gamma}$ impacts the optimal cash price and the point value. For simplicity, we consider a uniformly distributed customer valuation $\tilde{v}$, a deterministic loyalty threshold ξ, and a linear reward function f. Based on empirical evidence, we furthermore assume that the point balance $\tilde{\omega}$ is exponentially distributed.[1]

As discussed in the paper, empirical evidence suggests that the point value perceived by consumers may systematically differ from the monetary point value set by the firm, due to a variety of cognitive and behavioral effects. To study this bias, we parameterize the consumers' perceived point value $\tilde{\gamma}$ as follows:

$$\tilde{\gamma} = \begin{cases} 1 - \Gamma^N & \text{with probability } 1 - z, \\ 1 + \Gamma^P & \text{with probability } z, \end{cases} \tag{1}$$

where $z \in [0, 1]$. A fraction z of customers overvalue points by an additive amount $\Gamma^P \geq 0$, and the remaining $1 - z$ undervalue points by an additive amount $\Gamma^N \in [0, 1]$. We refer to the former (latter) class of customers as *positively (negatively) biased.* For tractability, we analyze the limiting case $\Gamma^P, \Gamma^N \to 0$.

Lemma 1. *As a larger fraction of consumers are positively biased (i.e., as z increases), the optimal cash price $p^\star$ decreases, and the optimal value of points $L^\star$ increases.*

When a larger fraction of the population is positively biased (i.e., z increases), more consumers would consider purchases with the firm, since the perceived value of points more readily meets or exceeds the loyalty thresholds (i.e., $\tilde{\gamma}\theta\tilde{\omega}_i \geq \tilde{\xi}$). The result is surprising, as one might expect the firm to *raise* its cash price under increased demand, and also *lower* the points' value given the higher perceived value.

2 Proofs

Proof of Theorem 1. The representation holds at $t = T+1$, since $J_{T+1}(w_{T+1}, p_{T+1}, \theta_{T+1}) = \mathbb{E}[f_{T+1}(\kappa_{T+1} + L_{T+1})] \overset{\text{def}}{=} \mathbb{E}[V_{T+1}(y_{T+1})]$. Assume this holds at time $t + 1$, and consider the Bellman recursion (5) at t:

$$\begin{aligned} J_t(w_t, p_t, \theta_t) &= \mathbb{E}_{\tilde{\varepsilon}_t}\Big[\max_{p_{t+1},\theta_{t+1}} \Big\{ f_t\big(\kappa_t + L_t - L_{t+1}(w_{t+1}, p_{t+1}, \theta_{t+1})\big) + \alpha\, J_{t+1}(w_{t+1}, p_{t+1}, \theta_{t+1}) \Big\}\Big] \\ &= \mathbb{E}_{\tilde{\varepsilon}_t}\Big[\max_{p_{t+1},\theta_{t+1}} \Big\{ f_t\big(y_t - L_{t+1}(w_{t+1}, p_{t+1}, \theta_{t+1})\big) + \alpha\, \mathbb{E}_{\tilde{\varepsilon}_{t+1}}[V_{t+1}(y_{t+1})] \Big\}\Big] \\ &= \mathbb{E}_{\tilde{\varepsilon}_t}\Big[\max_{p_{t+1},L_{t+1}} \Big\{ f_t\big(y_t - L_{t+1}\big) + \alpha\, \mathbb{E}_{\tilde{\varepsilon}_{t+1}}[V_{t+1}(y_{t+1})] \Big\}\Big]. \end{aligned}$$

[1] We collected customer point balance data from an industry partner with a large established loyalty program. Our statistical analysis revealed the exponential distribution to provide an excellent fit. The industry partner's management team had expertise with other established loyalty programs, and confirmed our view that the exponential distribution would likely remain appropriate to model consumer point balances more broadly.

The last step is justified by Assumption 1 and Assumption 2 (see §3). These ensure that y_{t+1} only depends on $(p_{t+1}, L_{t+1}, \tilde{\varepsilon}_{t+1})$, and that one can equivalently maximize over L_{t+1} instead of θ_{t+1}. The latter follows since w_{t+1} is known and fixed at the time when the decisions (p_{t+1}, θ_{t+1}) are taken, and L_{t+1} is strictly increasing in $w_{t+1}\theta_{t+1}$ (in view of Assumption 2). In particular, there exists a strictly increasing bijection $\phi_{t+1}(p_{t+1}, \cdot) : [0, \infty) \to [0, \infty)$ so that $\theta_{t+1} = \frac{\phi_{t+1}(p_{t+1}, L_{t+1})}{w_{t+1}}$, for any fixed p_{t+1}. This also shows how one can recover the optimal prices $(p^\star_{t+1}, \theta^\star_{t+1})$, proving part (b). Part (c) readily follows. □

