Online Appendix for "Monitoring with Limited Information" by Dan A. Iancu, Nikolaos Trichakis, and Do Young Yoon

Appendix A: Proofs

Proof of Proposition 1.

It suffices to show that for any $\boldsymbol{x}^{\{k\}}$ and monitorings $\boldsymbol{t}^{\{r\}}$, $\boldsymbol{t}^{\{r'\}}$ $(k<r<r'\leq n)$,

$$\Pi_{:,i:j}\mathcal{U}(\boldsymbol{t}^{\{r\}},\boldsymbol{x}^{\{k\}})=\Pi_{:,i:j}\mathcal{U}(\boldsymbol{t}^{\{r'\}},\boldsymbol{x}^{\{k\}}),\,\forall\, i\leq j<r-k+1.$$

Note that because introducing more monitoring times imposes more constraints and thereby shrinks the uncertainty sets and their projections, we get that $\Pi_{:,i:j}\mathcal{U}(\boldsymbol{t}^{\{r\}},\boldsymbol{x}^{\{k\}})\supseteq\Pi_{:,i:j}\mathcal{U}(\boldsymbol{t}^{\{r'\}},\boldsymbol{x}^{\{k\}})$, for any $r<r'$.

To prove the opposite direction, we need to show that if $\boldsymbol{B}\in\Pi_{:,i:j}\mathcal{U}(\boldsymbol{t}^{\{r\}},\boldsymbol{x}^{\{k\}})$, then $\boldsymbol{B}\in\Pi_{:,i:j}\mathcal{U}(\boldsymbol{t}^{\{r'\}},\boldsymbol{x}^{\{k\}})$, i.e., $\exists\,\boldsymbol{A}\in\mathbb{R}^{d\times(i-1)}$, $\boldsymbol{C}\in\mathbb{R}^{d\times(r'-k+1-j)}$ such that $[\boldsymbol{A},\boldsymbol{B},\boldsymbol{C}]\in\mathcal{U}(\boldsymbol{t}^{\{r'\}},\boldsymbol{x}^{\{k\}})$. But, $\boldsymbol{B}\in\Pi_{:,i:j}\mathcal{U}(\boldsymbol{t}^{\{r\}},\boldsymbol{x}^{\{k\}})$ implies that $\exists\,\overline{\boldsymbol{A}}\in\mathbb{R}^{d\times(i-1)}$, $\overline{\boldsymbol{C}}\in\mathbb{R}^{d\times(r-k+1-j)}$ such that $[\overline{\boldsymbol{A}},\boldsymbol{B},\overline{\boldsymbol{C}}]\in\mathcal{U}(\boldsymbol{t}^{\{r\}},\boldsymbol{x}^{\{k\}})$. Let us denote the columns of these matrices as $\overline{\boldsymbol{A}}=[\boldsymbol{x}_{k+1},\ldots,\boldsymbol{x}_{k+i-1}]$, $\boldsymbol{B}=[\boldsymbol{x}_{k+i},\ldots,\boldsymbol{x}_{k+j}]$ and $\overline{\boldsymbol{C}}=[\boldsymbol{x}_{k+j+1},\ldots,\boldsymbol{x}_r,\boldsymbol{x}_{n+1}]$.

Let $\boldsymbol{A}=\overline{\boldsymbol{A}}$, and construct $\boldsymbol{C}$ by starting off from $\overline{\boldsymbol{C}}$, replacing its last column with $\boldsymbol{x}_r$, and then augmenting it with another $r'-r$ copies of $\boldsymbol{x}_r$, that is

$$\boldsymbol{C}=[\boldsymbol{x}_{k+j+1},\ldots,\boldsymbol{x}_r,\boldsymbol{x}_r,\underbrace{\boldsymbol{x}_r,\ldots,\boldsymbol{x}_r}_{r'-r\text{ times}}].$$

To complete the proof we shall show that $[\boldsymbol{A},\boldsymbol{B},\boldsymbol{C}]\in\mathcal{U}(\boldsymbol{t}^{\{r'\}},\boldsymbol{x}^{\{k\}})$. To that end, we need to verify that $f(t_p,t_q,\boldsymbol{x}_p,\boldsymbol{x}_q)\leq 0$, for all $p,q\in\{0,1,\ldots,r',n+1\}$ and $p<q$, where by the construction of $\boldsymbol{C}$ we have that $\boldsymbol{x}_v=\boldsymbol{x}_r$, for all $v>r$. First, for $q\leq r$, $f(t_p,t_q,\boldsymbol{x}_p,\boldsymbol{x}_q)\leq 0$ is trivially satisfied since $[\overline{\boldsymbol{A}},\boldsymbol{B},\overline{\boldsymbol{C}}]\in\mathcal{U}(\boldsymbol{t}^{\{r\}},\boldsymbol{x}^{\{k\}})$. Second, for $p\leq r<q$, we need to check that $f(t_p,t_q,\boldsymbol{x}_p,\boldsymbol{x}_r)\leq 0$, which is true given that $f(t_p,t_r,\boldsymbol{x}_p,\boldsymbol{x}_r)\leq 0$ holds by $[\overline{\boldsymbol{A}},\boldsymbol{B},\overline{\boldsymbol{C}}]\in\mathcal{U}(\boldsymbol{t}^{\{r\}},\boldsymbol{x}^{\{k\}})$, in conjunction with t_q being a later monitoring time than t_r and assumption (2). Therefore, we also get that $f(t_q,t_q,\boldsymbol{x}_r,\boldsymbol{x}_r)\leq 0$ for $q>r$. Finally, for $r<p$, we need to check that $f(t_p,t_q,\boldsymbol{x}_r,\boldsymbol{x}_r)\leq 0$, which follows from the previous point and (2). ■

Proof of Theorem 1. It is trivial that $J\geq V$, since any optimal solution for the static model is also feasible for the dynamic model. To show the opposite direction, define

$$J_k(\boldsymbol{t}^{\{k\}},\boldsymbol{x}^{\{k\}})=\max\Big(g\big(t_k,\boldsymbol{x}_k\big),\,C_k(\boldsymbol{t}^{\{k\}},\boldsymbol{x}^{\{k\}})\Big)$$
$$C_k(\boldsymbol{t}^{\{k\}},\boldsymbol{x}^{\{k\}})\stackrel{\text{def}}{=}\max_{t_{k+1}\in[t_k,T]}G_{k+1}(\boldsymbol{t}^{\{k+1\}},\boldsymbol{x}^{\{k\}})$$
$$G_{k+1}(\boldsymbol{t}^{\{k+1\}},\boldsymbol{x}^{\{k\}})\stackrel{\text{def}}{=}\min_{\boldsymbol{x}_{k+1}\in\mathcal{U}_{k+1}(\boldsymbol{t}^{\{k+1\}},\boldsymbol{x}^{\{k\}})}J_{k+1}(\boldsymbol{t}^{\{k+1\}},\boldsymbol{x}^{\{k+1\}}).$$

The proof relies on the following auxiliary result, which we prove in Lemma 1: for any $\boldsymbol{t}^{\{k+1\}}$ and $\boldsymbol{x}^{\{k-1\}}$, with $\underline{\boldsymbol{x}}_k\stackrel{\text{def}}{=}\min\big\{\boldsymbol{x}\,:\,\boldsymbol{x}\in\mathcal{U}_k(\boldsymbol{t}^{\{k\}},\boldsymbol{x}^{\{k-1\}})\big\}$, we have

$$\underline{\boldsymbol{x}}_k\in\operatorname*{arg\,min}_{\boldsymbol{x}_k\in\mathcal{U}_k(\boldsymbol{t}^{\{k\}},\boldsymbol{x}^{\{k-1\}})}g(t_k,\boldsymbol{x}_k)\;\cap\operatorname*{arg\,min}_{\boldsymbol{x}_k\in\mathcal{U}_k(\boldsymbol{t}^{\{k\}},\boldsymbol{x}^{\{k-1\}})}C_k(\boldsymbol{t}^{\{k\}},\boldsymbol{x}^{\{k\}})\;\cap\operatorname*{arg\,min}_{\boldsymbol{x}_k\in\mathcal{U}_k(\boldsymbol{t}^{\{k+1\}},\boldsymbol{x}^{\{k-1\}})}G_{k+1}(\boldsymbol{t}^{\{k+1\}},\boldsymbol{x}^{\{k\}}).\quad(21)$$

