Online Appendix
Boosting Sales and Customer Welfare from Premade Foods
(Let the Freshest Chicken Fly off the Shelf First)

In this online appendix, we present proofs for all the results in the paper.

EC.1. Notation

Throughout the appendix, we omit the dependency on the demand rate and the replenishment rate whenever it is clear from the context. We denote the retailer's policy by the pair (I,T), where $I \in \{F, L\}$ denotes the issuance rule and $T \in [0,\infty)$ denotes the shelf life, and use superscripts to denote the dependency of quantities of interest on the policy.

We introduce notations to describe inventory status at a given time. For a given policy (I,T), we let $z_t^{I,T}[a,b)$ denote the level of inventory whose age is in the interval $[a,b)$ at time t and $z_t^{I,T} := z_t^{I,T}[0,T)$ denote the total inventory level at time t, with the convention $z_t^{I,T}[a,a) = 0$. When a batch or a customer arrives at time t, we use $z_{t^-}^{I,T}[a,b)$ to denote the inventory level with age in $[a,b)$ under policy (I,T) right before a batch or a customer arrives at time t. For example, if a batch arrives at time t, the inventory level with age in $[0,x)$ is updated according to $z_t^{I,T}[0,x) = z_{t^-}^{I,T}[0,x) + B$ for $0 < x \le T$; if a customer arrives at time t, the inventory level evolves according to $z_t^{I,T}[0,x) = \left[z_{t^-}^{I,T}[0,x) - \left[1 - z_{t^-}^{I,T}[x,T)\right]^+\right]^+$ under FIFO and $z_t^{I,T}[0,x) = \left[z_{t^-}^{I,T}[0,x) - 1\right]^+$ under LIFO. In order to describe the performance of the inventory system, we introduce notations for long-run average measures. Suppose all customers who find an item on offer buy it under policy (I,T). For a given policy (I,T), we define the following cumulative measures: $Z_t^{I,T}[a,b) := \int_0^t z_t^{I,T}[a,b)dt$, $S_t^{I,T}[a,b) := \sum_{i=1}^{N_c(t)} \mathbf{1}_{\{z_{t_i}^{I,T}>0 \text{ and } \tau_{t_i}^{I,T}\in[a,b)\}}$ and $R_t^{I,T} := \sum_{i=1}^{N_c(t)} q(\tau_{t_i}^{I,T})\mathbf{1}_{\{z_{t_i}^{I,T}>0\}}$. We define notations for long-run average quantities as the accumulated quantities during time period $[0,t]$ divided by t as the time horizon t goes to ∞. Specifically, $Z^{I,T}[a,b) := \lim_{t\to\infty} \frac{1}{t} Z_t^{I,T}[a,b)$ denotes the inventory level with age in $[a,b)$, $S^{I,T}[a,b) := \lim_{t\to\infty} \frac{1}{t} S_t^{I,T}[a,b)$ denotes the sales of items of age in $[a,b)$ (which gives $S^{I,T} = S^{I,T}[0,T)$), and $R^{I,T} := \lim_{t\to\infty} \frac{1}{t} R_t^{I,T}$ denotes the *total* quality of *all* purchased items. Note that we may write $Q^{I,T} = R^{I,T}/S^{I,T}$.

The following table summarizes the main symbols that are used in the proofs.

EC.2. Supporting Lemmas

We start by introducing several lemmas that will enable us to prove the main results. We note that the lemmas hold regardless of whether customers are homogeneous or heterogeneous.

Table EC.1 Table of Notation

Notation	Description
B	Batch size
$I \in \{F, L\}$	Issuance order
T	Shelf life
$z_t^{I,T}[a,b)$	Level of inventory whose age is in the interval $[a,b)$ under policy (I,T) at time t
$z_t^{I,T}$	Total inventory level under policy (I,T) at time t
$N_c(t)$	Total number of customers who arrive during time period $[0,t]$
$N_r(t)$	Total number of items replenished during time period $[0,t]$
$\tau_t^{I,T}$	Age of the item on offer under policy (I,T) at time t if there is any
$S^{I,T}$	Sales under policy (I,T)
$R^{I,T}$	Total quality of all purchased items under policy (I,T)
$Q^{I,T}$	Purchased quality of an item under policy (I,T)
$W^{I,T}$	Customer welfare under policy (I,T)
$D^{I,T}$	Inventory disposal under policy (I,T)
$\bar{S}^I$	Maximal sales that can be achieved under issuance I
$T_I(S)$	Unique shelf life that yields sales S under issuance I
$R^I(S)$	Total quality of all purchased items under issuance I when the sales is S
$V^{I,T}$	Retailer's objective under policy (I,T)
$V^I(S)$	Retailer's objective under issuance I when the sales is S
T_I^*	Optimal shelf life when the retailer does not use timestamps under issuance I
T_{ts}^*	Optimal shelf life when the retailer uses timestamps under issuance I
$T_{I,\text{ts}}^*$	Optimal shelf life when the retailer uses timestamps under issuance I

The following lemma compares the inventory position under different shelf lives and issuance rules.

LEMMA EC.1. *Suppose all arriving customers purchase the product if it is available. Then, for any sample path of batch and customer arrivals, the following results hold almost surely:*

(a) Under FIFO and for any time t and age x, the inventory level of items of age at least x at time t is increasing in the shelf life. That is, for any shelf lives $T_1 < T_2$,

$$z_t^{F,T_1}[x,T_1) \le z_t^{F,T_2}[x,T_2).$$

(b) Under FIFO, the inventory level of items of age strictly less than x increases with the shelf life, whereas under LIFO it is independent of the shelf life. That is, for any time t, shelf lives $T_1 \le T_2$ and age level $x \le T_1$,

$$z_t^{F,T_1}[0,x) \le z_t^{F,T_2}[0,x) \text{ and } z_t^{L,T_1}[0,x) = z_t^{L,T_2}[0,x).$$

(c) For any time t and for any age $x \in [0,T]$, the shelf holds more items of age strictly less than x under FIFO than under LIFO. That is,

$$z_t^{F,T}[0,x) \ge z_t^{L,T}[0,x). \tag{EC.2.1}$$

(d) Under LIFO, for any sequences of customer arrivals $\{\tilde{t}_n\} \subseteq \{\tilde{t}'_n\}$, time t and age $x \in [0,T]$, the shelf holds less items of age strictly less than x with customer arrivals $\{t_n\}$ than with $\{t'_n\}$. That is,

$$z_t^{L,T}([0,x);\{\tilde{t}_n\}) \geq z_t^{L,T}([0,x);\{\tilde{t}'_n\}). \tag{EC.2.2}$$

We remark that Part (b) implies that the inventory level is increasing with the shelf life at any given time and for any issuance rule. That is, $z_t^{I,T_1} \leq z_t^{I,T_2}$ for any time t, shelf lives $T_1 \leq T_2$ and any $I \in \{F, L\}$.

Proof of Lemma EC.1. (a) Because the retailer begins with an empty shelf regardless of the policy used, the claim holds at $t = 0$. Note that if the claim holds at some t and there is no further replenishment or customer arrival during $[t,t')$, $t < t'$, the claim continues to hold at t' because only aging and disposal of existing inventory occur and inventory is disposed of at an older age with shelf life T_2 than T_1. Thus, we only need to demonstrate the claim for the sequence of times $\{t_n\}_{n\geq 0}$ at which batches or customers arrive. (Recall that a batch arrives at $t = 0$ so $t_0 = 0$.) The proof is by induction on n. Suppose that the claim holds at t_n for some $n \geq 0$. If a batch arrives at t_{n+1}, clearly the claim continues to hold at t_{n+1} because the items in the batch are of age 0 under both (F,T_1) and (F,T_2). If a customer arrives at t_{n+1}, we have

$$\begin{aligned} z_{t_{n+1}}^{F,T_1}[x,T_1) &= \left[z_{t_{n+1}^-}^{F,T_1}[x,T_1) - 1\right]^+ = \left[z_{t_n}^{F,T_1}[x-(t_{n+1}-t_n),T_1) - 1\right]^+ \\ &\leq \left[z_{t_n}^{F,T_2}[x-(t_{n+1}-t_n),T_2) - 1\right]^+ = \left[z_{t_{n+1}^-}^{F,T_2}[x,T_2) - 1\right]^+ = z_{t_{n+1}}^{F,T_2}[x,T_2), \end{aligned}$$

where the inequality follows from the induction hypothesis. Because the claim continues to hold at t_{n+1}, the induction principle implies that the claim holds at $\{t_n\}_{n\geq 0}$ and thus, at any $t \geq 0$.

(b) Using similar arguments as in the proof of Part (a), it can be shown that we only need to prove the claim for the sequence of times $\{t_n\}_{n\geq 0}$ when batches or customers arrive. The proof is by induction on n. Suppose that the claim holds at t_n for some $n \geq 0$. If a batch arrives at t_{n+1}, by the induction hypothesis, we have

$$z_{t_{n+1}}^{F,T_1}[0,x) = B + z_{t_n}^{F,T_1}[0,x-(t_{n+1}-t_n)) \leq B + z_{t_n}^{F,T_2}[0,x-(t_{n+1}-t_n)) = z_{t_{n+1}}^{F,T_2}[0,x)$$

under FIFO and

$$z_{t_{n+1}}^{L,T_1}[0,x) = B + z_{t_n}^{L,T_1}[0,x-(t_{n+1}-t_n)) = B + z_{t_n}^{L,T_2}[0,x-(t_{n+1}-t_n)) = z_{t_{n+1}}^{L,T_2}[0,x)$$

under LIFO for any $x \in [0, T_1]$. Suppose a customer arrives at t_{n+1}; under FIFO, we obtain

$$\begin{aligned} z^{F,T_1}_{t_{n+1}}[0,x) &= \left[z^{F,T_1}_{t^-_{n+1}}[0,x) - \left[1 - z^{F,T_1}_{t^-_{n+1}}[x,T_1) \right]^+ \right]^+ \\ &= \left[z^{F,T_1}_{t_n}[0,x-(t_{n+1}-t_n)) - \left[1 - z^{F,T_1}_{t_n}[x-(t_{n+1}-t_n),T_1) \right]^+ \right]^+ \\ &\overset{(i)}{\le} \left[z^{F,T_2}_{t_n}[0,x-(t_{n+1}-t_n)) - \left[1 - z^{F,T_2}_{t_n}[x-(t_{n+1}-t_n),T_2) \right]^+ \right]^+ \\ &= \left[z^{F,T_2}_{t^-_{n+1}}[0,x) - \left[1 - z^{F,T_2}_{t^-_{n+1}}[x,T_1) \right]^+ \right]^+ = z^{F,T_2}_{t_{n+1}}[0,x), \end{aligned}$$

where (i) follows from observing that the function $[x-(1-y)^+]^+$ is increasing in both x and y, our induction hypothesis that $z^{F,T_1}_{t_n}[0,x-(t_{n+1}-t_n)) \le z^{F,T_2}_{t_n}[0,x-(t_{n+1}-t_n))$, and $z^{F,T_1}_{t_n}[x-(t_{n+1}-t_n),T_1) \le z^{F,T_2}_{t_n}[x-(t_{n+1}-t_n),T_2)$ from Part (a); under LIFO, we have

$$\begin{aligned} z^{L,T_1}_{t_{n+1}}[0,x) &= \left[z^{L,T_1}_{t^-_{n+1}}[0,x) - 1 \right]^+ = \left[z^{L,T_1}_{t_n}[0,x-(t_{n+1}-t_n)) - 1 \right]^+ \\ &\overset{(ii)}{=} \left[z^{L,T_2}_{t_n}[0,x-(t_{n+1}-t_n)) - 1 \right]^+ = \left[z^{L,T_2}_{t^-_{n+1}}[0,x) - 1 \right]^+ = z^{L,T_2}_{t_{n+1}}[0,x), \end{aligned}$$

where (ii) follows from our induction hypothesis that $z^{L,T_1}_{t_n}[0,x-(t_{n+1}-t_n)) = z^{L,T_2}_{t_n}[0,x-(t_{n+1}-t_n))$. Because the claim continues to hold at t_{n+1}, the induction principle implies that the claim holds at $\{t_n\}_{n\ge 0}$ and thus, at any $t \ge 0$.

(c) As before, let $\{t_n\}_{n\ge 0}$ denote the sequence of times at which either a batch or a customer arrives with $t_0 = 0$. Note that if the claim holds at time $t \ge 0$ and no batch or customer arrives during $(t, t']$, the claim continues to hold at t' because

$$z^{F,T}_{t'}[0,x) = z^{F,T}_{t}[0,x-(t'-t)) \ge z^{L,T}_{t}[0,x-(t'-t)) = z^{L,T}_{t'}[0,x).$$

Thus, it is sufficient to prove that the claim holds at the discrete times $\{t_n\}_{n\ge 0}$. The rest of the proof is by induction on n. The claim clearly holds at $t_0 = 0$ because the shelf initially holds the same inventory. Suppose now that the claim holds at t_n for some $n \in \mathbb{N} \cup \{0\}$. If a batch arrives at t_{n+1}, we have:

$$\begin{aligned} z^{F,T}_{t_{n+1}}[0,x) &= z^{F,T}_{t_n}[0,x-(t_{n+1}-t_n)) + B \cdot \mathbf{1}_{\{t_{n+1}-t_n<x\}} \\ &\ge z^{L,T}_{t_n}[0,x-(t_{n+1}-t_n)) + B \cdot \mathbf{1}_{\{t_{n+1}-t_n<x\}} = z^{L,T}_{t_{n+1}}[0,x), \end{aligned}$$

where the inequality follows from the induction hypothesis. If a customer arrives at t_{n+1}:

$$z^{F,T}_{t_{n+1}}[0,x) = \left\{ z^{F,T}_{t}[0,x-(t_{n+1}-t_n)) - \left(1 - z^{F,T}_{t}[x-(t_{n+1}-t_n), T-(t_{n+1}-t_n)) \right)^+ \right\}^+$$

$$
\begin{aligned}
&= \Big\{ z_t^{F,T}\left[0, x-(t_{n+1}-t_n)\right) - \\
&\qquad\qquad \Big[1-\left(z_t^{F,T}\left[0, T-(t_{n+1}-t_n)\right) - z_t^{F,T}\left[0, x-(t_{n+1}-t_n)\right)\right)\Big]^+\Big\}^+ \\
&\geq \Big\{ z_t^{L,T}\left[0, x-(t_{n+1}-t_n)\right) - \\
&\qquad\qquad \Big[1-\left(z_t^{L,T}\left[0, T-(t_{n+1}-t_n)\right) - z_t^{L,T}\left[0, x-(t_{n+1}-t_n)\right)\right)\Big]^+\Big\}^+ \\
&= \Big\{ z_t^{L,T}\left[0, x-(t_{n+1}-t_n)\right) - \Big(1 - z_t^{L,T}\left[x-(t_{n+1}-t_n), T-(t_{n+1}-t_n)\right)\Big)^+\Big\}^+ \\
&\geq \Big(z_t^{L,T}\left[0, x-(t_{n+1}-t_n)\right) - 1\Big)^+ = z_{t_{n+1}}^{L,T}\left[0, x\right).
\end{aligned}
$$

The first inequality follows from the induction hypothesis and because the function $\{x-[1-(y-x)]^+\}^+$ is increasing in both x and y for any $x \leq y$; the second inequality follows because

$$z_t^{L,T}\left[x-(t_{n+1}-t_n), T-(t_{n+1}-t_n)\right) \geq 0$$

and the function $[x-(1-y)^+]^+$ is increasing in y. Therefore, (EC.2.1) holds for all n by the induction principle and the claim follows. Hence, if the shelf is in stock under LIFO, it is also in stock under FIFO.