Proof of Theorem 2. The Bellman recursion in Theorem 1 for period $t-1$ can be rewritten as:

$$V_{t-1}(y) = \max_{L_t} \phi_t(y, L_t), \tag{2a}$$

$$\phi_t(y, L) \stackrel{\text{def}}{=} f_{t-1}\big(y - L\big) + \alpha\, G_t(L) \tag{2b}$$

$$G_t(L) \stackrel{\text{def}}{=} \max_{p_t \geq 0} \mathbb{E}_{\tilde{\varepsilon}_t}\Big[V_t\big(\kappa_t(p_t, L, \tilde{\varepsilon}_t) + L\big)\Big]. \tag{2c}$$

Since f is concave increasing and $\kappa_t(p, L, \tilde{\varepsilon}_t)$ is jointly concave in (p, L) for any $\tilde{\varepsilon}_t$ (in view of Assumption 3), a simple inductive argument can be used to show that $G_t(L)$ and $V_t(y)$ are concave, $\phi_t(y, L)$ is jointly concave, and V_t and ϕ_t are increasing in y.

To prove (a) and (b), note that ϕ_t is supermodular in (y, L) on the lattice $\mathbb{R}^2_+$, since f_{t-1} is concave. Thus, the maximizer $L^\star_t(y)$ in (2a) must be increasing in y. Furthermore, by changing variables to $x \stackrel{\text{def}}{=} y - L_t$, problem (2a) can be rewritten: $V_t(y) = \max_x\big[f_{t-1}(x) + \alpha\, G_t(y - x)\big]$. The maximand is supermodular in (x, y) on the lattice $\mathbb{R}^2_+$, since G_t is concave. Thus, $x^\star(y) = y - L^\star_t(y)$ increases in y. □

Proof of Theorem 3. In view of Assumption 3, the Bellman recursions at time $t-1$ can be written as:

$$V_{t-1}(y, \sigma) = \max_{L_t} \phi_t(y, L_t, \sigma), \tag{3a}$$

$$\phi_t(y, L, \sigma) = f(y - L) + \alpha\, \mathbb{E}\Big[V_t\big(\rho(L) + \sigma\tilde{\varepsilon}_t + L, \sigma\big)\Big], \text{ where} \tag{3b}$$

$$\rho(L) \stackrel{\text{def}}{=} \max_{p \geq 0} \bar{\kappa}(p, L). \tag{3c}$$

We first prove several useful intermediate results. To ease notation, we omit explicitly showing the dependency on σ here, and we use V'_t to denote $\frac{\partial V_t(y,\sigma)}{\partial y}$. Also, we omit the argument for some functions that are evaluated repeatedly at the same argument (as are their derivatives). In particular, $L^\star_t$ is repeatedly evaluated at y in the expressions below; thus $L^\star_t$ will denote $L^\star_t(y)$. Similarly, the functions f, V_t and ρ (as well as their derivatives) are evaluated at $y - L^\star_t(y)$, $\rho(L^\star_t) + L^\star_t + \sigma\tilde{\varepsilon}_t$ and $L^\star_t$ respectively. In such instances, we will similarly omit their respective argument; for instance, $f^{(2)} = f^{(2)}(y - L^\star_t)$.

Property (O). At optimality, $1 + \rho'(L^\star_t(y)) \geq 0$, for all $t = 2, \ldots, T+1$ and y.

To prove this, consider the first-order condition (FOC) yielding $L^\star_t$ in (3a). By an application of the Envelope Theorem, we have: $f'(y - L^\star_t) = \alpha\left(1 + \rho'(L^\star_t)\right)\mathbb{E}\big[V'_t\big(\rho(L^\star_t) + \sigma\tilde{\varepsilon}_t + L^\star_t\big)\big]$. Since f is strictly increasing, and V_t is increasing, we must have that $1 + \rho'(L^\star_t) \geq 0$, completing the proof.

Property (I). $\frac{\partial V_t(y,\sigma)}{\partial y}$ is convex in y for all $t = 1, \ldots, T+1$ and $\sigma \geq 0$.