We then have:

$$\begin{aligned} G_k(\boldsymbol{t}^{\{k\}},\boldsymbol{x}^{\{k-1\}}) &= \min_{\boldsymbol{x}_k\in\mathcal{U}_k(\boldsymbol{t}^{\{k\}},\boldsymbol{x}^{\{k-1\}})} J_k(\boldsymbol{t}^{\{k\}},\boldsymbol{x}^{\{k\}}) \\ &\overset{(21)}{=} \max\Big(g\big(t_k,\underline{\boldsymbol{x}}_k\big),\, C_k(\boldsymbol{t}^{\{k\}},[\boldsymbol{x}^{\{k-1\}},\underline{\boldsymbol{x}}_k])\Big) \\ &= \max\Big(g\big(t_k,\underline{\boldsymbol{x}}_k\big),\, \max_{t_{k+1}\in[t_k,T]} G_{k+1}(\boldsymbol{t}^{\{k+1\}},[\boldsymbol{x}^{\{k-1\}},\underline{\boldsymbol{x}}_k])\Big) \\ \text{(since } \underline{\boldsymbol{x}}_k \text{ independent of } t_{k+1}) &= \max_{t_{k+1}\in[t_k,T]} \max\Big(g\big(t_k,\underline{\boldsymbol{x}}_k\big),\, G_{k+1}(\boldsymbol{t}^{\{k+1\}},[\boldsymbol{x}^{\{k-1\}},\underline{\boldsymbol{x}}_k])\Big) \\ &\overset{(21)}{\leq} \max_{t_{k+1}\in[t_k,T]} \min_{\boldsymbol{x}_k\in\mathcal{U}_k(\boldsymbol{t}^{\{k+1\}},\boldsymbol{x}^{\{k-1\}})} \max\Big(g\big(t_k,\boldsymbol{x}_k\big),\, G_{k+1}(\boldsymbol{t}^{\{k+1\}},\boldsymbol{x}^{\{k\}})\Big). \end{aligned}$$

The argument then follows by induction. ■

LEMMA 1. *Consider any $1\leq k\leq n$, any $\boldsymbol{t}^{\{k+1\}}$ and $\boldsymbol{x}^{\{k-1\}}$, and let*

$$\underline{\boldsymbol{x}}_k \overset{\text{def}}{=} \min\big\{\boldsymbol{x} : \boldsymbol{x}\in\mathcal{U}_k(\boldsymbol{t}^{\{k\}},\boldsymbol{x}^{\{k-1\}})\big\}.$$

Then,

$$\underline{\boldsymbol{x}}_k \in \operatorname*{arg\,min}_{\boldsymbol{x}_k\in\mathcal{U}_k(\boldsymbol{t}^{\{k\}},\boldsymbol{x}^{\{k-1\}})} g(t_k,\boldsymbol{x}_k) \cap \operatorname*{arg\,min}_{\boldsymbol{x}_k\in\mathcal{U}_k(\boldsymbol{t}^{\{k\}},\boldsymbol{x}^{\{k-1\}})} C_k(\boldsymbol{t}^{\{k\}},\boldsymbol{x}^{\{k\}}) \cap \operatorname*{arg\,min}_{\boldsymbol{x}_k\in\mathcal{U}_k(\boldsymbol{t}^{\{k+1\}},\boldsymbol{x}^{\{k-1\}})} G_{k+1}(\boldsymbol{t}^{\{k+1\}},\boldsymbol{x}^{\{k\}}).$$

Proof of Lemma 1. Note that $\underline{\boldsymbol{x}}_k$ is well defined since $\mathcal{U}_k(\boldsymbol{t}^{\{k\}},\boldsymbol{x}^{\{k-1\}})$ is a lattice, by Assumption 1(i). Since $g(t_k,\boldsymbol{x}_k)$ is increasing in $\boldsymbol{x}_k$ for any t_k, it suffices to prove that $C_k(\boldsymbol{t}^{\{k\}},\boldsymbol{x}^{\{k\}})$ and $G_{k+1}(\boldsymbol{t}^{\{k+1\}},\boldsymbol{x}^{\{k\}})$ are increasing functions, for any k and any $\boldsymbol{t}^{\{k+1\}}$. To the latter point, we claim that it suffices to show that $J_{k+1}(\boldsymbol{t}^{\{k+1\}},\boldsymbol{x}^{\{k+1\}})$ is increasing in $\boldsymbol{x}^{\{k+1\}}$; when this holds, we readily have that:

$$G_{k+1}(\boldsymbol{t}^{\{k+1\}},\boldsymbol{x}^{\{k\}}) \overset{\text{def}}{=} \min_{\boldsymbol{x}_{k+1}\in\mathcal{U}(\boldsymbol{t}^{\{k+1\}},\boldsymbol{x}^{\{k\}})} J_{k+1}(\boldsymbol{t}^{\{k+1\}},\boldsymbol{x}^{\{k+1\}}) = J_{k+1}(\boldsymbol{t}^{\{k+1\}},[\boldsymbol{x}^{\{k\}},\underline{\boldsymbol{x}}_{k+1}])$$

is increasing in $\boldsymbol{x}^{\{k\}}$, since $\underline{\boldsymbol{x}}_{k+1}$ is increasing in $\boldsymbol{x}^{\{k\}}$, by Assumption 1(ii). And thus

$$C_k(\boldsymbol{t}^{\{k\}},\boldsymbol{x}^{\{k\}}) \overset{\text{def}}{=} \max_{t_{k+1}\in[t_k,T]} G_{k+1}(\boldsymbol{t}^{\{k+1\}},\boldsymbol{x}^{\{k\}})$$

is also increasing in $\boldsymbol{x}^{\{k\}}$, as a maximum of increasing functions.

To complete our proof, it thus suffices to show that $J_k(\boldsymbol{t}^{\{k\}},\cdot)$ is increasing, for any k and any $\boldsymbol{t}^{\{k+1\}}$. We prove this by induction. For $k=n+1$, we have that $J_{n+1}(\boldsymbol{t}^{\{n+1\}},\boldsymbol{x}^{\{n+1\}}) \overset{\text{def}}{=} g(t_{n+1},\boldsymbol{x}_{n+1})$, so $J_{n+1}(\boldsymbol{t}^{\{n+1\}},\cdot)$ is increasing. Assume the property holds for J_{k+1}. Then, $G_{k+1}(\boldsymbol{t}^{\{k+1\}},\boldsymbol{x}^{\{k\}})$ and $C_k(\boldsymbol{t}^{\{k\}},\boldsymbol{x}^{\{k\}})$ are both increasing in $\boldsymbol{x}^{\{k\}}$, by the argument above. And since $g(t_k,\boldsymbol{x}_k)$ is also increasing in $\boldsymbol{x}_k$ by Assumption 2, we have that

$$J_k(\boldsymbol{t}^{\{k\}},\boldsymbol{x}^{\{k\}}) = \max\Big(g\big(t_k,\boldsymbol{x}_k\big),\, C_k(\boldsymbol{t}^{\{k\}},\boldsymbol{x}^{\{k\}})\Big)$$

is also increasing in $\boldsymbol{x}^{\{k\}}$, completing our inductive step. ■

Proof of Proposition 2. According to Lemma 1, the worst-case process value at time t_k, which we denote by $\underline{\boldsymbol{x}}_k(\boldsymbol{t}^{\{n+1\}},\underline{\boldsymbol{x}}^{\{k-1\}})$, can be obtained by choosing the smallest element of the corresponding uncertainty set, i.e.,

$$\underline{\boldsymbol{x}}_k(\boldsymbol{t}^{\{n+1\}},\boldsymbol{x}^{\{k-1\}}) \overset{\text{def}}{=} \min\Big\{\boldsymbol{\xi}\in\mathbb{R}^d : \boldsymbol{\xi}\in\mathcal{U}_k\big(\boldsymbol{t}^{\{n+1\}},\boldsymbol{x}^{\{k-1\}}\big)\Big\}, \forall k\in\{1,\dots,n+1\}.$$

The result follows by induction. ■

Proof of Theorem 2. We claim that there always exists an optimal choice of $\boldsymbol{t}^{\{n+1\}}$ such that the inner maximum in (7) is reached at t_n or $t_{n+1}=T$ (i.e., for $k=n$ or $n+1$, respectively). To see this, assume that the maximum occurs at $\bar{k}<n$, and introduce a new set of monitoring times $\boldsymbol{z}$ where all times are identical except for the $\bar{k}$-th and the $(\bar{k}+1)$-th, which now equal $t_{\bar{k}-1}$ and $t_{\bar{k}}$, respectively, i.e., $\boldsymbol{z}=[t_0,t_1,\ldots,t_{\bar{k}-1},t_{\bar{k}-1},t_{\bar{k}},t_{\bar{k}+2},\ldots,t_n,t_{n+1}]$. Then,

$$V \geq \max_{k\in\{1,\ldots,n+1\}} g(z_k,\underline{\boldsymbol{x}}_k(\boldsymbol{z})) \overset{(7)}{\geq} g(z_{\bar{k}+1},\underline{\boldsymbol{x}}_{\bar{k}+1}(\boldsymbol{z})) = g(t_{\bar{k}},\underline{\boldsymbol{x}}_{\bar{k}}(\boldsymbol{t}^{\{n+1\}})) = V.$$