(d) If we let $\{\hat{t}_n\}_{n\geq 0}$ denote the sequence of times at which a batch arrives, then $\{t_n\} := \{\hat{t}_n\} \cup \{\tilde{t}_n\}$ and $\{t'_n\} := \{\hat{t}_n\} \cup \{\tilde{t}'_n\}$ are the sequences of times at which either a batch or a customer arrives when the customer arrival times are $\{\tilde{t}_n\}$ and $\{\tilde{t}'_n\}$, respectively. By a similar argument as in the proof of Part (b), we only need to prove the claim for the sequence of times $\{t'_n\}(\supset \{t_n\})$. The rest of the proof is by induction on n. The claim clearly holds at $t'_0 = 0$. Suppose (EC.2.2) holds at t'_n. With either sequence of customer arrivals, the inventory level of items of age strictly less than x evolves according to

$$
z_{t'_{n+1}}^{L,T}[0,x) = \begin{cases} z_{t'_n}^{L,T}[0,x) + B & \text{if a batch arrives at } t'_{n+1}, \\ \left(z_{t'_n}^{L,T}[0,x) - 1\right)^+ & \text{if a customer arrives at } t'_{n+1}, \\ z_{t'_n}^{L,T}[0,x) & \text{if neither customer nor batch arrives at } t'_{n+1}. \end{cases}
$$

From this, one can infer that (EC.2.2) continues to hold at t'_{n+1}. Therefore, (EC.2.2) holds at all of $\{t'_n\}$ by the induction principle and the claim follows. Q.E.D.

The next result proves that all the relevant long-run average quantities exist and are finite.

LEMMA EC.2. *The inventory process is regenerative under any policy (I,T). Furthermore, $Z^{I,T}$, $S^{I,T}$ and $R^{I,T}$ exist and are finite.*

Proof of Lemma EC.2. Let us fix a policy (I,T) and assume that the first batch arrives at $t=0$ to an empty shelf; the proof can be easily modified for the general case where the shelf is nonempty at time $t=0$ and the first batch arrives at some time $t>0$. We define a cycle to start each time an arriving batch finds an empty shelf. We denote the length of the n-th cycle by X_n and consider the following three types of total reward earned during the n-th cycle: the integral of the inventory level over the cycle Z_n, the number of items sold during the cycle S_n and the total quality purchased during the cycle R_n. We additionally define $Y_n := R_n - q(T)S_n$ for ease of exposition. Because batches arrive according to a stochastic point process with independent and identically distributed interarrival times and customers arrive according to a Poisson process and the two processes are independent of each other, one can infer that the inventory process is regenerative and the pairs in each of $\{(X_n, Z_n)\}_{n\geq 1}$, $\{(X_n, S_n)\}_{n\geq 1}$ and $\{(X_n, Y_n)\}_{n\geq 1}$ are independent and identically distributed.

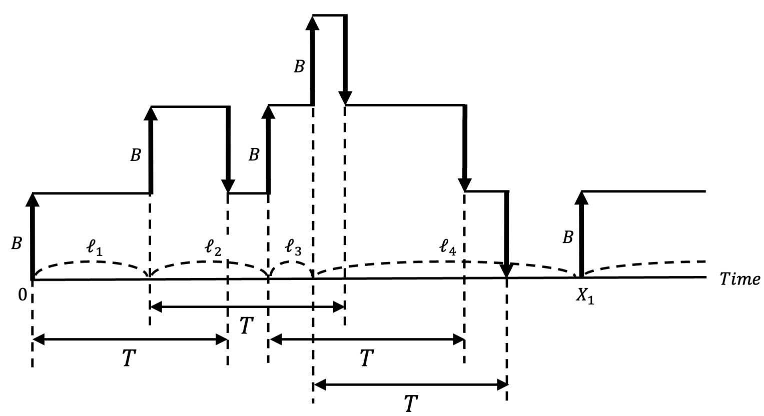


Figure EC.1 Inventory Level Over Time without Customer Arrivals.

We argue that $\mathbb{E}\,X_n < \infty$. Without loss of generality, we consider the first cycle. Clearly, the cycle would last longer without demand because the shelf would always hold more inventory than with demand. Thus, it is sufficient to prove that the claim holds when there is no demand. Suppose there is no demand. Note that the cycle continues if and only if a new batch arrives before the last batch is disposed since there is no demand. Suppose the cycle ends and a new cycle starts at the $m+1$-th arrival from the batch process for some integer $m\geq 1$, in which case the ending cycle has length $\sum_{i=1}^{m}\ell_i$. This implies that the i-th batch (for each $2\leq i\leq m$) arrived within T since the arrival of the $(i-1)$-th batch (i.e., $\ell_i\leq T$), whereas the $(m+1)$-th batch arrived after T since the m-th batch ($\ell_{m+1}>T$). It follows that the length of the cycle is bounded by mT ($\sum_{i=1}^{m}\ell_i < mT$). Let us denote the probability that the cycle ends after the first batch by $p_0 \equiv P(\ell_1\geq T)\in(0,1)$. Observe that the probability that the cycle ends after m replenishments is $(1-p_0)^{m-1}p_0$. Therefore, the expected length of the cycle satisfies

$$\begin{aligned}\mathbb{E}\,A_1 &= \sum_{m=1}^{\infty}\mathbb{E}\left[\sum_{i=1}^{m}\ell_i\middle|\text{the cycle ends after the } m\text{-th batch}\right]\\ &\qquad\qquad\cdot\mathbb{P}\,(\text{The cycle ends after the } m\text{-th batch})\\ &\le \sum_{m=1}^{\infty}\mathbb{E}\left[mT\middle|\text{the cycle ends after the } m\text{-th batch}\right] = T\sum_{m=1}^{\infty}(1-p_0)^{m-1}p_0 = \frac{T}{p_0} < \infty,\end{aligned}$$

and the claim follows.

We also claim that $\mathbb{E}\,|Z_n|$, $\mathbb{E}\,|S_n|$, $\mathbb{E}\,|Y_n| < \infty$. Because both sales S_n and the average inventory level during the n-th cycle Z_n are bounded by the number of replenished items during the n-th cycle, we have $\mathbb{E}\,S_n, \mathbb{E}\,Z_n \le \mathbb{E}\,[\mu B X_n] = \mu B\cdot\mathbb{E}\,X_n < \infty$. From the fact that the quality of any purchased item lies in $(q(T), q(0)]$, we obtain that $Y_n \in (0, (q(0)-q(T))S_n]$. Combining this observation with $\mathbb{E}\,S_n < \infty$ yields $\mathbb{E}\,|Y_n| < \infty$.

Let us denote by $N(t)$ the number of completed cycles up to time t and define $\underline{Z}_t^{I,T} := \sum_{n=1}^{N(t)} Z_n$, $\bar{Z}_t := \sum_{n=1}^{N(t)+1} Z_n$, $\underline{S}_t^{I,T} := \sum_{n=1}^{N(t)} S_n$, $\bar{S}_t := \sum_{n=1}^{N(t)+1} S_n$, $\underline{Y}_t^{I,T} := \sum_{n=1}^{N(t)} Y_n$, $\bar{Y}_t := \sum_{n=1}^{N(t)+1} Y_n$ and $Y_t^{I,T} := \sum_{i=1}^{N_c(t)} q(\tau_{t_i}^{I,T})\mathbf{1}_{\{z_{t_i}^{I,T}>0\}}$. Because $Z_t^{I,T}$, $S_t^{I,T}$ and $Y_t^{I,T}$ are increasing in t, we have $\underline{Z}_t^{I,T} \le Z_t^{I,T} \le \bar{Z}_t^{I,T}$, $\underline{S}_t^{I,T} \le S_t^{I,T} \le \bar{S}_t^{I,T}$ and $\underline{Y}_t^{I,T} \le Y_t^{I,T} \le \bar{Y}_t^{I,T}$. Given $\mathbb{E}\,X_n < \infty$, $\mathbb{E}\,|Z_n| < \infty$, $\mathbb{E}\,|S_n| < \infty$ and $\mathbb{E}\,|Y_n| < \infty$, the renewal reward theorem (see Theorem 3.6.1 in Ross 1996) implies that $\lim_{t\to\infty}\frac{1}{t}\underline{Z}_t^{I,T} = \lim_{t\to\infty}\frac{1}{t}\bar{Z}_t^{I,T} = \mathbb{E}[Z_n]/\mathbb{E}[X_n]$, $\lim_{t\to\infty}\frac{1}{t}\underline{S}_t^{I,T} = \bar{S}_t^{I,T} = \mathbb{E}[S_n]/\mathbb{E}[X_n]$ and $\lim_{t\to\infty}\frac{1}{t}\underline{Y}_t^{I,T} = \frac{1}{t}\bar{Y}_t^{I,T} = \mathbb{E}[Y_n]/\mathbb{E}[X_n]$. Applying the squeeze theorem gives that

$$Z^{I,T} = \lim_{t\to\infty}\frac{1}{t}Z_t^{I,T} = \frac{\mathbb{E}[Z_1]}{\mathbb{E}[X_1]} \quad\text{and}\quad S^{I,T} = \lim_{t\to\infty}\frac{1}{t}S_t^{I,T} = \frac{\mathbb{E}[S_1]}{\mathbb{E}[X_1]},$$

from which we get

$$R^{I,T} = \lim_{t\to\infty}\frac{1}{t}R_t^{I,T} = \lim_{t\to\infty}\frac{1}{t}\left(Y_t^{I,T} + q(T)S_t^{I,T}\right) = \frac{\mathbb{E}[Y_1] + q(T)\,\mathbb{E}[S_1]}{\mathbb{E}[X_1]}.$$

The proof of the existence and finiteness of $Z^{I,T}[a,b)$ and other long-run average values follows similar arguments and is omitted for conciseness. Q.E.D.

We make the dependency of functions on the customer arriving rate λ and replenishment rate μ explicit whenever it clarifies arguments. For example, $S^{I,T}(\lambda,\mu)$, $Q^{I,T}(\lambda,\mu)$, $R^{I,T}(\lambda,\mu)$ and $W^{I,T}(\lambda,\mu)$ denote the sales, purchased quality, the total quality of sold items and customer welfare, respectively, under policy (I,T) and with customer arriving rate λ and replenishment rate μ.

The next Lemma provides structural results concerning the sales $S^{I,T}$.

LEMMA EC.3. *Suppose all arriving customers purchase the product if it is available. Then, the following results hold:*

(a) $S^{I,T}$ *is continuous and strictly increasing in* T *for* $I \in \{F, L\}$.

(b) $S^{L,T}[0,x) = S^{L,x}$ *for any* $x \in [0,T]$. *Moreover,* $S^{L,T}$ *is concave in* T.

(c) $S^{I,T}(\lambda,\mu)$ *is increasing in* λ *for* $I \in \{F, L\}$.

(d) $S^{L,T}(\lambda,\mu)$ *is continuous in* (T,λ).

(e) $0 < S^{I,T}(\lambda,\mu) \leq \min(\lambda,\mu B)$ *for* $I \in \{F, L\}$. *Moreover, we have* $0 < \bar{S}^I := \lim_{T\to\bar{T}^I} S^{I,T} \leq \min(\lambda,\mu B)$ *for* $I \in \{F, L\}$.

Proof of Lemma EC.3. (a) Fix the initial inventory status and a sample path of batch and customer arrivals. Recall that $N_c(t)$ denotes the total number of customer arrivals during $[0,t]$. Let us denote the total number of items replenished during $[0,t]$ by $N_r(t)$. Clearly, both $N_c(t)$ and $N_r(t)$ are counting processes. We first prove that $S^{I,T}$ is strictly increasing in T. Take any two shelf lives $T_1 < T_2$. Recall from Lemma EC.1 that the inventory position is greater under (I,T_2) than under (I,T_1) at any given time. Thus, (I,T_2) generates greater sales than (I,T_1) during $[0,t]$ for any $t > 0$, implying that $S^{I,T_1} \leq S^{I,T_2}$.

We argue that the inequality is indeed strict, i.e., $S^{I,T_1} < S^{I,T_2}$. We use the definition of cycle in the proof of Lemma EC.2. That is, a new cycle starts when a batch arrives to an empty shelf. Suppose a new cycle starts at t under (I,T_2). Then, Lemma EC.1 implies that a new cycle starts at t under (I,T_1), too. Note that there is positive probability that no batch arrives during $[t,t+T_2)$ (so that the shelf becomes empty at $t+T_2$ under both policies) and at least one customer arrives during $[t+T_1,t+T_2)$. When this occurs, because no item is sold during $[t,t+T_2)$ under (I,T_1) whereas at least one item is sold during $[t+T_1,t+T_2)$ under (I,T_2), we see strictly higher sales under (I,T_2) than (I,T_1) during this cycle. This proves that long-run sales are strictly larger under (I,T_2) than (I,T_1).