We prove by induction. Note that this holds at $T+1$ since $V_{T+1}(y) = f(y)$, and f' is convex by assumption. For $t < T$, by the Envelope Theorem and taking the second order derivative we have:

$$V_{t-1}^{(3)}(y) = f^{(3)}\left(1 - L_{t,y}^\star\right)^2 - f^{(2)} L_{t,yy}^\star, \tag{4}$$

where $L_{t,y}^\star$ denotes the partial derivative of $L_t^\star$ with respect to y. The first-order optimality condition that $L_t^\star$ satisfies can be written as $F_t(y, L) = 0$, where

$$F_t(y, L) \stackrel{\text{def}}{=} -f'(y - L) + \alpha\big(1 + \rho'(L)\big)\mathbb{E}\big[V_t'(\rho(L) + L + \sigma\tilde{\varepsilon}_t)\big]. \tag{5}$$

The maximand ϕ_t in (3a) is strictly concave, hence $F_{t,L}(y, L_t^\star) < 0$. To obtain expressions for the derivatives of $L_t^\star$ we apply the Implicit Function Theorem to the above equation, yielding

$$F_{t,y}(y, L_t^\star) + L_{t,y}^\star F_{t,L}(y, L_t^\star) = 0.$$

Applying the Implicit Function Theorem again we get

$$F_{t,yy}(y, L_t^\star) + L_{t,yy}^\star F_{t,L}(y, L_t^\star) + (L_{t,y}^\star)^2 F_{t,LL}(y, L_t^\star) + 2L_{t,y}^\star F_{t,yL}(y, L_t^\star) = 0.$$

By using this expression to substitute for $L_{t,yy}^\star$ in (4), we get:

$$V_{t-1}^{(3)}(y) = \frac{1}{F_{t,L}}\left[f^{(3)}(1 - L_{t,y}^\star)^2 F_{t,L} + f^{(2)}\Big(F_{t,yy} + (L_{t,y}^\star)^2 F_{t,LL} + 2L_{t,y}^\star F_{t,yL}\Big)\right]. \tag{6}$$

We show that this is non-negative, which proves (I). To that end, note that from (5) we have:

$$F_{t,L} = f^{(2)} + \alpha\rho^{(2)}\mathbb{E}[V_t'] + \alpha(1+\rho')^2\mathbb{E}[V_t^{(2)}] \tag{7a}$$

$$F_{t,y} = -f^{(2)} \tag{7b}$$

$$F_{t,yy} = -f^{(3)} \tag{7c}$$

$$F_{t,yL} = f^{(3)} \tag{7d}$$

$$F_{t,LL} = -f^{(3)} + \alpha\rho^{(3)}\mathbb{E}[V_t'] + 3\alpha\rho^{(2)}(1+\rho')\mathbb{E}[V_t^{(2)}] + \alpha(1+\rho')^3\,\mathbb{E}[V_t^{(3)}]. \tag{7e}$$

We now use these to rewrite (6). First, note that the parenthesis in the second term of (6) can be written:

$$\begin{aligned}
&F_{t,yy} + (L_{t,y}^\star)^2 F_{t,LL} + 2L_{t,y}^\star F_{t,yL} \\
&= -f^{(3)} + (L_{t,y}^\star)^2\Big(-f^{(3)} + \alpha\rho^{(3)}\mathbb{E}[V_t'] + 3\alpha\rho^{(2)}(1+\rho')\mathbb{E}[V_t^{(2)}] + \alpha(1+\rho')^3\,\mathbb{E}[V_t^{(3)}]\Big) + 2L_{t,y}^\star f^{(3)} \\
&= -f^{(3)}\left(1 - L_{t,y}^\star\right)^2 + (L_{t,y}^\star)^2\Big(\alpha\rho^{(3)}\mathbb{E}[V_t'] + 3\alpha\rho^{(2)}(1+\rho')\mathbb{E}[V_t^{(2)}] + \alpha(1+\rho')^3\,\mathbb{E}[V_t^{(3)}]\Big).
\end{aligned}$$

Using this expression and (7a) to replace $F_{t,L}$, we can rewrite (6) as follows:

$$\begin{aligned}
V_{t-1}^{(3)}(y) &= \frac{1}{F_{t,L}}\Big\{f^{(3)}\big(1-L^\star_{t,y}\big)^2\Big(f^{(2)}+\alpha\rho^{(2)}\mathbb{E}[V_t']+\alpha(1+\rho')^2\mathbb{E}[V_t^{(2)}]\Big)\\
&\qquad -f^{(2)}f^{(3)}\big(1-L^\star_{t,y}\big)^2+f^{(2)}(L^\star_{t,y})^2\Big(\alpha\rho^{(2)}\mathbb{E}[V_t']+3\alpha\rho^{(2)}(1+\rho')\mathbb{E}[V_t^{(2)}]+\alpha(1+\rho')\mathbb{E}[V_t^{(3)}]\Big)\Big\}\\
&= \frac{1}{F_{t,L}}\Big\{\underbrace{f^{(3)}\Big[\alpha\rho^{(2)}\mathbb{E}[V_t']+\alpha(1+\rho')^2\mathbb{E}[V_t^{(2)}]\Big]}_{:=A}\\
&\qquad +\underbrace{f^{(2)}(L^\star_{t,y})^2\Big(\alpha\rho^{(3)}\mathbb{E}[V_t']+3\alpha\rho^{(2)}(1+\rho')\mathbb{E}[V_t^{(2)}]+\alpha(1+\rho')^3\,\mathbb{E}[V_t^{(3)}]\Big)}_{:=B}\Big\}.
\end{aligned}$$