The penultimate equality holds because $\mathcal{U}_{\bar{k}}\big(\boldsymbol{t}^{\{n+1\}},\underline{\boldsymbol{x}}^{\{\bar{k}-1\}}\big)=\mathcal{U}_{\bar{k}+1}\big(\boldsymbol{z},\underline{\boldsymbol{x}}^{\{\bar{k}\}}\big)$, by the construction of $\boldsymbol{z}$ and by the definition of $\mathcal{U}$ in (1). Thus, under the monitoring times $\boldsymbol{z}$, the same optimum is reached at $\bar{k}+1$. Repeating the argument inductively yields the result. ■

Proof of Theorem 3 For the convex case in (i), please refer to the proof of Theorem 4 for a more general result. For the concave case in (ii), consider problem (9), $\max_{\boldsymbol{t}^{\{n+1\}}} g(t_n,\underline{\boldsymbol{x}}_n(\boldsymbol{t}^{\{n+1\}}))$. By Theorem 4, $\underline{\boldsymbol{x}}_n(\boldsymbol{t}^{\{n+1\}})=\underline{\boldsymbol{x}}_0+\sum_{i=1}^n \ell(t_i-t_{i-1})\mathbf{1}$ when ℓ is concave. Thus, for a fixed t_n, maximizing $g(t_n,\underline{\boldsymbol{x}}_n(\boldsymbol{t}^{\{n+1\}}))$ is equivalent to maximizing $\sum_{i=1}^n \ell(t_i-t_{i-1})$, since $g(t,\boldsymbol{x})$ is increasing in $\boldsymbol{x}$. Due to the concavity of $\ell(\delta)$, the maximum is achieved when $t_i-t_{i-1}=t_n/n$, $i=1,\ldots,n$ (by Jensen's inequality). Therefore, problem (9) becomes equivalent to $\max_{t_n} g(t_n,\boldsymbol{\phi}_n(t_n))$. Similarly, $\max_{\boldsymbol{t}^{\{n+1\}}} g(t_{n+1},\underline{\boldsymbol{x}}_{n+1}(\boldsymbol{t}^{\{n+1\}}))=g(T,\boldsymbol{\phi}_{n+1}(T))$. ■

Proof of Theorem 4 For case (i), we first show that $\underline{\boldsymbol{x}}_k(\boldsymbol{t}^{\{n+1\}})=\underline{\boldsymbol{x}}_0+\ell(t_0,t_k-t_0)\mathbf{1}$. To that end, note that for any $j\in\{1,\ldots,d\}$ and $1\leq p<q\leq n$:

$$x_0^j+\ell(t_0,t_p-t_0)+\ell(t_p,t_q-t_p)\leq x_0^j+\ell(t_0,t_p-t_0)+\ell(t_0,t_q-t_p)\leq x_0^j+\ell(t_0,t_q-t_0), \tag{22}$$

where the first inequality follows since $\ell(t,\delta)$ is decreasing in t, and the second inequality follows because ℓ is superadditive in δ (since ℓ is convex in δ and $\ell(\cdot,0)=0$). Therefore,

$$\begin{aligned}\underline{\boldsymbol{x}}_k(\boldsymbol{t}^{\{n+1\}}) &\overset{\text{def}}{=} \min\Big\{\boldsymbol{\xi}:\boldsymbol{\xi}\in\mathcal{U}_k\big(\boldsymbol{t}^{\{n+1\}},[\boldsymbol{x}_0,\underline{\boldsymbol{x}}_1,\ldots,\underline{\boldsymbol{x}}_{k-1}]\big)\Big\}\\ &=\underline{\boldsymbol{x}}_0+\left[\max_{\{0=k_1\leq\cdots\leq k_r=k\}\in\{1,\ldots,k\}}\sum_{i=1}^r \ell(t_{k_i-1},t_{k_i}-t_{k_i-1})\right]\mathbf{1}=\underline{\boldsymbol{x}}_0+\ell(t_0,t_k-t_0)\mathbf{1}.\end{aligned}$$

For case (ii), note that when $\ell(t,\delta)$ is increasing in t and concave in δ, the reverse inequalities hold in (22). Thus, nature's optimal (worst-case) response is given by:

$$\underline{\boldsymbol{x}}_k(\boldsymbol{t}^{\{n+1\}})=\underline{\boldsymbol{x}}_0+\left[\sum_{i=1}^k \ell(t_{i-1},t_i-t_{i-1})\right]\mathbf{1}.$$

Since $g(t,\boldsymbol{x})$ is jointly concave and increasing in $\boldsymbol{x}$, and $\ell(t,\delta)$ is jointly concave, the composition $g\big(t_n,\underline{\boldsymbol{x}}_n(\boldsymbol{t}^{\{n+1\}})\big)$ is concave in $\boldsymbol{t}^{\{n+1\}}$, so that problem (9) requires maximizing a concave function over the convex set $0\leq t_1\leq\cdots\leq t_n\leq T$. ■

Proof of Proposition 3. Our proof relies on the following known result from lattice programming.

Lemma 2 **(Topkis 1998)**. *If h is a convex, increasing (i.e., isotone) function, and f is supermodular and either isotone or antitone, then $h\circ f$ is supermodular.*

Note that every component of $\underline{\boldsymbol{x}}_n$ can be written as:

$$\underline{\boldsymbol{x}}_n^k = \underline{\boldsymbol{x}}_0^k + \max_{s\in\mathcal{S}} f_s(\boldsymbol{t}^{\{n\}})$$
$$f_s(\boldsymbol{t}^{\{n\}}) \stackrel{\text{def}}{=} \sum_{i=0}^{p-1} \ell(t_{s(i)}, t_{s(i+1)} - t_{s(i)}),$$

where $\mathcal{S}$ denotes the set of all ordered subsets of $\{0,\ldots,n\}$ that include 0 and n. By assumption (ii), for every $s \in \mathcal{S}$, f_s is supermodular in $\boldsymbol{t}^{\{n\}}$ and decreasing (i.e., antitone) in $\boldsymbol{t}^{\{n\}}$. Since the max function is convex and increasing (i.e., isotone), we can invoke Proposition 2 to conclude that $\underline{\boldsymbol{x}}_n^k$ is supermodular in $\boldsymbol{t}^{\{n\}}$. Since $\underline{\boldsymbol{x}}_n^k$ is also decreasing in $\boldsymbol{t}^{\{n\}}$, and $g(t,\boldsymbol{x})$ is increasing and component-wise convex in $\boldsymbol{x}$, we can again invoke Proposition 2 to conclude that $g\big(t_n, \underline{\boldsymbol{x}}_n(\boldsymbol{t}^{\{n\}})\big)$ is supermodular in $\boldsymbol{t}^{\{n-1\}}$ for any fixed t_n. ■

Proof of Proposition 4. Consider any patient with starting age $A \in [33,62]$. We first make two simple observations: (i) any transition from disease progression constitutes a worsening of the patient's condition; and (ii) for any disease stage transition, a worst-case reward is obtained by transitioning as quickly as possible. Both observations follow in view of expression (14) for the total QALYs: transitioning always results in the same or worse QALY factors (by Table 2), and a worsening in the survival rate (by Table 3). Thus, to minimize the reward from re-transplantation (given by the product of the total QALYs from (14) and the survival rate from), it is best to transition as quickly as possible from states and thus spending more time in later states, which are "worse" in terms of both QALY factors and survival rate.

If the patient is in stage 2L, there are three possible transitions, to 3L, 2H and 3H. We argue that for any future time, a transition to 2H can always be dominated by a transition to 3L, in the sense that it leaves the patient worse off, i.e., it results in lower worst-case rewards for any possible future re-transplantation time. First, consider future times small enough so that only a transition 2L to 2H is possible (and not 2L to 2H to 3H). We argue that 2L to 3L is worse. In view of the two observations above, since the survival rate is lower in state 3L than in state 2H (by Remark 1), it suffices to show that the transition 2L-3L results in spending less time in state 2L than the transition 2L-2H. To that end, assume the patient is in state $2L$ at time t, and recall that the worst-case (i.e., lowest-possible) transition time $L_{iji'j'}(t,A)$ in our robust model is $\frac{-\ln(\rho)}{\lambda_{iji'j'}(t,A)}$. By using Table 4, we can check that $1/\lambda_{2L3L}(t,A) \le 1/\lambda_{2L2H}(t,A)$ holds for all $A \le 62.77$, which gives the desired result.