Next, we prove the continuity of $S^{I,T}$ with respect to T under $I \in \{F, L\}$. Let $T > 0$ and $\epsilon > 0$ be given and define $\delta := \min\left\{\frac{\epsilon}{2\lambda\mu}, T\right\}$. We shall show that we have $|S^{I,T'} - S^{I,T}| < \epsilon$ for any $T' \in (T-\delta, T+\delta)$. First, suppose that $T' \in [T, T+\delta)$. Because we have $S^{I,T'} - S^{I,T} \geq 0$ by the argument above, it is sufficient to show that $S^{I,T'} - S^{I,T} < \epsilon$. It is useful to conceptually divide the shelf under (I,T') into two separate blocks: block $[0,T)$ contains items of age less than T and block $[T,T')$ contains items of age at least T. Thus, all items first stay in block $[0,T)$ then if not sold, move on to the older block $[T,T')$. Observe that block $[0,T)$ under (I,T') receives batches at the same rate under (I,T) and receives customers at a smaller rate than λ (because customers purchase from the other block $[T, T')$ before they reach block $[0, T)$ under FIFO). On the other hand, block $[T, T')$ under (I, T') receives batches that are smaller than B because some items in a batch may be sold while staying in block $[0, T)$ and receives customers at a smaller rate than λ (because customers reach the newer block first under LIFO). Combining these observations, we have

$$S^{I,T'} = S^{I,T'}[0,T) + S^{I,T'}[T,T') \le S^{I,T} + S^{I,T'-T} \le S^{I,T} + S^{I,\delta},$$

where the last inequality follows from the monotonicity of $S^{I,T}$ because $T' - T < \delta$. Hence, it remains to show that $S^{I,\delta} < \epsilon$. Note that each batch stays on the shelf for at most δ under (I, δ). Thus, the shelf is in stock for at most $\delta N_r(t)$ during $[0, t)$ considering potential overlaps. Then, the Strong Law of Large Numbers for renewal theory (see, e.g., Ross 1996) implies that assuming no customer arrivals, the shelf is in stock with probability at most $\frac{\delta N_r(t)}{t}$ which converges to $\delta \mu B$ as $t \to \infty$ almost surely. The PASTA ("Poisson Arrivals See Time Averages") property then implies that each arriving customer finds the shelf in stock with probability at most $\delta\mu \le \frac{\epsilon}{2\lambda\mu} \cdot \mu = \frac{\epsilon}{2\lambda}$, yielding $S^{I,\delta} \le \frac{\epsilon}{2\lambda} \cdot \lambda = \frac{\epsilon}{2} < \epsilon$. The proof for the other case $T' \in (T - \delta, T)$ is similar and omitted.

(b) Lemma EC.1(b) implies that the dynamics of the inventory system up to age x under (L, T) is identical to that of (L, x), yielding $S^{L,T}[0, x) = S^{L,x}$.

For the latter result, let us show that the incremental sales $S^{L,T+\epsilon} - S^{L,T}$ are decreasing in T for any given $\epsilon > 0$. This can be readily seen from the fact that the demand for an item decreases with its age because customers buy the youngest item first under LIFO.

(c) Under either FIFO or LIFO, with a higher demand rate, each item has a greater chance to be presented to a customer before being discarded. Hence, there is less inventory disposal with a higher demand rate.

(d) We first prove that $S^{L,T}(\lambda, \mu)$ is Lipschitz continuous in λ. Specifically, we will show that for any shelf life T and demand rates λ and λ',

$$\left|S^{L,T}(\lambda', \mu) - S^{L,T}(\lambda, \mu)\right| \le |\lambda' - \lambda|. \tag{EC.2.3}$$

Due to Part (c), it is sufficient to show the following for $\lambda < \lambda'$:

$$S^{L,T}(\lambda', \mu) - S^{L,T}(\lambda, \mu) \le \lambda' - \lambda \quad \Leftrightarrow \quad \lambda - S^{L,T}(\lambda, \mu) \le \lambda' - S^{L,T}(\lambda', \mu). \tag{EC.2.4}$$

That is, stockouts occur more frequently with demand rate λ' than with λ. Lemma EC.1(d) implies that the shelf becomes empty more frequently with demand rate λ' than λ. The inequality (EC.2.4) then follows from the PASTA ("Poisson Arrivals See Time Averages") property.

We next show that $S^{L,T}(\lambda,\mu)$ is continuous in (T,λ). Let (T,λ) be given and take an arbitrary $\epsilon > 0$. By Part (a), there exists $\delta_2 > 0$ such that we have

$$\left|S^{L,T'}(\lambda,\mu) - S^{L,T}(\lambda,\mu)\right| < \frac{\epsilon}{2} \text{ if } |T' - T| < \delta_2. \tag{EC.2.5}$$

Thus, if we define $\delta_1 := \frac{\epsilon}{2}$ and $\delta := \min(\delta_1,\delta_2)$, for any (T',λ') such that $\|(T',\lambda') - (T,\lambda)\| < \delta$ we have

$$\begin{aligned}\left|S^{L,T'}(\lambda',\mu) - S^{L,T}(\lambda,\mu)\right| &\overset{(i)}{\le} \left|S^{L,T'}(\lambda',\mu) - S^{L,T'}(\lambda,\mu)\right| + \left|S^{L,T'}(\lambda,\mu) - S^{L,T}(\lambda,\mu)\right| \\ &\overset{(ii)}{\le} |\lambda' - \lambda| + \frac{\epsilon}{2} \overset{(iii)}{<} \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,\end{aligned}$$

where (i) follows from the triangle inequality and (ii) follows from (EC.2.3) and (EC.2.5) where we used $|T' - T| \le \|(T',\lambda') - (T,\lambda)\| < \delta \le \delta_2$ and (iii) follows because $|\lambda' - \lambda| \le \|(T',\lambda') - (T,\lambda)\| < \delta \le \delta_1 = \frac{\epsilon}{2}$. This concludes the proof.

(e) Lastly, we derive the bounds for $S^{I,T}$, $I \in \{F, L\}$. Obviously, $S^{I,T}$ is non-negative. It is easy to see that the number of items sold during $[0,t]$ cannot exceed the number of customers arriving during $[0,t]$ ($N_c(t)$) or the number of items replenished during $[0,t]$ ($N_r(t)$). Furthermore, we have $\frac{N_c(t)}{t} \to \lambda$ and $\frac{N_r(t)}{t} \to \mu B$ as $t \to \infty$ from renewal theory (see Proposition 3.3.1 in Ross 1996). Therefore, we obtain that the sales is bounded above by $\min(\lambda,\mu B)$. As the shelf life tends to infinity, the same number of items are sold during any time period under both FIFO and LIFO because items are never disposed of. Thus, we have the same limiting sales under both issuance rules, and it is finite and bounded by $\min(\lambda,\mu B)$ by the argument above. Q.E.D.

In view of the one-to-one correspondence between the shelf life T and sales $S^{I,T}$ established in Lemma EC.3, we introduce the following definition that is useful subsequently.

DEFINITION EC.1. Assuming that customers buy the product if it is in stock, for any issuance I and any sales $S \le \bar{S}^I$, we define $T^I(S)$ as the (unique) shelf life that yields sales S and $R^I(S)$ as the total quality of purchased items when sales are S, i.e., $S^{I,T^I(S)} = S$ and $R^I(S) := R^{I,T}$ where $T = T_I(S)$ is the shelf life satisfying $S^{I,T} = S$.

Obviously, we have $T^I(S^{I,T}) = T$ for $T > 0$ and $I \in \{F, L\}$.

Finally, our last lemma provides additional structural results on the quality of sold items.

LEMMA EC.4. *Suppose all arriving customers purchase the product if it is available. Then, we have the following:*

(a) $R^F(S)$ *is continuous in* S.

(b) $R^L(S)$ *is differentiable and strictly concave in* S*. Furthermore, its derivative is* $R^{L'}(S) = q(T_L(S))$.

(c) $R^I(S;\lambda_1,\mu) \le R^I(S;\lambda_2,\mu)$ *for any sales* S*, customer arriving rates* $\lambda_1 < \lambda_2$*, batch replenishment rate* μ *and issuance* $I \in \{F, L\}$.

(d) $Q^{F,T}(\lambda,\mu)$ *increases with* λ *for any shelf life* T.

(e) $Q^{L,T}(\lambda,\mu)$ *is continuous in* (T,λ).

Proof of Lemma EC.4. (a) Let S be given and let T be the shelf life such that $S^{L,T} = S$. Take $S' > S$ and let T' be the shelf life such that $S^{L,T'} = S'$. By Lemma EC.3(a), we have $T < T'$. We show that $R^F(S') \to R^F(S)$ as S' tends to S from above. Note that when a customer arrives, Lemma EC.1 implies that there are three cases to consider.

Case 1. When both policies (F,T) and (F,T') have sales, the difference in the purchased quality between the two policies is at most $q_0 + |q(T')|$.

Case 2. When only (F,T') has a sale, the purchased quality is at most $\max(|q(T')|, q_0) \le q_0 + |q(T')|$.

Case 3. When none of the policies has a sale, the purchased quality is 0.

Hence, combining these observations gives that

$$|R(S') - R(S)| \le (q_0 + |q(T')|) \cdot |S' - S|.$$

Because $\lim_{T'\to T^+} q(T') = q(T)$ and $\lim_{T'\to T^+} |S' - S| = \lim_{T'\to T^+} |S^{F,T'} - S^{F,T}| = 0$, the Squeeze Theorem implies that $\lim_{S'\to S^+} R^F(S') = R^F(S)$. The proof of $\lim_{S'\to S^-} R^F(S') = R^F(S)$ is similar and thus omitted.

(b) Let S be given and let T_1 be the shelf life such that $S^{L,T_1} = S$. We show that $R^L(S)$ is differentiable with respect to S, and its derivative at S is equal to $q(T_1)$. Take any shelf life $T_2 > T_1$. Recall that the sale of an item does not affect the dynamics afterwards (i.e., the sales of later items). This implies that the dynamics of items with age in $[T_1, T_2)$ does not affect the dynamics of items with age in $[0, T_1)$. Hence, we have $S^{L,T_2} - S^{L,T_1} = S^{L,T_2}[T_1,T_2)$ and $R^{L,T_2} - R^{L,T_1} = R^{L,T_2}[T_1,T_2)$. Because each item that is sold at an age in $[T_1,T_2)$ has quality in $(q(T_2), q(T_1)]$, we have

$$q(T_2)S^{L,T_2}[T_1,T_2) < R^{L,T_2} - R^{L,T_1} = R^{L,T_2}[T_1,T_2) \le q(T_1)S^{L,T_2}[T_1,T_2).$$

Dividing both sides by $S^{L,T_2} - S^{L,T_1} = S^{L,T_2}[T_1,T_2)$, we obtain

$$q(T_2) < \frac{R^L(S^{L,T_2}) - R^L(S^{L,T_1})}{S^{L,T_2} - S^{L,T_1}} = \frac{R^{L,T_2} - R^{L,T_1}}{S^{L,T_2} - S^{L,T_1}} \le q(T_1)$$

Since both $q(T_2)$ and S^{L,T_2} are continuous in T_2, sending T_2 to T_1 and applying the Squeeze Theorem give that $R^L(S)$ is differentiable at S and the derivative is given as $R^{L'}(S^{L,T_1}) = q(T_1)$. Since $q(T)$ is strictly decreasing in T and $S^{L,T}$ is strictly increasing in T, it follows that $R^L(S)$ is strictly concave in S.

(c) According to Lemma EC.3, sales is increasing in both shelf life and demand rate. Suppose that we achieve the same sales $S^{I,T_1}(\lambda_1, \mu) = S^{I,T_2}(\lambda_2, \mu)$ under issuance $I \in \{F, L\}$ for some customer arrival rates $\lambda_1 < \lambda_2$ and shelf lives $T_1 > T_2$.

Under FIFO, with the same shelf life T, each item stays on the shelf for a longer time (until it is sold to a customer or disposed of) with a smaller demand rate. Further, when customers arrive at the same rate, each item stays on the shelf for a longer time with a longer shelf life. These two observations together imply that each customer receives an older item under (F, T_1) with rate λ_1 than under (F, T_2) with rate λ_2: $Q^{F,T_1}(\lambda_1, \mu) \leq Q^{F,T_2}(\lambda_2, \mu)$. The claim follows from multiplying $S^{I,T_2}(\lambda_1, \mu) = S^{I,T_1}(\lambda_2, \mu)$ on both sides.

Under LIFO, because $R^L(0; \lambda_i, \mu) = 0$ for $i = 1, 2$, it is sufficient to compare the derivatives and show that $R^{L'}(S; \lambda_1, \mu) < R^{L'}(S; \lambda_2, \mu)$ for any $S > 0$. Take an arbitrary S and let $T_1 > T_2$ be shelf lives such that $S = S^{L,T_1}(\lambda_1, \mu) = S^{L,T_2}(\lambda_2, \mu)$. Then, Part (b) implies that $R^{L'}(S; \lambda_1, \mu) = q(T_1) < q(T_2) = R^{L'}(S; \lambda_2, \mu)$.

(d) Let a shelf life T and customer arriving rates $\lambda_1 < \lambda_2$ be given. According to Lemma EC.1(c), the oldest item is always younger when customers arrive with rate λ_2 than with rate λ_1. The PASTA ("Poisson Arrivals See Time Averages") property then implies that customers on average purchase a younger item with rate λ_2 than with rate λ_1.

(e) Because $Q^{L,T}(\lambda, \mu) = R^{L,T}(\lambda, \mu)/S^{L,T}(\lambda, \mu)$ and due to Lemma EC.3(d), it remains to show that $R^{L,T}(\lambda, \mu)$ is continuous in (T, λ). We begin by showing that $R^{L,T}(\lambda, \mu)$ is Lipschitz continuous in λ. Following the proof of Lemma 3, we let $G(x) := q(0) - q(x)$, $x > 0$. Note that under LIFO, due to Lemma EC.3(b), (EC.3.3) becomes

$$R^{L,T}(\lambda, \mu) = q(T) S^{L,T}(\lambda, \mu) + \int_0^T S^{L,T}(x) dG(x) = q(T) S^{L,T}(\lambda, \mu) + \int_0^T S^{L,x} dG(x).$$

Hence, for demand rates $\lambda < \lambda'$, we have

$$\begin{aligned}
&\left| R^{L,T}(\lambda', \mu) - R^{L,T}(\lambda, \mu) \right| \\
=& \left| q(T) \left(S^{L,T}(\lambda', \mu) - S^{L,T}(\lambda, \mu) \right) + \int_0^T \left(S^{L,x}(\lambda', \mu) - S^{L,x}(\lambda, \mu) \right) dG(x) \right|
\end{aligned}$$

$$\overset{(i)}{\leq} |q(T)| \cdot |S^{L,T}(\lambda',\mu) - S^{L,T}(\lambda,\mu)| + \int_0^T |S^{L,x}(\lambda',\mu) - S^{L,x}(\lambda,\mu)| dG(x)$$
$$\overset{(ii)}{\leq} |q(T)| \cdot |\lambda' - \lambda| + (G(T) - G(0)) \cdot |\lambda' - \lambda| = (|q(T)| + q(0) - q(T)) \cdot |\lambda' - \lambda|, \qquad \text{(EC.2.6)}$$

where (i) follows because $G(x)$ is an increasing function of x and (ii) follows because $S^{L,T}(\lambda,\mu)$ is Lipschitz continuous in λ (see the proof of Lemma EC.3(d)).