To conclude the argument, recall the following properties for the functions of interest:

$$f \text{ is concave and } f' \text{ is convex} \Rightarrow f^{(2)}\le 0,\quad f^{(3)}\ge 0$$

$$\rho \text{ is concave, } \rho' \text{ is convex, and property (O)} \Rightarrow \rho^{(2)}\le 0,\quad \rho^{(3)}\ge 0,\quad (1+\rho')\ge 0$$

$$V_t \text{ increasing and concave, and the induction hypothesis} \Rightarrow V_t'\ge 0,\quad V_t^{(2)}\le 0,\quad V_t^{(3)}\ge 0.$$

The induction and the proof for (I) follow since: $\Big\{f^{(2)}\le 0,\ \rho^{(2)}\le 0,\ V_t'\ge 0,\ V_t^{(2)}\le 0\Big\}\Rightarrow F_{t,L}\le 0$

$$\Big\{f^{(3)}\ge 0,\ \rho^{(2)}\le 0,\ V_t'\ge 0,\ (1+\rho')\ge 0,\ V_t^{(2)}\le 0\Big\}\Rightarrow A\le 0$$

$$\Big\{f^{(2)}\le 0, \big\{\rho^{(3)}\ge 0, V_t'\ge 0\big\}, \big\{\rho^{(2)}\le 0, (1+\rho')\ge 0, V_t^{(2)}\le 0\big\}, \big\{(1+\rho')\ge 0, V_t^{(3)}\ge 0\big\}\Big\}\Rightarrow B\le 0.$$

Property (II). If X is a continuous random variable with zero mean and $f:\mathbb{R}\to\mathbb{R}$ is a differentiable, strictly concave (convex) and increasing (decreasing) function, then $\mathbb{E}[Xf'(X)]<0$ (>0).

We prove this for f concave, increasing (the argument for convex, decreasing is similar). Let h denote the probability density function of X. We have:

$$\begin{aligned}
\mathbb{E}[Xf'(X)] &= \int_{-\infty}^{0} xf'(x)h(x)dx+\int_{0}^{\infty} xf'(x)h(x)dx\\
&< \int_{-\infty}^{0} xf'(0)h(x)dx+\int_{0}^{\infty} xf'(x)h(x)dx && [f \text{ is strictly concave and increasing}]\\
&= -\int_{0}^{\infty} xf'(0)h(x)dx+\int_{0}^{\infty} xf'(x)h(x)dx && [X \text{ is zero mean}]\\
&= \int_{0}^{\infty} x(f'(x)-f'(0))h(x)dx<0. && [f \text{ is strictly concave}]
\end{aligned}$$

(b) Consider the simplified recursion as in (3a-3c). We have for all $t=1,\ldots,T$, y and $\sigma\ge 0$:

$$V_t(y,\sigma)=\max_{L_{t+1}}\Big[f\big(y-L_{t+1}\big)+\alpha\mathbb{E}\big[V_{t+1}(\rho(L_{t+1})+L_{t+1}+\sigma\tilde{\varepsilon}_{t+1},\sigma)\big]\Big]. \tag{8}$$

We have: $$\frac{\partial V_T(y,\sigma)}{\partial\sigma} = \alpha\mathbb{E}\big[\tilde{\varepsilon}_{T+1}\, f'(\rho(L^\star_{T+1}(y,\sigma)) + L^\star_{T+1}(y,\sigma) + \sigma\tilde{\varepsilon}_{T+1})\big] \qquad \text{[by the Envelope Theorem]}$$
$$< 0. \qquad [f \text{ is concave increasing + (II)}]$$

To complete the proof via induction, assume that $\frac{\partial V_{t+1}(y,\sigma)}{\partial\sigma} < 0$ for all y and $\sigma \geq 0$. Then

$$\begin{aligned}
\frac{\partial V_t(y,\sigma)}{\partial\sigma} &= \alpha\mathbb{E}\Big[\tilde{\varepsilon}_{t+1}\,\frac{\partial}{\partial y}V_{t+1}(\rho(L^\star_{t+1}(y,\sigma)) + L^\star_{t+1}(y,\sigma) + \sigma\tilde{\varepsilon}_{t+1},\sigma)\Big] \\
&\quad + \alpha\mathbb{E}\big[\frac{\partial}{\partial\sigma}V_{t+1}(\rho(L^\star_{t+1}(y,\sigma)) + L^\star_{t+1}(y,\sigma) + \sigma\tilde{\varepsilon}_{t+1},\sigma)\big] && \text{[by the Envelope Theorem]} \\
&< \alpha\mathbb{E}\big[\tilde{\varepsilon}_{t+1}\,\frac{\partial}{\partial y}V_{t+1}(\rho(L^\star_{t+1}(y,\sigma)) + L^\star_{t+1}(y,\sigma) + \sigma\tilde{\varepsilon}_{t+1},\sigma)\big] && \text{[induction hypothesis]} \\
&< 0. && [V_{t+1} \text{ is concave, increasing in } L]
\end{aligned}$$

We next prove another useful intermediate result.