For longer future times so that 2L to 3L to 3H is also possible, note that 2L to 3L to 3H would still provide worse outcomes. To see this, we can again check that $1/\lambda_{2L3L}(t,A) + 1/\lambda_{3L3H}(t,A) \le 1/\lambda_{2L2H}(t,A) + 1/\lambda_{2H3H}(t,A)$, for the age groups we are considering.

Similarly, we can rule out transitions from 1L to 1H and from 1L to 2H.

Result (ii) follows by replacing the variables along the worst-case progression path. ■

Appendix B: Extensions

We explore two extensions of our model that may be relevant in practice.

B.1. Costly Monitoring & Optimal Number of Monitoring Times

Consider our base model, but assume that each monitoring incurs a fixed cost $c \geq 0$. In particular, the reward from stopping at the k-th monitoring time t_k with a state of $\boldsymbol{x}_k$ becomes $g(t_k, \boldsymbol{x}_k) - kc$, for all $k = 1, \ldots, n$. When monitoring is costly, the DM need not opt for monitoring n times; instead, the number of monitoring times becomes an implicit decision.

In this extension, it can be readily checked that Theorem 1 continues to hold, so that the worst-case optimal rewards under dynamic and static monitoring are the same. Thus, it suffices again to focus on the static monitoring problem. Let $\tilde{V}_k^c(\boldsymbol{t}^{\{n+1\}}, \boldsymbol{x}^{\{k\}})$ be the worst-case value-to-go function for the static monitoring problem at time t_k, with the first k observations made. The Bellman equations become:

$$\tilde{V}_k^c(\boldsymbol{t}^{\{n+1\}}, \boldsymbol{x}^{\{k\}}) = \max\left(g(t_k, \boldsymbol{x}_k) - ck, \min_{\boldsymbol{x}_{k+1} \in \tilde{\mathcal{U}}_{k+1}(\boldsymbol{t}^{\{n+1\}}, \boldsymbol{x}^{\{k\}})} \tilde{V}_{k+1}^c(\boldsymbol{t}^{\{n+1\}}, \boldsymbol{x}^{\{k+1\}}) \right),$$
$$\tilde{V}_{n+1}^c(\boldsymbol{t}^{\{n+1\}}, \boldsymbol{x}^{\{n+1\}}) = g(t_{n+1}, \boldsymbol{x}_{n+1}) - c\,(n+1).$$

It can be readily checked that, under a given set of monitoring times $\boldsymbol{t}^{\{n+1\}}$, nature's optimal (worst-case) response $\underline{\boldsymbol{x}}_k(\boldsymbol{t}^{\{n+1\}})$ in period t_k is still given by Proposition 2. Thus, the DM's problem can again be reformulated as:

$$\tilde{V}_0^c(t_0, \boldsymbol{x}_0) = \max_{\boldsymbol{t}^{\{n+1\}}} \max_{k \in \{1,\ldots,n+1\}} \Big[g\big(t_k, \underline{\boldsymbol{x}}_k(\boldsymbol{t}^{\{n+1\}})\big) - c\,k \Big]. \tag{23}$$

A key difference from our earlier results lies in the DM's optimal stopping strategy. Recall that in our base model, it was worst-case optimal to either stop at the last monitoring time t_n or continue until the end of the horizon (see Theorem 2). This is no longer the case here, as an optimal policy may require stopping at an earlier time due to the monitoring cost c. Thus, the optimal k^* in (23) may be strictly smaller than n. In other words, in the above problem (23), by choosing the worst-case stopping strategy, the DM also implicitly chooses the optimal number of monitoring times, k^*. Therefore, n should only be interpreted here as an upper bound on monitoring opportunities, and k as the number of monitorings in the monitoring schedule that the DM needs to decide.

To solve (23), one can switch the order of the maximization operators. Since finding the optimal $\boldsymbol{t}^{\{n+1\}}$ for a fixed k requires solving problems that are structurally identical to Problem (9), our results in §3.3.1 and §3.3.2 can be directly leveraged. By iterating over k, one can then recover the optimal number of monitoring times.

Moreover, when the bounds are stationary, the problem of finding the optimal number of monitoring times is also tractable under mild conditions, as summarized next.

Proposition 5. *Under Assumption 2 and for the uncertainty set in (10) with stationary lower bounds $\ell(t_q - t_p)$,*

(i) if $\ell(\cdot)$ is convex, then a single monitoring time is sufficient for achieving the worst-case optimal reward;

(ii) if $\ell(\cdot)$ is concave, and $g(t, \boldsymbol{x})$ is jointly concave, then the optimal number of monitoring times can be done by solving convex optimization problems.

Proof. Part (i) follows directly from Theorem 3(i). For part (ii), recall from Theorem 3(ii) that finding the optimal stopping time under a fixed number of monitoring times n requires solving the problem $\max_{t\in[0,T]} g\big(t, \underline{\boldsymbol{x}}_0 + n\ell\big(t/n\big)\mathbf{1}\big)$. The function to be maximized is jointly concave in (t,n), since $g(t,\boldsymbol{x})$ is jointly concave and increasing in $\boldsymbol{x}$, and the functions $n\ell(t/n)$ are jointly concave in (t,n) since ℓ is concave. Thus, one can find an optimal t and n by first maximizing a concave function over a convex feasible set (considering n continuous), and then checking the nearest integers (possibly solving two additional one dimensional convex optimization problems to determine the corresponding t). ■

B.2. More General Decision Process

Some of our results concerning the monitoring policy also extend to a more general decision problem, where the DM, instead of simply stopping, can modify the processes by increasing the state values (an action we refer to as "injection") or decreasing them ("extraction"). This allows capturing several applications of interest. In chronic disease monitoring, "injections" could denote interventions that are costly or have immediate side-effects in the short run, but carry long-term benefits, while "extractions" could capture relaxing a strict treatment, leading to immediate relief but carrying potential long-term consequences. In collateralized lending, "injections" could denote the costly addition of new collateral, which improves the borrowing base, and "extractions" could denote immediate collateral liquidations, which generate cash but reduce the borrowing base.

To formalize this, consider our setup in §2, but assume that at the k-th monitoring time t_k, upon observing the state value $\boldsymbol{x}_k \overset{\text{def}}{=} \boldsymbol{x}(t_k)$, the DM decides an action $\boldsymbol{y}_k \in A(\boldsymbol{x}_k) \subseteq \mathbb{R}^d$, which results in an immediately updated state $\boldsymbol{z}_k \overset{\text{def}}{=} \boldsymbol{x}_k - \boldsymbol{y}_k$, and a net reward $r(t_k, \boldsymbol{x}_k, \boldsymbol{y}_k)$ accruing to the DM. When $\boldsymbol{y}_k \geq 0$ (<0), the action can be thought of as extracting value from (injecting value into) the processes, in which case the corresponding net reward would typically be positive (respectively, negative). Not all actions are possible, and $A(\boldsymbol{x}_k)$ captures the feasible set when the initial state is $\boldsymbol{x}_k$.

Following the DM's action, the system subsequently evolves from time t_k to the next monitoring time t_{k+1}, where it takes a value of $\boldsymbol{x}_{k+1}$, chosen by nature from an uncertainty set. More precisely, for any $0 \leq k \leq r \leq n+1$, and given a fixed choice of monitoring times $\boldsymbol{t}^{\{r\}}$, observations $\boldsymbol{x}^{\{k\}}$ and post-action states $\boldsymbol{z}^{\{k\}}$ up to time t_k, the set of possible future values for $[\boldsymbol{x}_{k+1},\ldots,\boldsymbol{x}_r,\boldsymbol{x}_{n+1}]$ is given by:

$$\tilde{\mathcal{U}}(\boldsymbol{t}^{\{r\}}, \boldsymbol{x}^{\{k\}}, \boldsymbol{z}^{\{k\}}) \overset{\text{def}}{=} \Big\{ [\boldsymbol{x}_{k+1},\ldots,\boldsymbol{x}_r] \in \mathbb{R}^{d\times(r-k+1)} : \tilde{f}(t_p, t_q, \boldsymbol{x}_p, \boldsymbol{z}_p, \boldsymbol{x}_q), \quad \forall\, p,q \in \{0,\ldots,r,n+1\},\, p<q \Big\}. \tag{24}$$

As before, we consider two versions of the DM's problem—*static* and *dynamic*—depending on whether the monitoring times are chosen at inception or throughout the problem horizon. The DM's objective is to determine the monitoring times $\boldsymbol{t}^{\{n+1\}}$ and the optimal actions $\boldsymbol{y}^{\{n+1\}}$ that maximize his cumulative reward up to time t_{n+1}.

Assumptions. We assume that rewards and action sets are monotonic in states.