We obtain from Lemma EC.3(b) that $R^{L,T} = \int_0^T q(x) dS^{L,x}$. This implies that $R^{L,T}(\lambda,\mu)$ is continuous in T.

Finally, we prove that $R^{L,T}(\lambda,\mu)$ is continuous in (T,λ). Let (T,λ) be given and take an arbitrary $\epsilon > 0$. Because $|q(T)| + q(0) - q(T)$ is non-negative and continuous in T, there exists $\delta_1 > 0$ such that $|q(T')| + q(0) - q(T') < |q(T)| + q(0) - q(T) + 1$ if $|T' - T| < \delta_1$. We also define $\delta_2 := \frac{1}{|q(T)|+q(0)-q(T)+1} \cdot \frac{\epsilon}{2}$. From the continuity of $R^{L,T}(\lambda,\mu)$ with respect to T, there exists $\delta_3 > 0$ such that $|R^{L,T'}(\lambda,\mu) - R^{L,T}(\lambda,\mu)| < \frac{\epsilon}{2}$ if $|T' - T| < \delta_3$. Therefore, for any (T',λ') such that $\|(T',\lambda') - (T,\lambda)\| < \delta := \min(\delta_1,\delta_2,\delta_3)$, we have

$$\begin{aligned} |R^{L,T'}(\lambda',\mu) - R^{L,T}(\lambda,\mu)| &\overset{(iii)}{\leq} |R^{L,T'}(\lambda',\mu) - R^{L,T'}(\lambda,\mu)| + |R^{L,T'}(\lambda,\mu) - R^{L,T}(\lambda,\mu)| \\ &\overset{(iv)}{\leq} (|q(T')| + q(0) - q(T')) \cdot |\lambda' - \lambda| + |R^{L,T'}(\lambda,\mu) - R^{L,T}(\lambda,\mu)| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \end{aligned}$$

where (iii) follows from the triangle inequality and (iv) follows because $|T' - T|, |\lambda' - \lambda| \leq \|(T',\lambda') - (T,\lambda)\| < \delta \leq \delta_i$, $i = 1,2,3$. Q.E.D.

EC.3. Proofs of Results for Base Model

In this section, we provide proofs of the results in §4.

EC.3.1. Proof of Lemma 1

Fix a sample path of batch and customer arrivals, and let $T_1 < T_2$ be two shelf lives.

That sales $S^{I,T}$ are continuous and strictly increasing in T follows readily from Lemma EC.3(a). That purchased quality $Q^{I,T}$ is strictly decreasing in T follows by combining Lemmas EC.3 and EC.4.

We prove the monotonicity of purchased quality with respect to T. First, we consider FIFO issuance. According to Lemma EC.1(a), for an item of age x that exists on the shelf under both policies, (F,T_1) and (F,T_2), at the same time, (F,T_2) holds more items of age greater than x than (F,T_1). Thus, for each item that is sold under (F,T_2), we have that either: (i) the item was sold under (F,T_1) at an earlier time or (ii) the item was disposed of under (F,T_1), and it was only sold under (F, T_2). In the latter case, which occurs with positive probability, the item's age was in $[T_1, T_2)$ when it was purchased under (F, T_2), so its quality was strictly lower than $q(T_1)$ and thus strictly lower than any purchased quality under (F, T_1). These observations together imply that the long-run average purchased quality is strictly lower under (F, T_2) than under (F, T_1). We next consider LIFO issuance. According to Lemma EC.3(a), the retailer has greater sales under (L, T_2) than under (L, T_1). Lemma EC.1(b) then implies that the incremental sales under (L, T_2) come from selling items with age in $[T_1, T_2)$, whose quality is strictly lower than that of any items of age in $[0, T_1)$. Hence, it follows that the long-run average purchased quality is strictly lower under (L, T_2) than under (L, T_1). Therefore, if we define $\bar{T}_I := \sup\{T \geq 0 : Q^{I,T} \geq p\}$ for $I \in \{F, L\}$, constraint (3.3) is satisfied and customers buy the product if it is in stock if and only if $T \in (0, \bar{T}_I]$.

EC.3.2. Proof of Lemma 2

Let $T \leq \min(\bar{T}_F, \bar{T}_L)$ be given. We prove $S^{F,T} > S^{L,T}$ first and then prove $Q^{F,T} < Q^{L,T}$.

Let us fix a sample path of replenishment and customer arrivals. According to Lemma EC.1(c), the shelf holds more items under (F, T) than (L, T) at any given time (i.e., $z_t^{F,T} \geq z_t^{L,T}$ for any $t \geq 0$). This implies that FIFO issuance yields higher sales than LIFO issuance, i.e., $S^{F,T} \geq S^{L,T}$. We next claim that this inequality is indeed strict, i.e., $S^{F,T} > S^{L,T}$. To see this, note that during any renewal cycle (i.e., consecutive times when a batch arrives at an empty shelf), there is a positive probability that at some time the state of the inventory of different ages under FIFO and LIFO is different. Let t be the earliest such time during the cycle. By Lemma EC.1(c), the shelf must hold strictly more items under FIFO than under LIFO at t. Thus, with positive probability, FIFO can achieve strictly more sales than LIFO subsequently during the cycle (for example, there is a positive probability that enough customers arrive to completely clear the shelf under both policies, before any inventory is disposed of). Because the two policies achieved identical sales during the cycle up to t, it must be that FIFO achieves strictly more sales than LIFO overall during the cycle. And because this occurs with positive probability during any renewal cycle, we have $S^{F,T} > S^{L,T}$.

It follows from our argument above and Lemma EC.3(a) that there exists a shelf life $T' > T$ such that $S^{L,T'} = S^{F,T}$. Then, Lemma 3 gives that $R^{F,T} = R^F(S^{F,T}) \leq R^L(S^{F,T})$. Combining these observations, we obtain

$$Q^{F,T} = \frac{R^{F,T}}{S^{F,T}} \leq \frac{R^L\left(S^{F,T}\right)}{S^{F,T}} = \frac{R^L\left(S^{L,T'}\right)}{S^{L,T'}} = Q^{L,T'} < Q^{L,T},$$

where the last inequality follows from Lemma 1 because $T < T'$.

EC.3.3. Proof of Proposition 1

We fix a shelf life $T \le \min(\bar{T}_F, \bar{T}_L) = \bar{T}_F$ and compare two policies, (F,T) and (L,T).

(a) For convenience of exposition, let us denote the retailer's objective (3.4) under policy (I,T) by $V^{I,T}$. We obtain the following condition from (4.1) under which the FIFO policy outperforms the LIFO policy:

$$\begin{aligned} V^{L,T} \le V^{F,T} \quad &\Leftrightarrow \quad (p+d)S^{L,T} + fW^{L,T} \le (p+d)S^{F,T} + fW^{F,T} \\ &\Leftrightarrow \quad f(W^{L,T} - W^{F,T}) \le (p+d)(S^{F,T} - S^{L,T}) \\ &\Leftrightarrow \quad f \cdot \frac{W^{L,T} - W^{F,T}}{S^{F,T} - S^{L,T}} \le p + d, \end{aligned} \tag{EC.3.1}$$

where we used the fact that $S^{F,T} > S^{L,T}$ (Lemma 2). Because the RHS is strictly positive, it follows that there exists $\bar{f} > 0$ such that FIFO issuance outperforms LIFO issuance if and only if $f \le \bar{f}$.

(b) To show that $\bar{f}$ increases with d, we consider the following two cases. When $W^{L,T} \le W^{F,T}$ (i.e., the LHS of (EC.3.1) is negative), the inequality holds for all $d > -p$, implying that FIFO issuance dominates LIFO issuance regardless of the value of f. When $W^{L,T} > W^{F,T}$ (i.e., the LHS is strictly positive), the threshold $\bar{f}$ is given as

$$\bar{f} = (p+d) \cdot \underbrace{\frac{S^{F,T} - S^{L,T}}{W^{L,T} - W^{F,T}}}_{>0},$$

and it increases with d because $S^{F,T} > S^{L,T}$ and $W^{L,T} > W^{F,T}$. If we let $\bar{f} = 1$ and solve for d, we get $d = (Q^{L,T}S^{L,T} - Q^{F,T}S^{F,T})/(S^{F,T} - S^{L,T})$. This implies that we have $\bar{f} = 1$ for $d \ge (Q^{L,T}S^{L,T} - Q^{F,T}S^{F,T})/(S^{F,T} - S^{L,T})$.

EC.3.4. Proof of Lemma 3

Recalling the definition of $R^I(S)$ in Definition EC.1, we argue that it is sufficient to show that $R^L(S) > R^F(S)$ for any sales $S > 0$, assuming that customers who find items in stock purchase one. To see this, first note that the inequality implies that $R^L(S)/S = Q^{L,T_L(S)} > R^F(S)/S = Q^{F,T_F(S)}$. This in turn implies that if sales $S > 0$ satisfies $Q^{F,T_F(S)} \ge p$, then we have $Q^{L,T_L(S)} > p$. Hence, it follows from (4.2) that $S^{L,\bar{T}_L} > S^{F,\bar{T}_F}$. Furthermore, the inequality $R^L(S) > R^F(S)$ is equivalent to $W^L(S) > W^F(S)$ because $W^I(S) = S \cdot (Q^{I,T} - p) = R^I(S) - p \cdot S$. Figure EC.2 provides an illustration.

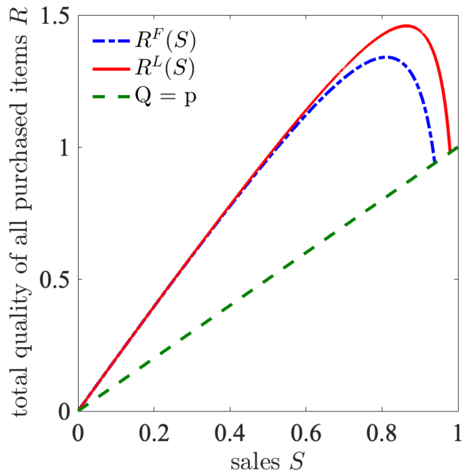


Figure EC.2 Comparison between $R^F(S)$ and $R^L(S)$.

The proof proceeds in five steps. Suppose that we have $S^{F,T_1} = S^{L,T_2} = S$ for some shelf lives $T_1 \le \bar{T}_F$ and $T_2 \le \bar{T}_L$. Note that Lemma 2 implies $T_1 < T_2$.[10] We argue that

$$R^{F,T_1} < R^{L,T_2}. \tag{EC.3.2}$$

Step 1. First, we derive an expression for $R^{I,T}$. For convenience of exposition, we denote the long-run average sales of inventory of age below x and the long-run average rate at which inventory reaches age x by $S^{I,T}(x) := S^{I,T}[0,x)$ and $D^{I,T}(x) := D^{I,T}[0,x)$, respectively, for each $x \in [0,T]$. In particular, we have $S^{I,T}(T) = S^{I,T}$ and $D^{I,T}(T) = D^{I,T}$. Then, it can be readily seen that the long-run average purchased quality can be written as

$$\begin{aligned} R^{I,T} &= \int_0^T q(x) dS^{I,T}(x) = \int_0^T \{q(0) - (q(0) - q(x))\} dS^{I,T}(x) \\ &= q(0) S^{I,T}(T) - \int_0^T \underbrace{(q(0) - q(x))}_{=G(x)} dS^{I,T}(x). \end{aligned}$$

Let us define $G(x) := q(0) - q(x)$, $x \ge 0$. Applying integration by parts to the second term and using $S^{I,T}(T) = S^{I,T}$ yield

$$\begin{aligned} R^{I,T} &= q(0) S^{I,T} - \int_0^T G(x) dS^{I,T}(x) \\ &= q(0) S^{I,T} - \left(G(T) S^{I,T}(T) - \underbrace{G(0)}_{=0} S^{I,T}(0) - \int_0^T S^{I,T}(x) dG(x) \right) \\ &= q(T) S^{I,T} + \int_0^T S^{I,T}(x) dG(x). \end{aligned} \tag{EC.3.3}$$

[10] Here, we used the monotonicity result regarding the sales only.

As each item is either sold prior to reaching age x or displayed on the shelf until it reaches age x for any $x \in [0, T]$, we must have $\mu B = S^{I,T}(x) + D^{I,T}(x)$ and in particular, $\mu B = S^{I,T} + D^{I,T}$. Applying these to (EC.3.3), we get

$$\begin{aligned} R^{I,T} &= q(T)\left(\mu B - D^{I,T}\right) + \int_0^T \left(\mu B - D^{I,T}(x)\right) dG(x) \\ &= q(0)\mu B - q(T) D^{I,T} - \int_0^T D^{I,T}(x) dG(x). \end{aligned} \tag{EC.3.4}$$

Step 2. If we apply (EC.3.4) to both sides of (EC.3.2), the inequality becomes

$$\begin{aligned} & q(0)\mu B - q(T_1) D^{F,T_1} - \int_0^{T_1} D^{F,T_1}(x) dG(x) \\ \le\ & q(0)\mu B - q(T_2) D^{L,T_2} - \int_0^{T_2} D^{L,T_2}(x) dG(x). \end{aligned} \tag{EC.3.5}$$

Note that we have the same inventory disposal under both policies because T_1 and T_2 were chosen to be such that $S^{F,T_1} = S^{L,T_2} = S$. Let us denote the inventory disposal under both policies by $D = D^{F,T_1} = D^{L,T_2} (= \mu B - S)$. Rearranging terms, it can be readily seen that (EC.3.5) is equivalent to

$$\begin{aligned} \int_0^{T_2} D^{L,T_2}(x) dG(x) &\le \int_0^{T_1} D^{F,T_1}(x) dG(x) + (q(T_1) - q(T_2)) D \\ &= \int_0^{T_1} D^{F,T_1}(x) dG(x) + D \int_{T_1}^{T_2} dG(x). \end{aligned} \tag{EC.3.6}$$

Now, based on the FIFO policy, let us define a "hidden-FIFO" policy", which we denote by $(\tilde{F}^{T_1}, T_2)$: under the hidden-FIFO policy, the retailer sells items of age at most T_1 to customers according to FIFO issuance but only discards items when they reach age T_2, so items of age $T \in [T_1, T_2)$ are kept on the shelf, but "hidden" from customers. In other words, under this policy, the retailer uses shelf life T_2, but customers are offered items of ages strictly below T_1 according to FIFO. Figure EC.3 illustrates the hidden-FIFO policy graphically along with (L, T_2).