Property (III). $\dfrac{\partial^2 V_t(y,\sigma)}{\partial y\partial\sigma} \geq 0$ for all $t = 1,\ldots,T$, y and $\sigma \geq 0$.

By using the expressions above we get

$$\begin{aligned}
\frac{\partial^2 V_T(y,\sigma)}{\partial\sigma\partial y} &= \frac{\partial}{\partial y}\alpha\mathbb{E}\big[\tilde{\varepsilon}_{T+1}\, f'(\rho(L^\star_{T+1}(y,\sigma)) + L^\star_{T+1}(y,\sigma) + \sigma\tilde{\varepsilon}_{T+1})\big] \\
&= \alpha\underbrace{(\rho'(L^\star_{T+1}(y,\sigma))+1)}_{\geq 0 \text{ by (O)}}\;\underbrace{\frac{\partial L^\star_{T+1}(y,\sigma)}{\partial y}}_{\geq 0 \text{ by Theorem 2(a)}}\;\underbrace{\mathbb{E}\big[\tilde{\varepsilon}_{T+1}\, f''(\rho(L^\star_{T+1}(y,\sigma)) + L^\star_{T+1}(y,\sigma) + \sigma\tilde{\varepsilon}_{T+1})\big]}_{\geq 0 \text{ by (II) for } f' \text{ convex, } f \text{ concave}} \;\geq\; 0.
\end{aligned}$$

To complete the proof via induction, assume that $\frac{\partial^2 V_{t+1}(y,\sigma)}{\partial\sigma\partial y} \geq 0$ for all y and $\sigma \geq 0$. Then

$$\begin{aligned}
\frac{\partial^2 V_t(y,\sigma)}{\partial\sigma\partial y} &= \alpha\underbrace{(\rho'(L^\star_{t+1}(y,\sigma))+1)}_{\geq 0 \text{ by (O)}}\;\underbrace{\frac{\partial L^\star_{t+1}(y,\sigma)}{\partial y}}_{\geq 0 \text{ by Theorem 2(a)}}\;\underbrace{\mathbb{E}\big[\tilde{\varepsilon}_{t+1}\,\frac{\partial^2}{\partial y^2}V_{t+1}(\rho(L^\star_{t+1}(y,\sigma)) + L^\star_{t+1}(y,\sigma) + \sigma\tilde{\varepsilon}_{t+1},\sigma)\big]}_{\geq 0 \text{ by (I), (II)}} \\
&\quad + \alpha\underbrace{(\rho'(L^\star_{t+1}(y,\sigma))+1)}_{\geq 0 \text{ by (O)}}\;\underbrace{\frac{\partial L^\star_{t+1}(y,\sigma)}{\partial y}}_{\geq 0 \text{ by Theorem 2(a)}}\;\underbrace{\mathbb{E}\big[\frac{\partial^2}{\partial\sigma\partial y}V_{t+1}(\rho(L^\star_{t+1}(y,\sigma)) + L^\star_{t+1}(y,\sigma) + \sigma\tilde{\varepsilon}_{t+1},\sigma)\big]}_{\geq 0 \text{ by the induction hypothesis}} \;\geq\; 0.
\end{aligned}$$

(a) Similarly with (I), the necessary and sufficient first-order optimality condition that $L^\star_t(y,\sigma)$ satisfies can be re-written in this case as $F_t(L,\sigma) = 0$, where

$$F_t(L,\sigma) \stackrel{\text{def}}{=} -f'(y-L) + \alpha(1+\rho'(L))\mathbb{E}\big[\frac{\partial}{\partial y}V_t(\rho(L) + L + \sigma\tilde{\varepsilon}_t,\sigma)\big]).$$