Assumption 3. *The net reward $r(t,\boldsymbol{x},\boldsymbol{y})$ is increasing in $\boldsymbol{x}$, and the action set $A(\boldsymbol{x})$ is increasing in $\boldsymbol{x}$ with respect to set inclusion, i.e., $\boldsymbol{x}_1 \leq \boldsymbol{x}_2 \Rightarrow A(\boldsymbol{x}_1) \subseteq A(\boldsymbol{x}_2)$.*

Several feasible sets satisfy our requirement; for instance, $A(\boldsymbol{x}) = \{\boldsymbol{y} : 0 \le \boldsymbol{y} \le \boldsymbol{x}\}$. Paralleling Assumption 1, we also require the uncertainty sets to be lattices, with suitable monotonicity and dynamic consistency properties.

ASSUMPTION 4. *For any $0 \le k < r \le n$, and given $\boldsymbol{t}^{\{r\}}, \boldsymbol{x}^{\{k\}}$ and $\boldsymbol{z}^{\{k\}}$,*

(i) (Lattice) $\tilde{\mathcal{U}}(\boldsymbol{t}^{\{r\}}, \boldsymbol{x}^{\{k\}}, \boldsymbol{z}^{\{k\}})$ is a lattice;

(ii) (Monotonicity) $\tilde{\mathcal{U}}(\boldsymbol{t}^{\{r\}}, \boldsymbol{x}^{\{k\}}, \boldsymbol{z}^{\{k\}})$ is increasing in $\boldsymbol{x}^{\{k\}}$ and $\boldsymbol{z}^{\{k\}}$;

(iii) (Dynamic Consistency) $\Pi_{:,i:j} \tilde{\mathcal{U}}(\boldsymbol{t}^{\{r\}}, \boldsymbol{x}^{\{k\}}, \boldsymbol{z}^{\{k\}}) = \Pi_{:,i:j} \tilde{\mathcal{U}}(\boldsymbol{t}^{\{r'\}}, \boldsymbol{x}^{\{k\}}, \boldsymbol{z}^{\{k\}})$, $\forall i \le j < r - k + 1$, $\forall r \le r' \le n$, and $\forall t_{r'} \ge t_{r'-1} \ge \ldots \ge t_r$.

These generalized sets allow future states to depend on historical state values both immediately before and immediately after the DM's actions. As before, we can prove that dynamic consistency is guaranteed when $\tilde{f}$ is monotonic in its second argument (details are omitted.)

Analysis. We first consider the dynamic problem. With $\tilde{J}_k(\boldsymbol{t}^{\{k\}}, \boldsymbol{x}^{\{k\}}, \boldsymbol{z}^{\{k-1\}})$ denoting[18] the DM's value-to-go function at time t_k, the Bellman recursions become:

$$\begin{aligned}
\tilde{J}_k(\boldsymbol{t}^{\{k\}}, \boldsymbol{x}^{\{k\}}, \boldsymbol{z}^{\{k-1\}}) &= \max_{\boldsymbol{y}_k \in A(\boldsymbol{x}_k)} \Bigg[r(t_k, \boldsymbol{x}_k, \boldsymbol{y}_k) + \\
&\qquad \max_{t_{k+1} \in [t_k, T]} \min_{\boldsymbol{x}_{k+1} \in \tilde{\mathcal{U}}(\boldsymbol{t}^{\{k+1\}}, \boldsymbol{x}^{\{k\}}, \boldsymbol{z}^{\{k\}})} \tilde{J}_{k+1}(\boldsymbol{t}^{\{k+1\}}, \boldsymbol{x}^{\{k+1\}}, \boldsymbol{z}^{\{k\}})) \Bigg], \\
\tilde{J}_{n+1}(\boldsymbol{t}^{\{n+1\}}, \boldsymbol{x}^{\{n+1\}}, \boldsymbol{z}^{\{n\}}) &= \max_{\boldsymbol{y}_{n+1} \in A(\boldsymbol{x}_{n+1})} r(t_{n+1}, \boldsymbol{x}_{n+1}, \boldsymbol{y}_{n+1}).
\end{aligned}$$

Let $\tilde{J}_0 \stackrel{\text{def}}{=} \tilde{J}_0(t_0, \boldsymbol{x}_0)$.

In the static problem, the DM chooses $\boldsymbol{t}^{\{n+1\}}$ at time t_0. With $\tilde{V}_k(\boldsymbol{t}^{\{k\}}, \boldsymbol{x}^{\{k\}}, \boldsymbol{z}^{\{k-1\}})$ denoting the value-to-go function at time t_k, the Bellman recursions become:

$$\begin{aligned}
\tilde{V}_k(\boldsymbol{t}^{\{n+1\}}, \boldsymbol{x}^{\{k\}}, \boldsymbol{z}^{\{k-1\}}) &= \max_{\boldsymbol{y}_k \in A(\boldsymbol{x}_k)} \Bigg[r(t_k, \boldsymbol{x}_k, \boldsymbol{y}_k) + \min_{\boldsymbol{x}_{k+1} \in \tilde{\mathcal{U}}(\boldsymbol{t}^{\{k+1\}}, \boldsymbol{x}^{\{k\}}, \boldsymbol{z}^{\{k\}})} \tilde{V}_{k+1}\big(\boldsymbol{t}^{\{n+1\}}, \boldsymbol{x}^{\{k+1\}}, \boldsymbol{z}^{\{k\}})\big) \Bigg], \\
\tilde{V}_{n+1}(\boldsymbol{t}^{\{n+1\}}, \boldsymbol{x}^{\{n+1\}}, \boldsymbol{z}^{\{n\}}) &= \max_{\boldsymbol{y}_{n+1} \in A(\boldsymbol{x}_{n+1})} r(t_{n+1}, \boldsymbol{x}_{n+1}, \boldsymbol{y}_{n+1}),
\end{aligned}$$

and the optimal choice of monitoring times yields a value of $\tilde{V}_0 \stackrel{\text{def}}{=} \max_{\boldsymbol{t}^{\{n+1\}}} \tilde{V}_0(\boldsymbol{t}^{\{n+1\}}, \boldsymbol{x}^{\{0\}})$.

In this context, we can confirm that an analogous result to Theorem 1 holds, and the dynamic problem yields the same worst-case optimal reward as the static problem.

THEOREM 5. *Under Assumption 3 and Assumption 4, $\tilde{J}_0 = \tilde{V}_0$.*

Proof. The Bellman recursion under dynamic monitoring can be written:

$$\begin{aligned}
\tilde{J}_k(\boldsymbol{t}^{\{k\}}, \boldsymbol{x}^{\{k\}}, \boldsymbol{z}^{\{k-1\}}) &= \max_{\boldsymbol{y}_k \in A(\boldsymbol{x}_k)} \max_{t_{k+1} \in [t_k, T]} \Big[r(t_k, \boldsymbol{x}_k, \boldsymbol{y}_k) + \tilde{G}_k(\boldsymbol{t}^{\{k+1\}}, \boldsymbol{x}^{\{k\}}, \boldsymbol{z}^{\{k\}}) \Big], \quad \text{where} \\
\tilde{G}_k(\boldsymbol{t}^{\{k+1\}}, \boldsymbol{x}^{\{k\}}, \boldsymbol{z}^{\{k\}}) &\stackrel{\text{def}}{=} \min_{\boldsymbol{x}_{k+1} \in \tilde{\mathcal{U}}(\boldsymbol{t}^{\{k+1\}}, \boldsymbol{x}^{\{k\}}, \boldsymbol{z}^{\{k\}})} \tilde{J}_{k+1}(\boldsymbol{t}^{\{k+1\}}, \boldsymbol{x}^{\{k+1\}}, \boldsymbol{z}^{\{k\}}), \; \forall k \in \{1, \ldots, n\}.
\end{aligned}$$

[18] To simplify notation, we define $\boldsymbol{z}^{\{-1\}} \stackrel{\text{def}}{=} \emptyset$.