The rate at which inventory reaches age $x \in [0, T_1]$ under policy $(\tilde{F}^{T_1}, T_2)$ is given as following because items are not available to customers once they reach age T_1.

$$D^{\tilde{F}^{T_1}, T_2}(x) = \begin{cases} D^{F,T_1}(x) & \text{if } 0 \le x \le T_1, \\ D^{F,T_1} = D & \text{if } T_1 < x \le T_2. \end{cases}$$

With this notation, we can simplify the RHS of (EC.3.6) and rewrite it as following.

$$\int_0^{T_2} D^{L,T_2}(x) dG(x) \le \int_0^{T_2} D^{\tilde{F}^{T_1}, T_2}(x) dG(x). \tag{EC.3.7}$$

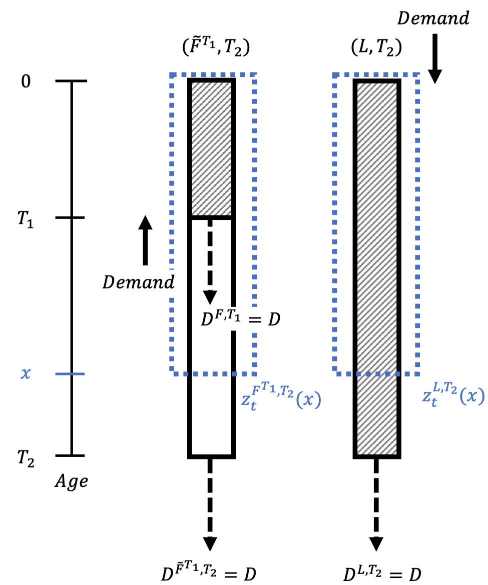


Figure EC.3 An Illustration of Policies $(\tilde{F}^{T_1}, T_2)$ and (L, T_2). In the figure, only items of ages less than x that are in the dashed areas are accessible to customers.

Recall that any convex or concave function is differentiable almost surely. We shall proceed the remaining analysis under the assumption that $q(\tau)$ is differentiable at *all* $\tau \geq 0$, and thus so is $G(x) = q(0) - q(x)$. The proof can be readily modified to include the set of non-differentiable points which is at most countable. Then, (EC.3.7) can be rewritten as following.

$$\int_0^{T_2} (-q'(x))\, D^{L,T_2}(x)dx \leq \int_0^{T_2} (-q'(x))\, D^{\tilde{F}^{T_1},T_2}(x)dx. \tag{EC.3.8}$$

Step 3. We claim that the following identity holds.

$$\int_0^x D^{I,T}(t)dt = Z^{I,T}(x), \quad \text{for any } x \in [0,T] \text{ and } I \in \{F, \tilde{F}^T, L\}. \tag{EC.3.9}$$

It is sufficient to prove the following finite-time version of the identity.

$$\frac{1}{M}\int_0^x d_{[0,M)}^{I,T}(y)dy = \frac{1}{M}\int_0^M z_t^{I,T}(x)dt, \quad \text{for any } M \geq 0 \text{ and } x \in [0,T], \tag{EC.3.10}$$

where $d_{[0,M)}^{I,T}(y)$ denotes the total number of items that reached age y under (I,T) during time interval $[0,M)$ and $z_t^{I,T}(x)$ denotes the inventory level under age x under (I,T) at time t. In other words, $d_{[0,M)}^{I,T}(y)$ and $z_t^{I,T}(x)$ respectively represent the finite-time versions of $D^{I,T}(y)$ and $Z^{I,T}(x)$. To see this, suppose an item is replenished at time t and stays on the shelf for τ time period where $t + \tau \leq M$. Because the item contributes 1 to $d_{[0,M)}^{I,T}(y)$ for each $y \in [0,\tau)$, it contributes $\frac{\tau}{T}$ to the

LHS in total. On the other hand, because the item is counted once in $z_t^{I,T}(x)$ for each $t \in [t, t+\tau)$, it contributes $\frac{\tau}{T}$ to the RHS. Thus, (EC.3.9) follows and we can rewrite (EC.3.8) as

$$\int_0^{T_2} (-q'(x))dZ^{L,T_2}(x) \le \int_0^{T_2} (-q'(x))dZ^{\tilde{F}^{T_1},T_2}(x). \tag{EC.3.11}$$

Step 4. We claim that $Z^{L,T_2}(x) \le Z^{\tilde{F}^{T_1},T_2}(x)$ for any $x \in [0,T_2]$. That is, on average, $(\tilde{F}^{T_1},T_2)$ holds more inventory of age below x than (L,T_2) does. It is sufficient to show that $(\tilde{F},T_2)$ holds larger inventory that is younger than age x than (L,T_2) does at any given time:

$$z_t^{L,T_2}(x) \le z_t^{\tilde{F}^{T_1},T_2}(x), \quad \text{for any } t \ge 0 \text{ and } x \in [0,T_2]. \tag{EC.3.12}$$

Let $\{t_n\}_{n\ge1}$ denote the increasing sequence of times at which the retailer replenishes inventory or customers arrive and let $t_0 = 0$. Note that if the claim holds at $t \ge 0$ and no batch or customer arrives during time period $(t,t']$, the claim continues to hold until t' because $z_{t'}^{\tilde{F},T_2}(x) = z_t^{\tilde{F},T_2}(x-(t'-t)) \ge z_t^{L,T_2}(x-(t'-t)) = z_{t'}^{L,T_2}(x)$. Thus, we only need to prove the claim at discrete times $\{t_n\}_{n\ge0}$. Clearly, the claim holds at $t_0 = 0$ when the shelf is empty. Suppose the claim holds until t_n for some integer $n \in \{0,1,2,\ldots\}$. Take any $x \in [0,T_2]$. If a batch arrived at t_{n+1}, we have

$$\begin{aligned} z_{t_{n+1}}^{\tilde{F}^{T_1},T_2}(x) &= z_{t_n}^{\tilde{F}^{T_1},T_2}(x-(t_{n+1}-t_n)) + B \\ &\ge z_{t_n}^{L,T_2}(x-(t_{n+1}-t_n)) + B = z_{t_{n+1}}^{L,T_2}(x). \end{aligned} \tag{EC.3.13}$$

On the other hand, if a customer arrived at t_{n+1}, we have

$$\begin{aligned} z_{t_{n+1}}^{\tilde{F}^{T_1},T_2}(x) &= \left(z_{t_n}^{\tilde{F}^{T_1},T_2}[0,\min(x-(t_{n+1}-t_n),T_1)) - 1\right)^+ \\ &\qquad + z_{t_n}^{\tilde{F}^{T_1},T_2}[\min(x-(t_{n+1}-t_n),T_1), x-(t_{n+1}-t_n)) \\ &\overset{(i)}{\ge} \left(z_{t_n}^{\tilde{F}^{T_1},T_2}[0,\min(x-(t_{n+1}-t_n),T_1)) \right. \\ &\qquad \left. + z_{t_n}^{\tilde{F}^{T_1},T_2}[\min(x-(t_{n+1}-t_n),T_1), x-(t_{n+1}-t_n)) - 1\right)^+ \\ &= \left(z_{t_n}^{\tilde{F}^{T_1},T_2}[0,x-(t_{n+1}-t_n)) - 1\right)^+ \\ &\overset{(ii)}{\ge} \left(z_{t_n}^{L,T_2}[0,x-(t_{n+1}-t_n)) - 1\right)^+ = z_{t_{n+1}}^{L,T_2}(x), \end{aligned} \tag{EC.3.14}$$

where (i) follows from the inequality $(a-1)^+ + b \ge (a+b-1)^+$ for $a,b \ge 0$ and (ii) follows from the induction hypothesis. Because x was chosen arbitrarily, it follows that (EC.3.12) continues to hold at t_{n+1}. Therefore, by the induction principle, (EC.3.12) holds for all t_n, $n = 0,1,2\ldots$, and thus it holds for any $t \ge 0$.

Step 5. Using the fact that $-q'(t)$ is positive and decreasing in t (because $q(t)$ is convex in t) and by **Step 4**, we can apply Lemma EC.6 to obtain inequality (EC.3.11). This proves that

$$R^F(S) \le R^L(S) \quad \text{for } S \le \min(\bar{S}^F, \bar{S}^L). \tag{EC.3.15}$$

Furthermore, we argue that the inequality is indeed strict, i.e., $R^F(S) < R^L(S)$. At any given time, there is positive probability that there are strictly more than one items with different ages below T_1 under both the hidden-FIFO policy and the forced-LIFO policy. (Imagine multiple batches arriving in a short period of time while no customer arrives.) In such a case, an arriving customer's purchased quality would be strictly larger under the forced-LIFO policy than the hidden-FIFO policy, i.e., $q(\tau_{t_n}^{\tilde{F}^{T_1},T_2}) < q(\tau_{t_n}^{L,T_2})$. This proves the claim.

Lastly, we show that $\bar{S}^F \le \bar{S}^L$. Due to the monotonicity of $S^{I,T}$ with respect to T (see Lemma EC.3(a)), we obtain from the definition of $\bar{T}^I$ that $\bar{S}^I = \sup\{S \ge 0 : R^I(S) \ge pS\}$, $I \in \{F, L\}$. So it remains to show that

$$\sup\{S \ge 0 : R^F(S) \ge pS\} \le \sup\{S \ge 0 : R^L(S) \ge pS\}.$$

This inequality holds because for any S that gives $R^F(S) \ge pS$, we have $R^L(S) \ge R^F(S) \ge pS$ by (EC.3.15). This concludes the proof.

EC.3.5. Proof of Theorem 1

Recall that under any issuance I, the optimal shelf life is chosen from the interval $[0, \bar{T}^I]$, without loss of optimality in maximizing the retailer's objective.

We first prove that LIFO dominates FIFO. Consider a policy (F, T_1) with $T_1 \le \bar{T}_F$ and define $T_2 := T_L(S^{F,T_1})$ to be the shelf life such that policies (F, T_1) and (L, T_2) have the same sales, $S^{F,T_1} = S^{L,T_2}$. (Lemma 3 guarantees that T_2 exists because the maximal sales achievable with LIFO are strictly larger than those achievable with FIFO, $S^{L,\bar{T}_L} > S^{F,\bar{T}_F}$, and sales are continuous in the shelf life by Lemma 1.) Then, it follows from Lemma 3 that

$$W^{L,T_2} = W^L(S^{L,T_2}) = W^L(S^{F,T_1}) > W^F(S^{F,T_1}) = W^{F,T_1}.$$

Because the objective is increasing in both $S^{I,T}$ and $W^{I,T}$ (see (4.1)), it follows that the optimal policy must use LIFO issuance.

Next, we solve for the retailer's optimal shelf life under LIFO. The retailer's objective under LIFO can be written as a function of S:

$$V^L(S) := \{(1-f)p + d\}\, S + fR^L(S) - d\mu B, \quad 0 < S \le \bar{S}^L, \tag{EC.3.16}$$

where $\bar{S}^L$ is defined in Lemma EC.3. Because $R^L(S)$ is differentiable and concave in S from Lemma EC.4, the first-order condition yields the a globally optimal unconstrained solution. The derivative of the objective with respect to S is

$$\frac{dV^L}{dS} = (1-f)p + d + f \cdot \frac{dR^L}{dS} = (1-f)p + d + f \cdot q\,(T_L(S)), \tag{EC.3.17}$$

where the second equality follows from Lemma EC.4(b). Thus, the critical point satisfies $T_L(S^*) = q^{-1}\big(p - (p+d)/f\big)$. This expression is trivially positive from the definition of the function q. Because T is chosen from the interval $[0, \bar{T}_L]$, we readily obtain that the optimal shelf life is $T_L^* = \min(q^{-1}\big(p-(p+d)/f\big), \bar{T}^L)$.

Lastly, we prove the comparative statics results. As the product quality degrades faster, the derivative in (EC.3.17) becomes smaller for each S. Because q is strictly decreasing, this implies that the critical point $T_L(S^*)$ decreases, and therefore the optimal shelf life T_L^* also decreases. That T_L^* decreases with f and increases with d follows from its expression.

LEMMA EC.5. *Consider any issuance I, shelf life T, and price p such that all customers who find an item in stock purchase it, and with a slight abuse of notation, let $\bar{T}_I(p) = \sup\{T \geq 0 : Q^{I,T} \geq p\}$ denote the largest shelf life with issuance I at which all customers purchase the item at price p. Then:*

$$Q^{I,T} > q(T) \tag{EC.3.18a}$$

$$\bar{T}_I(p) > q^{-1}(p). \tag{EC.3.18b}$$

Proof of Lemma EC.5. If I, T, p are such that all customers who find an item in stock purchase one, then the expression of purchased quality is given by $Q^{I,T}$ in (3.3), irrespective of whether or not there are timestamps, and the result in Lemma 1 is also applicable for any such T.

Because $q(\tau)$ is strictly decreasing in τ, it follows that the average purchased quality $Q^{I,T}$ is strictly larger than the lowest-possible quality of an item on the shelf $q(T)$, proving (EC.3.18a). Because $Q^{I,T}$ is continuous and strictly decreasing in T (by Lemma 1), we have that $Q^{I,\bar{T}_I(p)} = p$. Then, (EC.3.18a) implies that

$$Q^{I,\bar{T}_I(p)} = p > q\big(\bar{T}_I(p)\big) \;\Rightarrow\; \bar{T}_I(p) > q^{-1}(p), \tag{EC.3.19}$$

where the last inequality follows because q is strictly decreasing. Q.E.D.

EC.3.6. Proof of Proposition 2

Recall that the price p is chosen subject to the constraints

$$\max(0,-d) \leq p \leq Q^{I,T}, \tag{EC.3.20}$$

where the latter inequality restates condition (3.3) that guarantees strictly positive sales. With a slight abuse of notation, let $\bar{T}_L(p)$ denote the value of $\bar{T}_L$ from (4.2) for a given value of p, i.e., $\bar{T}_L(p) = \sup\{T \geq 0 : Q^{L,T} \geq p\}$. Note that the feasible range of values for p in (EC.3.20) implies that any feasible shelf life T must satisfy:

$$Q^{L,T} \geq \max(0,-d) \;\Leftrightarrow\; T \leq \bar{T}_L\big(\max(0,-d)\big) \;\Leftrightarrow\; T \leq \min\big(\bar{T}_L(0), \bar{T}_L(-d)\big), \tag{EC.3.21}$$

where the first equivalence follows from the definition of $\bar{T}_L(p)$ and the second equivalence follows because $\bar{T}_L(p)$ is decreasing in p.