Since the maximand of (8) is strictly concave in L_{t+1}, the partial derivative of F_t with respect to L is negative and we can apply the Implicit Function Theorem to obtain $\frac{\partial L^\star}{\partial\sigma} = -\big(\frac{\partial F_t}{\partial\sigma}\big|_{L^\star}\big)/\big(\frac{\partial F_t}{\partial L}\big|_{L^\star}\big)$. Thus, it suffices to show that the partial derivative of F_t with respect to σ, evaluated at $L^\star$ is non-negative:

$$\frac{\partial F_t}{\partial \sigma}\bigg|_{L^\star} = \alpha \underbrace{\left(\rho'(L_t^\star(y,\sigma)) + 1\right)}_{\geq 0 \text{ by (O)}} \underbrace{\mathbb{E}\Big[\tilde{\varepsilon}_t \frac{\partial^2}{\partial y^2} V_t(\rho(L_t^\star(y,\sigma)) + L_t^\star(y,\sigma) + \sigma\tilde{\varepsilon}_t, \sigma)\Big]}_{\geq 0 \text{ by (I), (II)}}$$
$$+ \alpha \underbrace{\left(\rho'(L_t^\star(y,\sigma)) + 1\right)}_{\geq 0 \text{ by (O)}} \underbrace{\mathbb{E}\Big[\frac{\partial^2}{\partial \sigma \partial y} V_t(\rho(L_t^\star(y,\sigma)) + L_t^\star(y,\sigma) + \sigma\tilde{\varepsilon}_t, \sigma)\Big]}_{\geq 0 \text{ by (III)}} \geq 0. \qquad \square$$

Proof of Theorem 4. We find it helpful to also prove that the value function has the following form:

$$V_t(y) = y - L_{t+1}^\star + \sum_{\tau=t+1}^{T+1} \alpha^{\tau-t} \Big\{ \mathbb{E}_{\tilde{\varepsilon}_\tau}\left[\kappa_\tau(p_\tau^\star, L_\tau^\star, \tilde{\varepsilon}_\tau)\right] + (L_\tau^\star - L_{\tau+1}^\star) \Big\}, \tag{9}$$

We prove all results by induction on t. Note that (9) holds trivially for $t = T+1$. Assume it holds at time t, so that $V_t(y) = y + K_t$, where K_t is a constant. Consider the Bellman recursion at $t-1$:

$$\begin{aligned} V_{t-1}(y) &= \max_{p_t, L_t} \Big\{ y - L_t + \alpha\, \mathbb{E}\big[V_t(y_t)\big] \Big\} \\ &= \max_{p_t, L_t} \Big\{ y - L_t + \alpha \left(\mathbb{E}_{\tilde{\varepsilon}_t}[\kappa_t(p_t, L_t, \tilde{\varepsilon}_t)] + L_t + K_t \right) \Big\} \\ &= y + \alpha \cdot K_t + \max_{p,L} \big\{ \alpha\, \mathbb{E}_{\tilde{\varepsilon}_t}[\kappa_t(p, L, \tilde{\varepsilon}_t)] - (1-\alpha) L \big\}. \end{aligned}$$

As such, letting $(p_t^\star, L_t^\star) \in \arg\max\big\{\alpha\, \mathbb{E}_{\tilde{\varepsilon}_t}[\kappa_t(p, L, \tilde{\varepsilon}_t)] - (1-\alpha)L\big\}$, one can see that the cash price and point value follow from (7a) and (7b), respectively, and the induction proof is completed as follows:

$$\begin{aligned} V_{t-1}(y) &= y + \alpha \cdot K_t + \alpha\, \mathbb{E}[\kappa_t(p_t^\star, L_t^\star, \tilde{\varepsilon}_t)] - (1-\alpha) L_t^\star \\ &= y + \alpha \cdot \left[L_{t+1}^\star - \sum_{k=t+1}^{T+1} \alpha^{k-t} \big[\mathbb{E}[\kappa_k(p_k^\star, L_k^\star, \tilde{\varepsilon}_k)] + (L_k^\star - L_{k+1}^\star) \big] \right] + \alpha\, \mathbb{E}\big[\kappa_t(p_t^\star, L_t^\star, \tilde{\varepsilon}_t)\big] - (1-\alpha) L_t^\star \\ &= y - L_t^\star + \sum_{k=t}^{T+1} \alpha^{k-t+1} \Big[\mathbb{E}_{\tilde{\varepsilon}_t}\left[\kappa_k(p_k^\star, L_k^\star, \tilde{\varepsilon}_k)\right] + (L_k^\star - L_{k+1}^\star) \Big]. \qquad \square \end{aligned}$$

Proof of Theorem 5. Note that the reward function is piecewise-linear, thus differentiable almost everywhere, except for a finite number of points. All quantities of interest (e.g., V_t and $L_t^\star$) will thus inherit this property. As a result, exchanging the order of integration and differentiation of V_t is possible under suitable continuity assumptions on the distribution of $\tilde{\varepsilon}_t$. To ease exposition, we use the standard derivative to denote either the derivative of a function, or any of its subgradients if it is not differentiable at the point it is evaluated.