First, using induction, we prove that $\tilde{J}_k$ and $\tilde{G}_k$ are increasing in all arguments except time. By Assumption 3, this is true[19] for $\tilde{J}_{n+1}(\boldsymbol{t}^{\{n+1\}}, \boldsymbol{x}^{\{n+1\}}, \boldsymbol{z}^{\{n\}})$. Assuming this is true at $k+1$, and using Assumption 4(i,ii), note that:

$$\begin{aligned}\operatorname{argmin}_{\boldsymbol{x}_{k+1}\in\tilde{\mathcal{U}}(\boldsymbol{t}^{\{k+1\}},\boldsymbol{x}^{\{k\}},\boldsymbol{z}^{\{k\}})}\tilde{J}_{k+1}(\boldsymbol{t}^{\{k+1\}}, \boldsymbol{x}^{\{k+1\}}, \boldsymbol{z}^{\{k\}}) &= \min_{\boldsymbol{x}_{k+1}\in\tilde{\mathcal{U}}(\boldsymbol{t}^{\{k+1\}},\boldsymbol{x}^{\{k\}},\boldsymbol{z}^{\{k\}})} \boldsymbol{x}_{k+1} \\ &\stackrel{\text{def}}{=} \underline{\boldsymbol{x}}_{k+1}(\boldsymbol{t}^{\{k+1\}}, \boldsymbol{x}^{\{k\}}, \boldsymbol{z}^{\{k\}}),\end{aligned}$$

and $\underline{\boldsymbol{x}}_{k+1}(\boldsymbol{t}^{\{k+1\}}, \boldsymbol{x}^{\{k\}}, \boldsymbol{z}^{\{k\}})$ is increasing in $\boldsymbol{x}^{\{k\}}$ and $\boldsymbol{z}^{\{k\}}$. Therefore,

$$\tilde{G}_k(\boldsymbol{t}^{\{k+1\}}, \boldsymbol{x}^{\{k\}}, \boldsymbol{z}^{\{k\}}) = \tilde{J}_{k+1}\Big(\boldsymbol{t}^{\{k+1\}}, \big[\boldsymbol{x}^{\{k\}}, \underline{\boldsymbol{x}}_{k+1}(\boldsymbol{t}^{\{k+1\}}, \boldsymbol{x}^{\{k\}}, \boldsymbol{z}^{\{k\}})\big], \boldsymbol{z}^{\{k\}}\Big)$$

is increasing in $(\boldsymbol{x}^{\{k\}}, \boldsymbol{z}^{\{k\}})$. But then, note that the maximand in the problem:

$$\tilde{J}_k(\boldsymbol{t}^{\{k\}}, \boldsymbol{x}^{\{k\}}, \boldsymbol{z}^{\{k-1\}}) = \max_{\boldsymbol{y}_k\in A(\boldsymbol{x}_k)} \max_{t_{k+1}\in[t_k,T]} \Big[r(t_k, \boldsymbol{x}_k, \boldsymbol{y}_k) + \tilde{G}_k\big(\boldsymbol{t}^{\{k+1\}}, \boldsymbol{x}^{\{k\}}, [\boldsymbol{z}^{\{k-1\}}, \boldsymbol{x}_k - \boldsymbol{y}_k]\big)\Big].$$

is increasing in $(\boldsymbol{x}^{\{k\}}, \boldsymbol{z}^{\{k-1\}})$, for any fixed value of $\boldsymbol{y}_k$ and t_{k+1}. And since the action set $A(\boldsymbol{x}_k)$ is increasing in $\boldsymbol{x}_k$ with respect to set inclusion by Assumption 3, this implies that $\tilde{J}_k$ is increasing in $(\boldsymbol{x}^{\{k\}}, \boldsymbol{z}^{\{k-1\}})$, which completes our induction.

Using these monotonicity properties, we then obtain:

$$\begin{aligned}&\tilde{G}_{k-1}(\boldsymbol{t}^{\{k\}}, \boldsymbol{x}^{\{k-1\}}, \boldsymbol{z}^{\{k-1\}}) &(25)\\ &= \min_{\boldsymbol{x}_k\in\tilde{\mathcal{U}}(\boldsymbol{t}^{\{k\}},\boldsymbol{x}^{\{k-1\}},\boldsymbol{z}^{\{k-1\}})} \max_{\boldsymbol{y}_k\in A(\boldsymbol{x}_k)} \max_{t_{k+1}\in[t_k,T]} \Big[r(t_k, \boldsymbol{x}_k, \boldsymbol{y}_k) + \tilde{G}_k\Big(\boldsymbol{t}^{\{k+1\}}, [\boldsymbol{x}^{\{k-1\}}, \boldsymbol{x}_k], \boldsymbol{z}^{\{k\}}\Big)\Big] \\ &= \max_{t_{k+1}\in[t_k,T]} \max_{\boldsymbol{y}_k\in A(\boldsymbol{x}_k)} \Big[r(t_k, \underline{\boldsymbol{x}}_k, \boldsymbol{y}_k) + \tilde{G}_k\Big(\boldsymbol{t}^{\{k+1\}}, [\boldsymbol{x}^{\{k-1\}}, \underline{\boldsymbol{x}}_k], \boldsymbol{z}^{\{k\}}\Big)\Big] \\ &= \max_{t_{k+1}\in[t_k,T]} \min_{\boldsymbol{x}_k\in\tilde{\mathcal{U}}(\boldsymbol{t}^{\{k+1\}},\boldsymbol{x}^{\{k-1\}},\boldsymbol{z}^{\{k-1\}})} \max_{\boldsymbol{y}_k\in A(\boldsymbol{x}_k)} \Big[r(t_k, \boldsymbol{x}_k, \boldsymbol{y}_k) + \tilde{G}_k\Big(\boldsymbol{t}^{\{k+1\}}, [\boldsymbol{x}^{\{k-1\}}, \boldsymbol{x}_k], \boldsymbol{z}^{\{k\}}\Big)\Big]. &(26)\end{aligned}$$

The second equality follows from the monotonicity of r and $\tilde{G}_k$ in $\boldsymbol{x}_k$; the last equality follows from the same monotonicity and the dynamic consistency Assumption 4(iii), which ensures that $\tilde{\mathcal{U}}(\boldsymbol{t}^{\{k+1\}}, \boldsymbol{x}^{\{k-1\}}, \boldsymbol{z}^{\{k-1\}}) = \tilde{\mathcal{U}}(\boldsymbol{t}^{\{k\}}, \boldsymbol{x}^{\{k-1\}}, \boldsymbol{z}^{\{k-1\}})$, so that the nature's worst-case response $\underline{\boldsymbol{x}}_k$ is independent of the choice t_{k+1}. Therefore, we can interchange the order of $\max_{t_{k+1}}$ and $\min_{\boldsymbol{x}_k\in\tilde{\mathcal{U}}_k}$. Repeating the argument inductively, we obtain $\tilde{J}_0 = \tilde{V}_0$. ■

This result again allows reconstructing the DM's optimal dynamic monitoring policy by (resolving) static versions of the monitoring problem. In fact, a further simplification is also possible here, as summarized in our next result.

PROPOSITION 6. *Consider the static monitoring problem. The DM can make all the injection decisions at time t_0 and recover the same worst-case reward, i.e.,*

$$\tilde{V}_0 = \max_{\boldsymbol{t}^{\{n+1\}}} \max_{\boldsymbol{y}^{\{n+1\}}\in\mathbb{R}^{d\times(n+1)}} \min_{\substack{\boldsymbol{x}^{\{n+1\}}:\,\forall k\in\{1,\dots,n+1\},\\ \boldsymbol{x}_k\in\tilde{\mathcal{U}}(\boldsymbol{t}^{\{n+1\}},\boldsymbol{x}^{\{k-1\}},\boldsymbol{z}^{\{k-1\}})}} \sum_{k=0}^{N} r(t_k, \boldsymbol{x}_k, \boldsymbol{y}_k).$$

[19] To see why this follows, consider $f(x) \stackrel{\text{def}}{=} \max_{y\in A(x)} g(x,y)$ where $g(\cdot, y)$ is increasing for any y, and let $y^*(x)$ denote a maximizer in the problem. Then, for $x_1 \le x_2$, we have $f(x_1) = g(x_1, y^*(x_1)) \le g(x_2, y^*(x_1)) \le \max_{y\in A(x_2)} g(x_2, y) = f(x_2)$, where the inequality in the second step relies on $y^*(x_1) \in A(x_2)$, which is guaranteed by Assumption 3.

Proof. Running through the same arguments as in the proof of Theorem 5, let $\boldsymbol{y}_k^*(\boldsymbol{t}^{\{k+1\}}, \boldsymbol{x}^{\{k\}}, \boldsymbol{z}^{\{k\}})$ denote an optimal policy for the DM in (26). It can be checked that the operators $\min_{\boldsymbol{x}_k \in \tilde{\mathcal{U}}(\boldsymbol{t}^{\{k+1\}}, \boldsymbol{x}^{\{k-1\}}, \boldsymbol{z}^{\{k-1\}})} \max_{\boldsymbol{y}_k \in A(\boldsymbol{x}_k)}$ in (26) can be interchanged under a choice $\boldsymbol{y}_k^*(\boldsymbol{t}^{\{k+1\}}, \underline{\boldsymbol{x}}^{\{k\}}, \boldsymbol{z}^{\{k-1\}})$, since this action remains feasible for any $\boldsymbol{x}_k$ by Assumption 3, and nature's worst-case response under knowledge of this action remains $\underline{\boldsymbol{x}}^{\{k\}}$. Repeating the argument by induction then yields the result. ■

In view of Proposition 6, for purposes of recovering the worst-case reward, the DM can restrict attention to static policies for *both* monitoring *and* extraction; this simplifies the problem, and allows reconstructing a dynamic policy by repeatedly finding static policies.