Because Theorem 1 proved the strict dominance of LIFO for any exogeneously fixed price $p > -d$ and because the objective is continuous in p, it readily follows that LIFO issuance is optimal even when p is optimally chosen subject to the stated constraints.

To characterize the optimal price and shelf life with LIFO, consider first the case $f < 1$. For any T satisfying (EC.3.21), it can be readily checked that the retailer's objective (3.4) is strictly increasing in p for any $p \leq Q^{L,T}$. To maximize the objective, the retailer should therefore charge a price $p = Q^{L,T}$.

To determine the optimal shelf life T, note that with price $p = Q^{L,T}$, the customer welfare is zero and the retailer's objective becomes:

$$Q^{L,T} \cdot S^{L,T} - d \cdot D^{L,T} + f \cdot 0 = (Q^{L,T} + d)S^{L,T} - d\mu B, \tag{EC.3.22}$$

which is identical to the objective of a social welfare-maximizing retailer ($f = 1$). Theorem 1 would therefore imply that an unconstrained optimal shelf life would be $q^{-1}(-d)$. To take account of the constraints (EC.3.21), note that the retailer's objective is pseudo-concave in the shelf life T and in particular, it is unimodal in T (because the objective is concave in S by Lemma EC.4 and S is strictly increasing in T). Therefore, the optimal shelf life is:

$$T_L^* = \min\big(\bar{T}_L(0),\, \bar{T}_L(-d),\, q^{-1}(-d)\big) = \min\big(\bar{T}_L(0), q^{-1}(-d)\big),$$

where the second equality follows because $\bar{T}_L(-d) \geq q^{-1}(-d)$ by Lemma EC.5.

If $f = 1$, the retailer maximizes social welfare. The retailer's objective (3.4) becomes

$$V^{I,T} = pS^{I,T} - dD^{I,T} + 1 \cdot W^{I,T} = dS^{I,T} + R^{I,T} - d\mu B,$$

which is equal to (EC.3.22). Hence, a similar line of arguments as above can be used to conclude that the shelf life is given by $T_L^* = \min\big(\bar{T}_L(0),\, q^{-1}(-d)\big)$. Because the objective is independent of price p, any price that satisfies (EC.3.20) for $T = T_L^*$ will be optimal.

EC.3.7. Proof of Proposition 3

(a) Fix the issuance I and a price p. With timestamps, recall that customers observe the quality of the item on offer before making a purchase decision, so to ensure non-zero sales, the shelf life and price must satisfy (3.1), or equivalently, $T \le q^{-1}(p)$.

For any shelf life $T \le q^{-1}(p)$, all customers who find an item in stock purchase it and derive positive ex-post utility from the purchase. This implies that the purchased quality expression is given by $Q^{I,T}$ in (3.3) and is the same as in the case without timestamps for shelf life T. Then, by Lemma EC.5, at shelf life $T = q^{-1}(p)$ we have that $Q^{I,q^{-1}(p)} > q\big(q^{-1}(p)\big) = p$ and $\bar{T}_I > q^{-1}(p)$.

(b) The argument above also implies that the feasible set of shelf life values with timestamps, $[0, q^{-1}(p)]$, is strictly contained in the feasible set of shelf life values without timestamps, $[0, \bar{T}_I]$, so the retailer's optimal objective value with timestamps is lower than without timestamps.

Finally, we prove that timestamps increase customer welfare under any issuance I (and under the optimal shelf life corresponding to that issuance).

With LIFO issuance, we first argue that the optimal shelf life with timestamps is exactly $q^{-1}(p)$ (irrespective of the weight f). This follows because $q^{-1}(p)$ exactly corresponds to the shelf life that maximizes customer welfare under LIFO (by Theorem 1 for the case $f \to \infty$), so the sales *and* the customer welfare are increasing in T for $T \le q^{-1}(p)$, which means that $T = q^{-1}(p)$ is optimal for any $f \in [0,1]$. In turn, because $q^{-1}(p)$ actually maximizes customer welfare, this readily implies that customer welfare with timestamps is always larger than without timestamps.

With FIFO issuance, we distinguish two cases depending on the value of the optimal shelf life without timestamps, which we denote by T_F^*. If $T_F^* \le q^{-1}(p)$, then T_F^* will be the optimal shelf life with timestamps as well (by Proposition 3 and because T_F^* would be feasible with timestamps in this case). In this case, timestamps will not result in any change in customer welfare. If $T_F^* > q^{-1}(p)$, then we claim that (i) FIFO issuance with timestamps and a shelf life of $q^{-1}(p)$ would always lead to a larger customer welfare than (ii) FIFO issuance without timestamps and a shelf life $T_F^* > q^{-1}(p)$.

To argue this, let us assign a unique identity (ID) to each item that is received by the retailer, as follows: we number the items with 1 to B in the first batch (with 1 being the first item on offer from that batch, 2 being the second, etc.), we number the items in the second batch with $B+1$ to $2B$ (with $B+1$ being the first on offer from the batch), etc. Then, a sample-path argument and induction can be used to prove the following properties:

1. If both policy (i) and policy (ii) have an item on offer, the item on offer under policy (i) has higher ID (and therefore lower age) than the item on offer under policy (ii);
2. If an item with a given ID is sold to customers under both policies, it is sold at a lower age under policy (i) than under policy (ii);
3. If an item with a given ID is sold under policy (ii) but is not sold under policy (i), it must have been disposed under policy (i) and therefore its age when it was sold under policy (ii) was strictly larger than $q^{-1}(p)$.

With these properties, it can be readily seen that customer welfare is strictly higher under policy (i), because any item sold under both policies results in higher ex post utility under policy (i), items sold under policy (i) but not under policy (ii) always result in positive ex-post utility (because policy (i) uses timestamps), and the only items sold under policy (ii) but not under policy (i) result in strictly negative ex post utility (because their age strictly is strictly above $q^{-1}(p)$ when sold).

To complete the argument, we distinguish two cases. If the optimal shelf life with timestamps is actually $q^{-1}(p)$, then the result is immediate from the argument above. If the optimal shelf life T with timestamps is strictly smaller than $q^{-1}(p)$, we claim that (under FIFO issuance and with timestamps) the customer welfare achieved with shelf life T must be strictly larger than the customer welfare achieved with shelf life $q^{-1}(p)$. This follows because if it were not the case, the latter policy (which achieves strictly higher sales) would be optimal instead. Therefore, in this case, FIFO issuance with timestamps and shelf life strictly below $q^{-1}(p)$ achieves strictly larger customer welfare than FIFO issuance without timestamps and with shelf life $T_F^* > q^{-1}(p)$.

EC.3.8. Proof of Proposition 4

(a) By Proposition 3, the shelf life with timestamps must satisfy $T \le q^{-1}(p) < \bar{T}_I$, under any issuance I. Without timestamps, the optimal shelf life is given by (4.3), namely $T_L^* = \min\big(\bar{T}_L,\, q^{-1}(p-(p+d)/f)\big)$. Because $p > -d$ and q is decreasing, we readily have that $q^{-1}(p-(p+d)/f) > q^{-1}(p)$ for any $f \in [0,1]$; and because $\bar{T}_L > q^{-1}(p)$, this implies that the optimal shelf life with timestamps is strictly smaller than T_L^*.

(b) Consider first the case $f = 0$, when the retailer maximizes direct profit or, equivalently, sales. Under any issuance I, because $q^{-1}(p) \le \bar{T}_I$ and sales strictly increase with the shelf life for $T \le \bar{T}_I$ by Lemma 1, it follows that the shelf life that maximizes sales (and the retailer's profit) is exactly $T = q^{-1}(p)$. But then, Lemma 2 implies that FIFO achieves strictly larger sales than LIFO for $T = q^{-1}(p)$, so FIFO is optimal.

To prove the threshold structure, note that the optimal objective value is continuous in f, so there exists a threshold $\hat{f} > 0$ such that FIFO issuance (with shelf life $\le q^{-1}(p)$) is optimal for $f \le \hat{f}$. It remains to show that if LIFO is the optimal issuance for some $\hat{f} < 1$, it is also optimal for any $f > \hat{f}$. To that end, consider the following facts:

- For any issuance I, sales $S^{I,T}$ is strictly increasing and continuous in T (Lemma 1), so optimizing over (I,T) with $T \le q^{-1}(p)$ is equivalent to optimizing over (I,S) with $S \in [0, S^{I,q^{-1}(p)}]$.
- For I and $S \in [0, S^{I,q^{-1}(p)}]$, the retailer's objective in (3.4) is a linear function of the form $a \cdot S + f \cdot W^I(S)$, where the coefficient a is independent of f.
- By Lemma 3, $W^L(S) > W^F(S)$ for any $S \in [0, S^{L,q^{-1}(p)}]$ and by Lemma EC.4, $W^L(S)$ is strictly concave in S.

Then, a convex analysis argument can be used to show that if LIFO issuance is strictly optimal for $\hat{f} > 0$, it must be strictly optimal for any $f > \hat{f}$. For any f and I, let $S_I^*(f)$ denote the sales and $V_I^*(f) := a \cdot S_I^*(f) + f \cdot W^I\big(S_I^*(f)\big)$ denote the objective value achieved under the optimal shelf life choice with issuance I. Also, for any function $f(S)$ defined over a closed, convex subset $D \subseteq [0, S^{F,q^{-1}(p)}]$, let $H(f)$ denote its hypograph, i.e., $H(f) := \big\{(S,t) : t \le f(S), S \in D\big\}$. Because FIFO is strictly suboptimal for $f = \hat{f}$, we have

$$H(W^F) \subset \{(S,t) : a \cdot S + \hat{f} \cdot t \le V_L^*(\hat{f})\}, \tag{EC.3.23}$$

that is, the hypograph of $W^F(S)$ is strictly contained in the half-space in the right-hand-side of (EC.3.23). Consider then the function $f(S)$ defined on $\big[0, S^{F,q^{-1}(p)}\big]$ as follows:

$$g(S) := \begin{cases} W^L(S), & \text{if } S \in [0, S_L^*(\hat{f})] \\ \big(-a \cdot S + V_L^*(\hat{f})\big)/\hat{f}, & \text{if } S \in \big(S_L^*(\hat{f}), S^{F,q^{-1}(p)}\big]. \end{cases}$$

By construction, this function satisfies the following two properties:

$$H(W^F) \subset H(g) \tag{EC.3.24a}$$

$$\max_{(S,t) \in H(g)} (a \cdot S + f \cdot t) = \max_{(S,t) \in H(W^L)} (a \cdot S + f \cdot t), \; \forall f > \hat{f}. \tag{EC.3.24b}$$

Eq. (EC.3.24a) states that the hypograph of W^F is strictly contained in the hypograph of $g(S)$, which follows from (EC.3.23) and because $W^F(S) < W^L(S)$ for any $S \in [0, S_L^*(\hat{f})]$. Eq. (EC.3.24b) states that maximizing any linear function $a \cdot S + f \cdot t$ with $f > \hat{f}$ over the hypograph of $W^L(S)$ yields the same value as maximizing it over the hypograph of $g(S)$.

But then, consider the problem of maximizing the retailer's objective for $f > \hat{f}$. With issuance I, the optimal value is the same as the optimal value when maximizing the linear function $a \cdot S + f \cdot t$ over the set $H(W^I)$. Therefore, (EC.3.24a) and (EC.3.24b) readily imply that LIFO issuance is strictly dominant.

For a proof that the optimal shelf life with LIFO and timestamps is $q^{-1}(p)$, we direct the reader to the argument used in the proof of part (b) in Proposition 3.

(c) Proposition 3 implies that the optimal objective without timestamps is larger than the optimal objective with timestamps. To prove the strict dominance, we distinguish two cases. If the optimal issuance with timestamps is LIFO, the result follows from part (a) and Proposition 3 (because the feasible set of shelf lives with timestamps is contained in the feasible set without timestamps). If the optimal issuance with timestamps is FIFO, the result follows because LIFO issuance without timestamps delivers a strictly larger optimal objective than FIFO issuance with timestamps (in view of Theorem 1), which in turn delivers a larger optimal objective that FIFO without timestamps (in view of Proposition 3).

EC.3.9. Proof of Proposition 5

We first derive the retailer's optimal policy with timestamps. Recall that with timestamps, each arriving customer observes the quality of the item on offer $q(\tau)$ and purchases it as long as it gives positive expected utility ex-ante. Therefore, an age-dependent price $p(\tau)$ should be chosen subject to the constraints:

$$\max(0, -d) \leq p(\tau) \leq q(\tau), \tag{EC.3.25}$$

where the latter inequality restates the condition that the ex-ante utility is positive. Note that the pricing constraint (EC.3.25) also implies that any feasible shelf life T must satisfy:

$$q(T) \geq \max(0, -d) \Leftrightarrow T \leq q^{-1}\big(\max(0, -d)\big) \Leftrightarrow T \leq \min\big(q^{-1}(0), q^{-1}(-d)\big), \tag{EC.3.26}$$

where the equivalences follow because q is decreasing.

Under any pricing policy that satisfies (EC.3.25) and shelf life that satisfies (EC.3.26), any arriving customer who finds an item in stock purchases it.

Consider first the case $f < 1$. It is easy to verify that the retailer's objective is increasing in the price values $p(\tau)$ provided that (EC.3.25) is satisfied. Therefore, the optimal pricing policy is $p(\tau) = q(\tau)$, which implies that each arriving customer who makes a purchase derives zero utility from that purchase. Because customer welfare is zero, the retailer's objective becomes equivalent to that of a social welfare-maximizing retailer ($f = 1$).

In the case $f = 1$, it is easy to verify that the retailer's objective does not depend on the price values $p(\tau)$ – so any pricing policy satisfying (EC.3.25) is optimal – and the objective trivially corresponds to that of a social welfare-maximizing retailer ($f = 1$).