(a) Consider the simplified recursion as in (3a-3c), where the dependency on σ is now replaced with a dependency on γ. The necessary and sufficient first-order optimality condition that $L_t^\star(y, \gamma)$ satisfies can be written as $F_t(L,\gamma) = f'(y-L)$, where

$$F_t(L,\gamma) \stackrel{\text{def}}{=} \alpha(1+\rho'(L))\mathbb{E}\Big[\frac{\partial V_t(y,\gamma)}{\partial y}\Big|_{(\rho(L)+L+\sigma\tilde{\varepsilon}_t,\gamma)}\Big].$$

Note that the left-hand side term $F_t(L,\gamma)$ is decreasing in L, since V_t is concave in L, whereas the right-hand side term $f'(y-L)$ is increasing in L. In particular, the right-hand side term takes the value of 1 for $L < y - \hat{\Pi}$, any value between 1 and γ for $L = y - \hat{\Pi}$, and γ for $L > y - \hat{\Pi}$. Consequently, there exist values $\underline{y}_t$ and $\overline{y}_t$ such that $L_t^\star(y,\gamma)$ satisfies

(i) $F_t(L_t^\star(y,\gamma),\gamma) = \gamma$, for $y < \underline{y}_t$,

(ii) $L_t^\star(y,\gamma) = y - \hat{\Pi}$, for $\underline{y}_t \le y \le \overline{y}_t$, and

(iii) $F_t(L_t^\star(y,\gamma),\gamma) = 1$, for $y > \overline{y}_t$.

Suppose that $y \le \overline{y}_t$. Then, either $L_t^\star(y,\gamma)$ is constant (case (ii)), or it satisfies the condition in (i). Using the notation as in the proof of Theorem 3, the Implicit Function Theorem yields

$$F_{t,\gamma}(L_t^\star(y,\gamma),\gamma) - 1 + \frac{\partial L_t^\star}{\partial \gamma} F_{t,L}(L_t^\star(y,\gamma),\gamma) = 0, \tag{10}$$

where $F_{t,L}(L_t^\star(y,\gamma),\gamma) < 0$ by the concavity of V_t. Also,

$$\begin{aligned} F_{t,\gamma}\big(L_t^\star(y,\gamma),\gamma\big) &= \alpha\Big(1+\rho'\big(L_t^\star(y,\gamma)\big)\Big)\mathbb{E}\Big[\frac{\partial}{\partial\gamma}\frac{\partial}{\partial y}V_t\Big(\rho\big(L_t^\star(y,\gamma)\big) + L_t^\star(y,\gamma) + \sigma\tilde{\varepsilon},\gamma\Big)\Big] \\ &= \frac{\mathbb{E}\big[\gamma\frac{\partial}{\partial\gamma}\frac{\partial}{\partial y}V_t(\rho(L_t^\star(y,\gamma)) + L_t^\star(y,\gamma) + \sigma\tilde{\varepsilon},\gamma)\big]}{\mathbb{E}\big[\frac{\partial}{\partial y}V_t(\rho(L_t^\star(y,\gamma)) + L_t^\star(y,\gamma) + \sigma\tilde{\varepsilon},\gamma)\big]} \le 1. \end{aligned} \tag{11}$$

The second equality above follows by substituting for $\alpha(1+\rho'(L_t^\star(y,\gamma)))$ using the condition in (i). For the inequality, note that at points at which the functions are differentiable (and these are the relevant ones for the expectations above) we have

$$\frac{\partial}{\partial y}V_t(y,\gamma) = f'(y - L_{t+1}^\star(y)). \tag{12}$$

The right-hand side above takes values 1 or γ. As such,

$$\gamma\frac{\partial}{\partial\gamma}\frac{\partial}{\partial y}V_t(y,\gamma) = \gamma\frac{\partial}{\partial\gamma}f'(y - L_{t+1}^\star(y)) \le f'(y - L_{t+1}^\star(y)) = \frac{\partial}{\partial y}V_t(y,\gamma).$$

Using the bounds $F_{t,L}(L_t^\star(y,\gamma),\gamma) < 0$ and $F_{t,\gamma}(L_t^\star(y,\gamma),\gamma) \le 1$, equation (10) yields that $\frac{\partial L_t^\star}{\partial\gamma} \le 0$ for case (i). For case (ii), the inequality still holds as $L_t^\star$ is constant. Thus, $\frac{\partial L_t^\star}{\partial\gamma} \le 0$ for $y \le \overline{y}_t$.

To complete the proof, note that for $y > \overline{y}_t$ and case (iii), the equivalent of equation (10) is

$$F_{t,\gamma}(L_t^\star(y,\gamma),\gamma) + \frac{\partial L_t^\star}{\partial\gamma}F_{t,L}(L_t^\star(y,\gamma),\gamma) = 0,$$

so it suffices to show $F_{t,\gamma}(L_t^\star(y,\gamma),\gamma) \ge 0$. Recall from (11) that the sign of $F_{t,\gamma}$ is given by $\alpha(1+$

$\rho')\mathbb{E}\Big[\frac{\partial}{\partial\gamma}\frac{\partial}{\partial y}V_t\Big]$. By (12), $\frac{\partial V_t}{\partial y} = f'\big(y - L^\star_{t+1}(y)\big)$; and since f' is trivially increasing in γ and $1+\rho' \geq 0$ by property (O), the result follows.