Appendix C: High acute rejections for CAV

In our data, the distribution of number of acute rejections is highly skewed. It ranges from 0 to 12, while 75% were less than or equal to 3. So in our analysis, we define a binary variable, indicating high or low number of acute rejections. The threshold for high acute rejections are chosen so as to maximize the statistical power of our analysis. In particular, we look at the statistical significance of the binary variable in the regression model for reward function (Table 3).

Appendix D: Tables

QOL_1	QOL_2	QOL_3	QOL_{re}
0.8583	0.7138	0.5774	0.6456

Table 2 Quality of Life Factors

	Estimate	Standard Error	Significance
Intercept	1.06412	0.02503	***
Starting age (A)	-0.00134	0.00048	**
CAV stage ($i \in \{1,2,3\}$)	-0.06510	0.00835	***
Acute rejections ($j \in \{L,H\}$)	-0.03503	0.01412	*

Table 3 Regression Model for Survival Probability

Appendix E: IP Formulation for CAV Dynamic Monitoring Policy

Problem (20) in §4.3 can be reformulated as an IP as follows. We start by defining binary variables $\boldsymbol{y}$ as follows:

$$y_k^{ij} \stackrel{\text{def}}{=} \begin{cases} 1, & \text{if } \bar{x}^{ij}(t_k) > 0, \\ 0, & \text{otherwise} \end{cases}, \quad \forall i \in \{1,2,3\}, j \in \{L,H\}, k \in \{0,\ldots,N\}. \tag{27}$$

Similarly, consider the binary variable u_n such that $u_n = 1$ if and only if $t_n > b = 5.060$. Then, we define multiplicative variables $\boldsymbol{z}, \boldsymbol{w}$ and $\boldsymbol{r}$ as

$$z_n^{iji'j'} \stackrel{\text{def}}{=} y_n^{ij} \bar{x}^{i'j'}(t_n), \qquad w_n^{ij} \stackrel{\text{def}}{=} y_n^{ij}\, t, \qquad r_n^{ij} \stackrel{\text{def}}{=} y_n^{ij}(t_n - b)^+, \tag{28}$$

$(i,j) \to (i',j')$	$\beta_0^{iji'j'}$	$\beta_1^{iji'j'}$	$\beta_2^{iji'j'}$
1L → 2L	20.48 (1.10)	-0.22 (0.02)	-0.69 (0.05)
1H → 3H	17.57 (3.98)	-0.086 (0.07)	-0.42 (0.46)
1L → 3L	46.02 (3.60)	-0.53 (0.07)	-1.15 (0.22)
2L → 3L	9.57 (1.53)	-0.10 (0.03)	-0.13 (0.1)
1L → 1H	7.39 (2.48)	0.299 (0.07)	29.109 (3.62)
2L → 2H	2126.97 (2377.73)	-33.84 (38.99)	54.16 (175.57)
1L → 2H	39.451 (14.53)	0.31 (0.34)	107.34 (22.45)
2L → 3H	3428.443 (5392.32)	-54.245 (87.69)	17.031 (186.05)
1L → 3H	73.70 (33.43)	0.38 (0.78)	168.36 (45.09)
2H → 3H	2.02 (0.95)	0.033 (0.02)	-0.219 (0.15)
1H → 2H	9.59 (1.38)	-0.073 (0.02)	-0.25 (1.15)
3L → 3H	9.96 ($1.04 \cdot 10^{-11}$)	-0.05 ($2.41 \cdot 10^{-11}$)	-1.07 ($2.03 \cdot 10^{-11}$)

Table 4 Coefficients for mean time spent in each state. (Standard errors are reported in parenthesis.)

Policy	Min	25% Quantile	Median	75% Quantile	Max	Mean
ISHLT	3.2345	7.1207	8.0081	9.3416	10.2690	7.9976
Uniform RO	5.9200	7.3592	7.9155	8.2316	8.2316	7.7749
Static RO	6.0596	7.3868	8.2096	8.9909	8.9909	8.1411
Dynamic RO	6.6691	7.6948	8.3207	8.9909	9.8833	8.3409

Table 5 Rewards under different monitoring policies
($T = 10$ years, initial age = 50, number of monitoring = 9, ρ = 90%, number of iterations = 10^3).

for all $i, i' \in \{1,2,3\}, j, j' \in \{L, H\}$. Then, for sufficiently large $M > 0$ and sufficiently small $\epsilon > 0$, maximizing $g(t_n, \bar{x}^{2L}(t_n), \bar{x}^{3L}(t_n), \bar{x}^{3H}(t_n))$ in Problem (20) can be reformulated as follows:

$$\begin{aligned}
\max_{\substack{\bar{\boldsymbol{x}}^{2L}, \bar{\boldsymbol{x}}^{3L}, \bar{\boldsymbol{x}}^{3H}, \boldsymbol{t}, \\ s_n, u_n, \boldsymbol{y}, \boldsymbol{z}, \boldsymbol{w}, \boldsymbol{r}}} \quad & (1.5269\, t_n - 0.1445\, \bar{x}_n^{2L} - 0.1364\, \bar{x}_n^{3L} + 1.3968 - 1.1445\, s_n)(0.9990 - 0.0013\, age) \\
& - 0.0994\, w_n^{2L} + 0.0094\, z_n^{2L2L} + 0.0089\, z_n^{2L3L} - 0.0909 + 0.0745\, r_n^{2L} \\
& - 0.0994\, w_n^{3L} + 0.0094\, z_n^{3L2L} + 0.0089\, z_n^{3L3L} - 0.0909 + 0.0745\, r_n^{3L} \\
& - 0.0534\, w_n^{3H} + 0.0051\, z_n^{3H2L} + 0.0048\, z_n^{3H3L} - 0.0489 + 0.0401\, r_n^{3H}
\end{aligned}$$

$$\text{subject to } \bar{x}_0^{ij} = 0, \quad \bar{x}_N^{ij} \le T, \quad (i,j) \in \{2L, 3L, 3H\}, \tag{29a}$$

$$t_k \le t_{k+1}, \quad \bar{x}_k^{ij} \le \bar{x}_{k+1}^{ij}, \quad (i,j) \in \{2L, 3L, 3H\}, k \in \{0, 1, \ldots, n+1\} \tag{29b}$$

$$\bar{x}_k^{ij} \le M y_k^{ij}, \quad \bar{x}_k^{ij} \ge \epsilon y_k^{ij}, \quad (i,j) \in \{2L, 3L, 3H\}, k \in \{0, 1, \ldots, n+1\} \tag{29c}$$

$$t_n \geq b\,u_n, \quad t_n \leq b + M u_n \tag{29d}$$

$$s_n \geq -M u_n, \quad s_n \geq t_n - b - M(1-u_n) \tag{29e}$$

$$s_n \leq M u_n, \quad s_n \leq t_n - b + M(1-u_n) \tag{29f}$$

$$z_n^{iji'j'} \leq \bar{x}_n^{i'j'} + M(1-y_n^{ij}),\, z_n^{iji'j'} \geq \bar{x}_n^{i'j'} - M(1-y_n^{ij}), \tag{29g}$$

$$z_n^{iji'j'} \leq M y_n^{ij}, \quad z_n^{iji'j'} \geq -M y_n^{ij}, \quad (i,j),(i'j') \in \{2L, 3L, 3H\} \tag{29h}$$

$$w^{ij} \leq t_n + M(1-y_n^{ij}), \quad w_n^{ij} \geq t_n - M(1-y_n^{ij}), \tag{29i}$$

$$w_n^{ij} \leq M y_k^{ij}, \quad w_n^{ij} \geq -M y_n^{ij}, \tag{29j}$$

$$r^{ij} \leq s_n + M(1-y_n^{ij}), \quad r_n^{ij} \geq s_n - M(1-y_n^{ij}), \tag{29k}$$

$$r_n^{ij} \leq M y_k^{ij}, \quad r_n^{ij} \geq -M y_n^{ij}, \quad (i,j) \in \{2L, 3L, 3H\}, \tag{29l}$$

$$\bar{x}_{k+1}^{2L} \geq t_{k+1} - t_k + \ln(\rho)(c^{1L2L} + \beta_2^{1L2L}\, t_k) - M y_k^{2L}, \tag{29m}$$

$$\bar{x}_{k+1}^{2L} \geq \bar{x}_k^{2L} + t_{k+1} - t_k - M + M y_k^{2L}, \tag{29n}$$