Therefore, under the optimal age-dependent pricing, the retailer's objective is equivalent to maximizing social welfare ($f = 1$). Theorem 1 and Proposition 4 would imply that without any other constraints, the optimal issuance would be LIFO and the optimal shelf life would be $T = q^{-1}(-d)$. However, the shelf life must satisfy constraint (EC.3.26), so we distinguish two cases, depending on the value of d:

- If $d \le 0$, the optimal shelf life is $q^{-1}(-d)$ both with and without timestamps. With timestamps, this follows because $d \le 0$ implies $q^{-1}(-d) \le q^{-1}(0)$, which in turn implies that the (unconstrained) optimal shelf life $T = q^{-1}(-d)$ is feasible in (EC.3.25) and is therefore optimal. Without timestamps, this follows from Proposition 2, because $T_L^* = \min\left(q^{-1}(-d), \bar{T}_L(0)\right)$ and in this case $q^{-1}(-d) \le q^{-1}(0) < \bar{T}_L(0)$ (where the last inequality follows from Lemma EC.5.) To summarize, in the case $d \le 0$, the optimal issuance and shelf life, and the resulting sales, purchased quality, and optimal objective value are identical in the case with and in the case without timestamps.
- If $d > 0$, note that in the case with timestamps, (EC.3.25) reduces to the constraint $T \le q^{-1}(0)$. The case therefore becomes equivalent to the case of a social-welfare maximizing retailer who prices items at $p = 0$ and is forced to used timestamps. The optimal policy is the one described in Proposition 4: the optimal issuance is LIFO and shelf life $q^{-1}(0)$ if and only if $f > \hat{f}$ (where the threshold $\hat{f}$ corresponds to the case $p = 0$) and it is FIFO otherwise. Moreover, by (EC.3.25), the retailer's objective is strictly smaller without timestamps, so the optimal policy is to *not* timestamp items and use the policy from Proposition 2.

EC.3.10. Proof of Proposition 6

We consider the case with a fixed price p first and prove both parts (i) and (ii). Recall that with a fixed price p, the retailer's optimal policy is to use LIFO issuance and a shelf life $T_L^* = \min\big(q^{-1}\big(p - (p+d)/f\big), \bar{T}_L\big)$, according to Theorem 1. Therefore, a change in the disposal cost d would impact the quantity and quality of disposed items and the customer welfare (henceforth referred to as the "quantities of interest" for the rest of this proof) only through changes in the optimal shelf life T_L^*.

If $f = 0$, the optimal shelf life is $T_L^* = \bar{T}_L$, which is independent of d. An increase in d would therefore leave all quantities of interest unchanged.

If $f > 0$, then note that the shelf life depends on d if and only if $q^{-1}\big(p - (p+d)/f\big) < \bar{T}_L$, which is equivalent to $d < \hat{d} := f \cdot \big(p - q(\bar{T}_L)\big) - p$. Thus, if $d \geq \hat{d}$, an increase in d leaves the shelf life and all the quantities of interest unchanged. When $d < \hat{d}$, increasing d strictly increases T_L^* because q is strictly decreasing. An increase in T_L^* strictly increases the sales S^{L,T_L^*} (by Lemma 1, which applies because $T_L^* < \bar{T}_L$ here), which strictly reduces the quantity of items disposed, $\mu - S^{L,T_L^*}$. The increase in T_L^* also strictly reduces the average quality of discarded items because all discarded items have strictly lower quality. To see that customer welfare also strictly decreases, note that customer welfare is maximized at a shelf life $T = q^{-1}(p)$ (by Theorem 1 for the case $f \to \infty$) and because the retailer's objective is pseudoconcave in T and $T_L^* > q^{-1}(p)$ (as argued in the proof of Proposition 4), it follows that the increase in d strictly reduces customer welfare.

For the case with an optimally chosen price, the optimal policy is characterized in Proposition 2. The shelf life is $T_L^* = \min\big(\bar{T}_L(0),\, q^{-1}(-d)\big)$ and depends on d if and only if $q^{-1}(-d) < \bar{T}_L(0)$, which is equivalent to $d < \hat{d} := -q\big(\bar{T}_L(0)\big)$. Thus, if $d \geq \hat{d}$, an increase in d leaves all the quantities of interest unchanged. If $d < \hat{d}$, increasing d strictly increases T_L^* because q is strictly decreasing and therefore strictly reduces the quantity and quality of disposed items. If $f < 1$, the optimal price is Q^{L,T_L^*} so the customer welfare is zero and is not affected by the change in d. If $f = 1$, the retailer could in principle use any price value from $[\max(0, -d)\, Q^{L,T_L^*}]$ because the price only serves to allocate the total welfare between the retailer and the customers. Because the total welfare is strictly lower when d increases, if the retailer uses a price that allocates a fixed or decreasing fraction of the total welfare to the customers, then the customer welfare will also strictly decrease.

EC.3.11. Technical Lemma

Throughout this section, we fix the initial inventory status and a sample path of batch arrivals, unless explicitly stated otherwise.

LEMMA EC.6. *Let $\phi : [0,\infty) \to [0,\infty)$ be a decreasing function and let $F_i : [0,\infty) \to \mathbb{R}$, $i = 1,2$ be two continuous functions such that $F_1(0) = F_2(0)$ and $F_1(x) \le F_2(x)$ for all $x \ge 0$. Then, we have*

$$\int_0^T \phi(x)dF_1(x) \le \int_0^T \phi(x)dF_2(x), \quad \textit{for any } T \ge 0. \tag{EC.3.27}$$

Proof of Lemma EC.6. Using integration by parts, we obtain

$$\int_0^T \phi(x)dF_i(x) = \phi(T)F_i(T) - \phi(0)\underbrace{F_i(0)}_{=0} - \int_0^T F_i(x)d\phi(x), \quad i = 1,2. \tag{EC.3.28}$$

By plugging in (EC.3.28) into both sides of (EC.3.27) and using $F_1(T) \le F_2(T)$, one can see that it is sufficient to show that

$$\int_0^T F_1(x)\,(-d\phi(x)) \le \int_0^T F_2(x)\,(-d\phi(x)), \quad \text{for any } T \ge 0.$$

This holds true because we have $F_1(x) \le F_2(x)$ for any $x \ge 0$ and $-\phi(x)$ is an increasing function of x. This concludes the proof. Q.E.D.

EC.4. Numerical Illustration of Results for Base Model

For the running example, this section illustrates results for the base model in §4. The running example has price $p = 1$, customer arrival rate $\lambda = 1$, Poisson replenishment with rate $\mu = 1$ and batch size $B = 1$, and quality $q(\tau) = 2 - 0.5\tau$.

Our two main results are that LIFO issuance is optimal and to *not* timestamp is optimal. Figure EC.4 helps to quantify the significance of each result. The first three rows of the figure show the optimal shelf life, the sales S, and the customer welfare W corresponding to several distinct policies (shown on columns). Specifically, the figure considers: FIFO issuance with an optimal shelf life and no timestamps (left column), LIFO issuance with an optimal shelf life and no timestamps (middle column), and an optimal issuance and shelf life when required to timestamp (right column). The LIFO policy in the middle column corresponds to the optimal policy overall, whereas the policies in the left and the right columns serve as useful benchmarks for isolating the benefits of using the optimal policy. To facilitate the discussion, the fourth row of the figure compares these policies: the middle panel depicts the improvement in the retailer's objective from using LIFO issuance rather than FIFO issuance (i.e., comparing the policies in the second and first column), and the right panel depicts the improvement from not using a timestamp (i.e., comparing the policies in the second and third column).

Using LIFO rather than FIFO greatly increases the retailer's objective, by two different mechanisms. When f is small, the shift from FIFO to LIFO greatly increases the shelf life, which increases sales and reduces disposals. In particular, at $f = 0$, that shift increases the shelf life by a factor of 3.75 - nearly quadruples the shelf life - which increases sales by 7% and reduces disposals by 27%. The corresponding improvements in the retailer's objective are monotonic increasing in the disposal cost d and can be as high as 12% for $d = 0.5$. When f is larger, the shift from FIFO to LIFO only slightly increases the optimal shelf life and sales, but substantially increases the customer welfare. The increase in customer welfare is monotonic decreasing (in both absolute and percentage terms) in the disposal cost d. For $f = 1$, the shift from FIFO to LIFO increases customer welfare by 13% at $d = -0.5$ and by 18% at $d = 0.5$. The increase in the retailer's objective is monotonic increasing (in both absolute and percentage terms) in d and f, to nearly 16% at $d = 0.5$ and $f = 1$.

Not timestamping items also greatly increases the retailer's objective. This occurs because a retailer that does not timestamp items can use a significantly larger shelf life: the optimal shelf life with timestamps is $q^{-1}(p) = 2$, whereas the optimal shelf life without timestamps always strictly exceeds that value and could be as high as $T = 15$, when maximizing direct profit ($f = 0$). This also translates in substantial increases in objective function value, and these increases are decreasing in the weight f on customer welfare and increasing in the disposal cost d. Specifically, a retailer who does not use timestamps could gain from 15% to 86% in objective when maximizing direct profit ($f = 0$) and could gain from 1% to 15% in objective when maximizing social welfare ($f = 1$).

EC.5. Proofs of Results for Model Extensions in §5

In this section, we provide proofs of the results in §5.

EC.5.1. Proof of Proposition 7

(a) The proof follows essentially the same arguments as those in the proof of Proposition 4 (b), so we omit the details for brevity.

(b) Because timestamps impose a constraint $T \leq q^{-1}(p)$, then cannot be strictly optimal. If the upper bound on shelf life satisfies $\bar{T} \leq q^{-1}(p)$, the constraint imposed by timestamps is redundant, so the retailer would optimally use the same issuance and shelf life with and without timestamps and achieve the same objective. If $\bar{T} > q^{-1}(p)$, timestamps impose a more restrictive upper bound on shelf life; several cases arise, depending on the value of f:

- If $f > f^{\text{UB}}$, LIFO issuance is optimal without timestamps. Because the retailer's objective is pseudoconcave in T, the optimal shelf life is $\min\left(T_L^*, \bar{T}\right) > q^{-1}(p)$, where the inequality follows

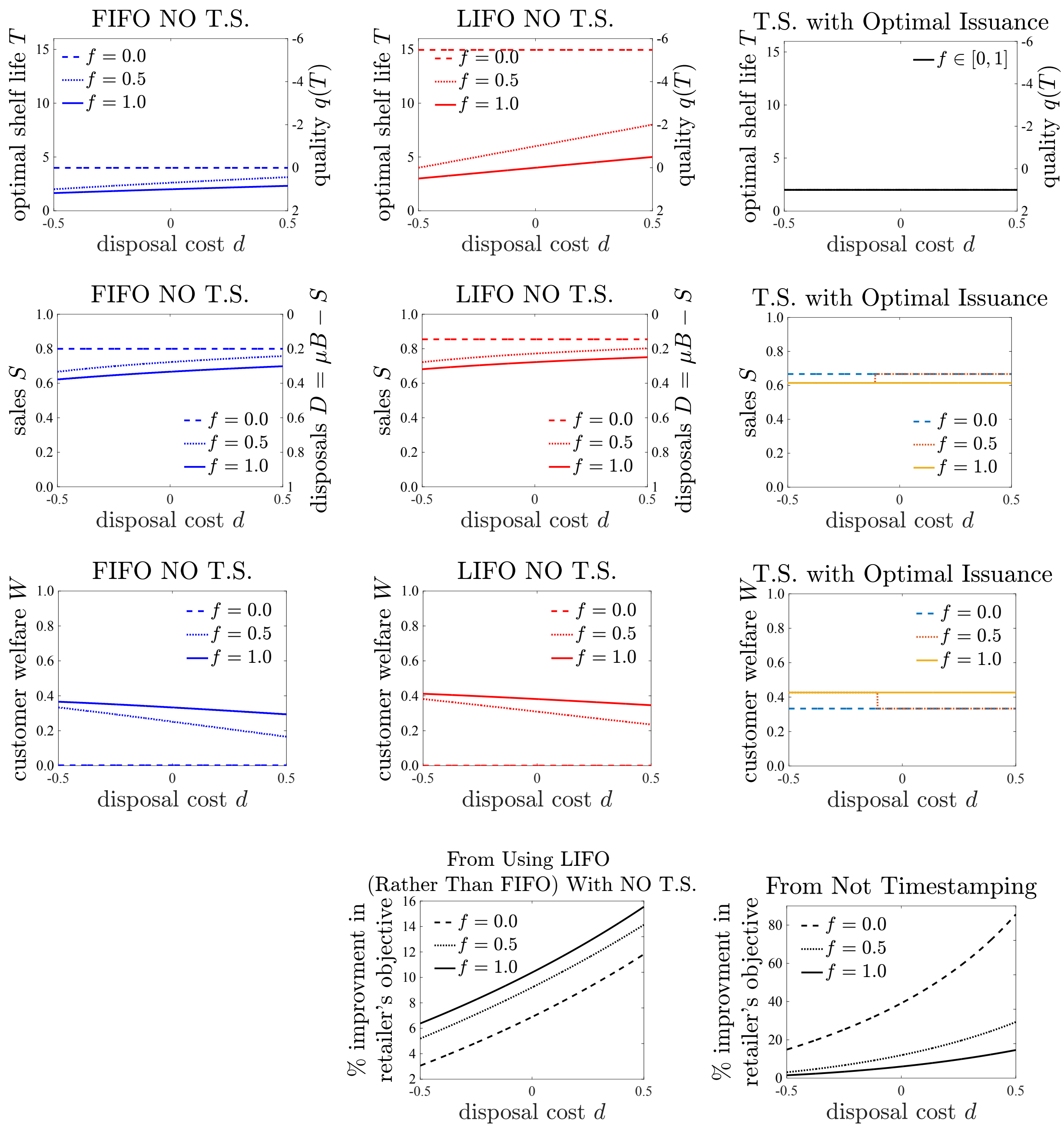


Figure EC.4 **Optimal shelf life (top row), sales (second row), customer welfare (third row), and improvement in retailer's objective (fourth row)**

because $T_L^* > q^{-1}(p)$ by Proposition 4. Timestamps would strictly reduce the shelf life and strictly reduce the retailer's objective.

- If $f = 0$, sales would be maximized with an optimal shelf life of $\min(\bar{T}_I, \bar{T}) > q^{-1}(p)$, where the inequality follows because $\bar{T}_I > q^{-1}(p)$ by Proposition 3. Because sales strictly increase with T, timestamps would strictly reduce the sales. Because the optimal objective is continuous in f, we readily obtain that there exists a threshold $\hat{f}^{\text{UB}} > 0$ such that timestamps would strictly decrease the retailer's objective if $f \in [0, \hat{f}^{\text{UB}})$.