(b) By Lemma 1(b), $\frac{\partial}{\partial y}V_t(y,\gamma)$ is decreasing in t. Thus, $F_t(L,\gamma)$ also decreases in t, and the result follows.

(c) As we remarked above, $F_t(L,\gamma)$ is increasing in γ and the result follows. □

Proof of Lemma 1. We prove both parts together, by backwards induction. Consider the Bellman recursions in (3a)-(3c), where we omit the dependency on σ. The Envelope Theorem for (3a) yields:

$$V'_t(y) = f'(y - L^\star_{t+1}),\ \forall\, t \in \{1,\ldots,T\}.$$

Since $V_{T+1}(y) = f(y)$, we readily have that $V'_T(y) = f'(y - L^\star_{T+1}) \geq V'_{T+1}(y) = f'(y)$, since f is strictly concave (so that f' is decreasing). Furthermore, we also have $L^\star_{T+1}(y) \geq L^\star_{T+2}(y) \equiv 0,\ \forall\, y$. Thus, the properties hold at time $T+1$. Assume they also hold at t, so that $V'_t(y) \geq V'_{t+1}(y)$. Then, consider the FOC for problem (3a) written at time $t-1$, yielding $L^\star_t$, and note that:

$$\begin{aligned}\frac{\partial\phi_t}{\partial L}\Big|_{L^\star_{t+1}} &= \Big\{f'(y-L) + \alpha\,\underbrace{\big(1+\rho'(L)\big)}_{\geq 0,\text{ by (c)}}\,\mathbb{E}\Big[V'_t\big(\rho(L)+\sigma\tilde{\varepsilon}_t+L\big)\Big]\Big\}\Big|_{L^\star_{t+1}}\\ &\geq \Big\{f'(y-L) + \alpha\,\big(1+\rho'(L)\big)\,\mathbb{E}\Big[V'_{t+1}\big(\rho(L)+\sigma\tilde{\varepsilon}_t+L\big)\Big]\Big\}\Big|_{L^\star_{t+1}} = \frac{\partial\phi_{t+1}}{\partial L}\Big|_{L^\star_{t+1}} = 0.\end{aligned}$$

As such, it must be that $L^\star_t \geq L^\star_{t+1}$. In turn, this implies that $V'_{t-1}(y) = f'(y-L^\star_t) \geq f'(y-L^\star_{t+1}) = V'_t(y)$, completing the proof of the inductive step. □

Proof of Lemma 2. The proof proceeds by induction, in an analogous fashion to Lemma 1. Details are omitted for space considerations, but are available from the authors upon request. □

Proof of Lemma 3. Consider recursions (3a)-(3b). If $\bar{\kappa}_t(p_t, L_t)$ is supermodular (submodular) in (p_t, L_t), the set of maximizers in problem (3c) is increasing (decreasing) in L_t. Since $L^\star_t \geq 0$, the result follows. □

Proof of Lemma 4. We argue for the case of linear reward. The proof for a concave reward function is similar. Note that $\frac{\partial^2\mathbb{E}[\kappa_t(p_t,L_t,c)]}{\partial c\,\partial p_t} = -\frac{\partial\mathbb{E}[r_t]}{\partial p_t} \leq 0$, $\frac{\partial^2\mathbb{E}[\kappa_t(p_t,L_t,c)]}{\partial c\,\partial L_t} = -\frac{\partial\mathbb{E}[r_t]}{\partial L_t} \leq 0$, and $\frac{\partial^2\mathbb{E}[\kappa_t(p_t,L_t,c)]}{\partial p_t\,\partial L_t} \geq 0$ by our assumption, so that $\mathbb{E}[\kappa_t]$ is supermodular in $(p_t, L_t, -c)$, and the optimal price $p^\star_t$ and LP value $L^\star_t$ will be decreasing in c. Lastly, recall that $L^\star_t = w_t\theta_t\, g_t(p_t, w_t\,\theta_t)$ is increasing in θ_t for any fixed w_t, p_t. Thus, consider increasing c: by the argument above, this would decrease $L^\star_t$ and $p^\star_t$, leading to (i) a decrease in the left-hand-side of the equation, and (if g_t decreases in p_t, by our assumption) (ii) an increase in the right-hand-side of the equation. Therefore, with w_t fixed, it must be that θ_t decreases with c. □

The proofs of all remaining results are available upon request.