$$\bar{x}_{k+1}^{3L} \geq t_{k+1} - t_k + \ln(\rho)(c^{1L3L} + \beta_2^{1L3L}\, t_k) - M y_k^{2L} - M y_k^{3L}, \tag{29o}$$

$$\bar{x}_{k+1}^{3L} \geq t_{k+1} - t_k + \ln(\rho)(c^{2L3L} + \beta_2^{2L3L}\, t_k) - M + M y_k^{2L} - M y_k^{3L}, \tag{29p}$$

$$\bar{x}_{k+1}^{3L} \geq \bar{x}_k^{3L} + t_{k+1} - t_k - 2M + M y_k^{2L} + M y_k^{3L}, \tag{29q}$$

$$\bar{x}_{k+1}^{3H} \geq t_{k+1} - t_k + \ln(\rho)(c^{1L3H} + \beta_2^{1L3H}\, t_k) - M y_k^{2L} - M y_k^{3L} - M y^{3H}, \tag{29r}$$

$$\bar{x}_{k+1}^{3H} \geq t_{k+1} - t_k + \ln(\rho)(c^{2L3H} + \beta_2^{2L3H}\, t_k) - M + M y_k^{2L} - M y_k^{3L} - M y^{3H}, \tag{29s}$$

$$\bar{x}_{k+1}^{3H} \geq t_{k+1} - t_k + \ln(\rho)(c^{3L3H} + \beta_2^{3L3H}\, t_k) - 2M + M y_k^{2L} + M y_k^{3L} - M y^{3H}, \tag{29t}$$

$$\bar{x}_{k+1}^{3H} \geq \bar{x}_k^{3H} + t_{k+1} - t_k - 3M + M y_k^{2L} + M y_k^{3L} + M y_k^{3H}, \quad k \in \{0, 1, \ldots, n\}. \tag{29u}$$

where $c^{iji'j'} = \beta_0^{iji'j'} + \beta_1^{iji'j'} age$. The inequalities in (29c) define the binary variables $\boldsymbol{y}$, the inequalities in (30d) define u_n, and the inequalities in (29e)-(29f) define s_n as $(t_n - b)^+$. Then, (29g)-(29h) and (29i)-(29l) define multiplicative variables $\boldsymbol{w}$ and $\boldsymbol{r}$ respectively. The inequalities (29m)-(30n), (29o)-(30q) and (29r)-(29u) correspond to (19a), (19b) and (19c), respectively. The problem of maximizing the term $g(t_{n+1}, \bar{x}^{2L}(t_{n+1}), \bar{x}^{3L}(t_{n+1}), \bar{x}^{3H}(t_{n+1}))$ in (20) can similarly be formulated as an IP.

When dynamically re-solving static problems, note that a patient could also be diagnosed as 1H, 2H or 3H. In such cases, the disease would progress from 1L to 1H, to 2H and to 3H. Then, we can follow a similar modeling approach and formulate the static monitoring problem as an IP in terms of $\bar{x}^{1H}, \bar{x}^{2H}, \bar{x}^{3H}$ as follows.

$$\begin{aligned}
\max_{\substack{\bar{\boldsymbol{x}}^{1H}, \bar{\boldsymbol{x}}^{2H}, \bar{\boldsymbol{x}}^{3H}, \boldsymbol{t}, \\ s_n, u_n, \boldsymbol{y}, \boldsymbol{z}, \boldsymbol{w}, \boldsymbol{r}}} \quad & (1.5269\, t_n - 0.1445\, \bar{x}_n^{2H} - 0.1364\, \bar{x}_n^{3H} + 1.3968 - 1.1445\, s_n)(0.9990 - 0.0013\, age) \\
& - 0.0994\, w_n^{2H} + 0.0094\, z_n^{2H2H} + 0.0089\, z_n^{2H3H} - 0.0909 + 0.0745\, r_n^{2H} \\
& - 0.0994\, w_n^{3H} + 0.0094\, z_n^{3H2H} + 0.0089\, z_n^{3H3H} - 0.0909 + 0.0745\, r_n^{3H} \\
& - 0.0534\, w_n^{1H} + 0.0051\, z_n^{1H2H} + 0.0048\, z_n^{1H3H} - 0.0489 + 0.0401\, r_n^{1H}
\end{aligned}$$

$$\text{subject to } \bar{x}_0^{ij} = 0, \quad \bar{x}_N^{ij} \leq T, \quad (i,j) \in \{1H, 2H, 3H\}, \tag{30a}$$

$$t_k \leq t_{k+1}, \quad \bar{x}_k^{ij} \leq \bar{x}_{k+1}^{ij}, \quad (i,j) \in \{1H, 2H, 3H\},\, k \in \{0, 1, \ldots, n+1\} \tag{30b}$$

$$\bar{x}_k^{ij} \le M y_k^{ij}, \quad \bar{x}_k^{ij} \ge \epsilon y_k^{ij}, \quad (i,j) \in \{1H, 2H, 3H\},\, k \in \{0,1,\dots,n+1\} \tag{30c}$$

$$t_n \ge b\, u_n, \quad t_n \le b + M u_n \tag{30d}$$

$$s_n \ge -M u_n, \quad s_n \ge t_n - b - M(1-u_n) \tag{30e}$$

$$s_n \le M u_n, \quad s_n \le t_n - b + M(1-u_n) \tag{30f}$$

$$z_n^{iji'j'} \le \bar{x}_n^{i'j'} + M(1-y_n^{ij}),\, z_n^{iji'j'} \ge \bar{x}_n^{i'j'} - M(1-y_n^{ij}), \tag{30g}$$

$$z_n^{iji'j'} \le M y_n^{ij}, \quad z_n^{iji'j'} \ge -M y_n^{ij}, \quad (i,j),(i'j') \in \{1H, 2H, 3H\} \tag{30h}$$

$$w^{ij} \le t_n + M(1-y_n^{ij}), \quad w_n^{ij} \ge t_n - M(1-y_n^{ij}), \tag{30i}$$

$$w_n^{ij} \le M y_k^{ij}, \quad w_n^{ij} \ge -M y_n^{ij}, \tag{30j}$$

$$r^{ij} \le s_n + M(1-y_n^{ij}), \quad r_n^{ij} \ge s_n - M(1-y_n^{ij}), \tag{30k}$$

$$r_n^{ij} \le M y_k^{ij}, \quad r_n^{ij} \ge -M y_n^{ij}, \quad (i,j) \in \{1H, 2H, 3H\}, \tag{30l}$$

$$\bar{x}_{k+1}^{1H} \ge t_{k+1} - t_k + \ln(\rho)(c^{1L1H} + \beta_2^{1L1H}\, t_k) - M y_k^{1H}, \tag{30m}$$

$$\bar{x}_{k+1}^{1H} \ge \bar{x}_k^{1H} + t_{k+1} - t_k - M + M y_k^{1H}, \tag{30n}$$

$$\bar{x}_{k+1}^{2H} \ge t_{k+1} - t_k + \ln(\rho)(c^{1L2H} + \beta_2^{1L2H}\, t_k) - M y_k^{1H} - M y_k^{2H}, \tag{30o}$$

$$\bar{x}_{k+1}^{2H} \ge t_{k+1} - t_k + \ln(\rho)(c^{1H2H} + \beta_2^{1H2H}\, t_k) - M + M y_k^{1H} - M y_k^{2H}, \tag{30p}$$

$$\bar{x}_{k+1}^{2H} \ge \bar{x}_k^{2H} + t_{k+1} - t_k - 2M + M y_k^{1H} + M y_k^{2H}, \tag{30q}$$

$$\bar{x}_{k+1}^{3H} \ge t_{k+1} - t_k + \ln(\rho)(c^{1L3H} + \beta_2^{1L3H}\, t_k) - M y_k^{1H} - M y_k^{2H} - M y^{3H}, \tag{30r}$$

$$\bar{x}_{k+1}^{3H} \ge t_{k+1} - t_k + \ln(\rho)(c^{1H3H} + \beta_2^{1H3H}\, t_k) - M + M y_k^{1H} - M y_k^{2H} - M y^{3H}, \tag{30s}$$

$$\bar{x}_{k+1}^{3H} \ge t_{k+1} - t_k + \ln(\rho)(c^{2H3H} + \beta_2^{2H3H}\, t_k) - 2M + M y_k^{1H} + M y_k^{2H} - M y^{3H}, \tag{30t}$$

$$\bar{x}_{k+1}^{3H} \ge \bar{x}_k^{3H} + t_{k+1} - t_k - 3M + M y_k^{1H} + M y_k^{2H} + M y_k^{3H}, \quad k \in \{0,1,\dots,n\}. \tag{30u}$$