- Note that the intermediate case above is only meaningful in its own right if without timestamps, the retailer optimally uses FIFO at $f = 0$ (otherwise, LIFO would be optimal for all $f \geq 0$ and timestamps would strictly reduce the objective). In this case, the threshold $\hat{f}^{\text{UB}}$ corresponds to the value of f at which the retailer would optimally set a shelf life of exactly $q^{-1}(p)$ without timestamps. Therefore, with $f \in [\hat{f}^{\text{UB}}, f^{\text{UB}})$, the retailer would optimally use FIFO issuance and use a shelf life $T \leq q^{-1}(p)$ without timestamps, and the presence of timestamps would have no effect on the optimal policy or on the objective.

EC.5.2. Proof of Proposition 8

The proof is by means of the instance documented in Figure 3.

EC.5.3. Proof of Proposition 9

The proof is by means of the instance documented in Figure 4. (b) Because timestamps translate into a constraint on the maximum shelf life, $T \leq q^{-1}(p)$, they cannot be strictly optimal. That they can strictly decrease the objective is apparent from the same numeric example in Figure 4.

EC.5.4. Proof of Proposition 10

(a) The result follows from the instance in Figure EC.5.

(b) With timestamps, an arriving customer purchases an item on offer if and only if $u(\tau) \geq 0$, which means that any feasible shelf life T must satisfy the constraint

$$T \leq u^{-1}(0). \tag{EC.5.1}$$

(Note that in the model with loss aversion, $u^{-1}(0) = q^{-1}(p)$, exactly like in our base model.)

Without timestamps, for any issuance I and shelf life T, we can readily define the (long-run) average utility $U^{I,T}$ that customers derive from a purchase and rewrite the condition that should be met to ensure that all arriving customers purchase an item on offer. Specifically, with issuance I and shelf life T, all customers would purchase an item on offer if and only if the average utility from purchases is positive, i.e.,

$$U^{I,T} := \lim_{t\to\infty} \frac{1}{t} \sum_{i=1}^{N_c(t)} u\left(\tau_{t_i}^{\pi}\right) \mathbf{1}_{\left\{z_{t_i}^{\pi} > 0\right\}} / S^{\pi} \geq 0, \tag{EC.5.2}$$

where the sales have the same expression as in (3.2).

Moreover, technical results mirroring those in Lemma EC.4 can be used to argue that $U^{I,T}$ is continuous and strictly decreasing in T, which allows us to define the maximum shelf life at which customers would purchase an item on offer with issuance I and without timestamps, $\bar{T}_I^\ell$, as:

$$\bar{T}_I^\ell := \sup\{T > 0 \,:\, U^{I,T} \geq 0\}. \tag{EC.5.3}$$

We claim that, mirroring our base model, the maximum shelf life without timestamps is strictly larger than with timestamps, $\bar{T}_I^\ell > u^{-1}(0)$. This follows because $U^{I,T}$ is continuous and strictly decreasing in T and every purchase of an item with age $\tau \leq u^{-1}(0)$ leads to positive ex-ante utility, so the shelf life T can be slightly extended beyond $u^{-1}(0)$ while still ensuring that the average utility $U^{I,T}$ is positive (albeit at the expense of generating strictly negative ex-post utility for some customers).

In this context, we can mirror the arguments from Proposition 3 to conclude that timestamps are never strictly optimal. Timestamps would require a choice of shelf life satisfying (EC.5.1), which strictly reduces the retailer's feasible options because $u^{-1}(0) < \bar{T}_I^\ell$.

Timestamps would strictly reduce the retailer's objective if $f = 0$. Without timestamps, that retailer would optimally set the maximum possible shelf life $\bar{T}_I^\ell$ under the optimal issuance I. By continuity of the optimal objectives in f, we can see that there exists a threshold $\hat{f}^\ell > 0$ such that a policy without timestamps is strictly better for any $f \leq \hat{f}^\ell$. For a sufficiently large f, it may be optimal to set a shelf life without timestamps that satisfies $\bar{T} \leq \bar{T}_I^\ell$ and at such a value of f, the policy with timestamps would achieve exactly the same objective as without timestamps.

EC.5.5. Proof of Theorem 2

First, consider an exogeneous price p satisfying $u(\theta_h, q(0)) > p > \max(0, -d)$. For the case without timestamps, introduce notation to formalize the equilibrium concept and we then prove that multiple equilibria exist and identify the Pareto-dominant equilibrium.

With heterogeneous customers, multiple rational equilibria could arise depending on the rate at which customers purchase items in stock and the resulting quality of the purchased items. Intuitively, the purchase rate and the purchased quality are positively related in equilibrium, because a higher purchased quality leads to a higher purchase rate, which in turn helps maintain a high purchased quality by preventing items from aging on the shelf.

To formalize this, we will define the rate with which items in stock are purchased and relate this to the minimum acceptable quality level that supports the demand. Note that when the purchased quality of items takes value Q, only those customers with type θ satisfying $\theta \geq \bar{\theta}(Q) := \inf\{\theta : u(\theta, Q) \geq p\}$ will buy an item on offer. This means that the rate at which customers arrive at the shelf and purchase an item on offer is $\Lambda := \lambda g(\bar{\theta}(Q))$. We subsequently refer to Λ as the *demand rate*, to distinguish it from the original arrival rate λ of customers. Additionally, note that the minimum quality level needed to ensure a demand rate of $\hat{\Lambda}$ is $\underline{Q}(\hat{\Lambda}) := \inf\{Q \in (-\infty, q(0)] : \lambda g(\bar{\theta}(Q)) \geq \hat{\Lambda}\}$.

These definitions allow us to formalize our equilibrium definition. An equilibrium is characterized by an equilibrium demand rate Λ_{eq} and an equilibrium purchased quality Q_{eq} that satisfy one of the following mutually exclusive conditions: (i) $\Lambda_{\text{eq}} = \lambda$ and $Q_{\text{eq}} \geq \underline{Q}(\Lambda_{\text{eq}})$, or (ii) $0 < \Lambda_{\text{eq}} = \lambda g(\bar{\theta}(Q_{\text{eq}}) < \lambda$ and $Q_{\text{eq}} = \underline{Q}(\Lambda_{\text{eq}})$, or (iii) $\Lambda_{\text{eq}} = 0$. Case (i) occurs when the purchased quality is sufficiently high that all arriving customers purchase items on offer. Case (ii) occurs when only those customers with sufficiently high θ purchase. Case (iii) occurs when no customers purchase items.

A few observations will allow us to leverage results derived in §4.1 in our analysis. First, note that for any policy π that the retailer follows, if the demand rate is Λ, then the stochastic process that governs the arrival of customers who purchase an item on offer is Poisson with rate Λ. So to characterize the sales and the purchased quality in the model with heterogeneous customers, we can replace the Poisson arrival process from our base model (with rate λ) with the Poisson arrival process of purchasing customers (with rate Λ) and assume that all customers who arrive according to this modified process purchase items on offer. This allows expressing the sales and purchased quality as a function of the retailer's policy π for any *hypothetical* demand rate $\hat{\Lambda}$ (including demand rates that may not occur in equilibrium). Specifically, if we considered a fixed demand rate $\hat{\Lambda}$, the sales $S^{\pi}(\hat{\Lambda})$ and purchased quality $Q^{\pi}(\hat{\Lambda})$ under policy π would be given by (3.2) and (3.3), respectively, where customer arrivals occur according to a Poisson process with rate $\hat{\Lambda}$ rather than rate λ as in our base model. Moreover, results analogous to Lemmas 1–3 and Proposition 1 are directly applicable under the *hypothetical* demand rate $\hat{\Lambda}$. To ensure self-consistency and that these results correspond to an equilibrium outcome with policy π, it then suffices to double check that the resulting purchased quality $Q^{\pi}(\hat{\Lambda})$ and the demand rate $\hat{\Lambda}$ satisfy one of the cases (i)-(iii).

Let us fix a policy $\pi = (I, T)$ and show that an equilibrium exists. Because $Q^{\pi}(\hat{\Lambda})$ and $\underline{Q}(\hat{\Lambda})$ are continuous functions of $\hat{\Lambda}$, one of the following three cases must arise:

- If $Q^{\pi}(\hat{\Lambda}) > \underline{Q}(\hat{\Lambda})$ for all $\hat{\Lambda} \in (0, \lambda]$, all customers would buy an item in stock and an equilibrium corresponding to Case (i) exists, i.e., $\Lambda_{\text{eq}} = \lambda$ and $Q_{\text{eq}} = Q^{\pi}(\lambda)$.

- If $Q^\pi(\hat{\Lambda}) < \underline{Q}(\hat{\Lambda})$ for all $\hat{\Lambda} \in (0, \lambda]$, no customer would buy and an equilibrium corresponding to Case (iii) exists, i.e., $\Lambda_{\text{eq}} = 0$.
- If $Q^\pi(\hat{\Lambda}) = \underline{Q}(\hat{\Lambda})$ for some $\hat{\Lambda} \in (0, \lambda]$, then only customers of types $\theta \geq \bar{\theta}(Q^\pi(\hat{\Lambda}))$ would buy and we have an equilibrium that corresponds to either Case (i) or (ii), i.e., $\Lambda_{\text{eq}} = \lambda$ and $Q_{\text{eq}} = Q^\pi(\lambda)$ or $\Lambda_{\text{eq}} = \hat{\Lambda}$ and $Q_{\text{eq}} = Q^\pi(\hat{\Lambda})$, $0 < \hat{\Lambda} < \lambda$.

Next, we argue that in case of multiple equilibria, the one with the largest demand rate yields the greatest objective for the retailer. Suppose there exist multiple equilibria (Λ_i, Q_i), $i = 1, \ldots, m$, with $0 \leq \Lambda_1 < \Lambda_2 < \cdots < \Lambda_m \leq \lambda$ and $Q_i = \underline{Q}(\Lambda_i)$. We claim that sales and purchased quality increase with the demand rate, i.e.,

$$S^\pi(\Lambda_i) \leq S^\pi(\Lambda_j) \text{ for any } \Lambda_i < \Lambda_j \tag{EC.5.4a}$$

$$Q^\pi(\Lambda_i) \leq Q^\pi(\Lambda_j) \text{ for any } \Lambda_i < \Lambda_j. \tag{EC.5.4b}$$

(EC.5.4b) follows because $Q^\pi(\Lambda_i) = Q_i = \underline{Q}(\hat{\Lambda})$ and $\underline{Q}(\hat{\Lambda})$ is increasing in $\hat{\Lambda}$ in the domain $[0, \lambda]$ because $\lambda g(\bar{\theta}(Q))$ is increasing in Q. Hence, $Q_1 \leq Q_2 \leq \cdots \leq Q_m$. (EC.5.4a) follows from a result that parallels Lemma EC.3(c). Therefore, (EC.5.4a) and (EC.5.4b) imply that higher sales and higher purchased quality are achieved in an equilibrium with a higher demand rate. This observation lets us characterize the Pareto-optimal equilibrium $(\Lambda_{\text{eq}}^{I,T}, Q_{\text{eq}}^{I,T})$ under policy (I, T):

$$\Lambda_{\text{eq}}^{I,T} = \sup\{\Lambda \in [0, \lambda] : Q^{I,T}(\Lambda) \geq \underline{Q}(\Lambda)\} \quad \text{and} \quad Q_{\text{eq}}^{I,T} = Q^{I,T}(\Lambda_{\text{eq}}^{I,T}). \tag{EC.5.5}$$

That the retailer's objective is highest in the Pareto-dominant equilibrium follows because the rate at which customer welfare is generated, $W^\pi = \mathbb{E}_\theta[u(\theta, Q) - p] \cdot S$, is increasing in both purchased quality Q and sales S.

The retailer's problem then involves choosing a policy (I, T) to maximize the objective:

$$V^{I,T}(d) := pS^{I,T}(\Lambda_{\text{eq}}^{I,T}) - d\big(\mu B - S^{I,T}(\Lambda_{\text{eq}}^{I,T})\big) + fW^{I,T}(\Lambda_{\text{eq}}^{I,T}), \tag{EC.5.6}$$

where

$$W^{I,T}(\Lambda_{\text{eq}}^{I,T}) = \mathbb{E}_\theta[u(\theta, Q_{\text{eq}}^{I,T}) - p] \cdot S^{I,T}(\Lambda_{\text{eq}}^{I,T}) \tag{EC.5.7}$$

is the rate of welfare generation for customers in equilibrium.

Finally, to show that LIFO dominates FIFO without timestamps, we prove that for any equilibrium under FIFO issuance, there exists an equilibrium with LIFO wherein the retailer achieves strictly higher objective value. Consider any shelf life T with FIFO and let $(\hat{\Lambda}, Q^{F,T}(\hat{\Lambda}))$ denote a pair of equilibrium demand rate and purchased quality values. By Lemma 3 applied with a fixed demand rate $\hat{\Lambda}$, there exists a shelf life $T' > T$ such that LIFO issuance with shelf life T' achieves the same sales as FIFO issuance with shelf life T but strictly higher customer welfare, i.e., $S^{L,T'}(\hat{\Lambda}) = S^{F,T}(\hat{\Lambda})$ and $W^{L,T'}(\hat{\Lambda}) > W^{L,T'}(\hat{\Lambda})$. Because sales are identical, this also implies that purchased quality is strictly higher with LIFO and T', i.e., $Q^{L,T'}(\hat{\Lambda}) > Q^{F,T}(\hat{\Lambda})$, and in particular, that the retailer's objective is strictly higher with (L,T') than with (F,T). To complete our proof, it remains to show that $(\hat{\Lambda}, Q^{L,T'}(\hat{\Lambda}))$ is an equilibrium with policy (L,T'). Two cases arise:

- If $\hat{\Lambda} = \lambda$, then $(\hat{\Lambda}, Q^{L,T'}(\hat{\Lambda}))$ is indeed an equilibrium according to case (i), because:

$$Q^{L,T'}(\lambda) > Q^{F,T}(\lambda) = \underline{Q}(\lambda).$$

- If $\hat{\Lambda} < \lambda$, we construct another policy with LIFO that leads to an equilibrium with strictly larger objective than (L,T'). Recall that $Q^{L,T}(\hat{\Lambda})$ is continuous and strictly decreasing in T (by Lemma 1). This implies that there exists a shelf life $T'' > T'$ such that $Q^{L,T''}(\hat{\Lambda}) = \underline{Q}(\hat{\Lambda})$, which implies that $(\hat{\Lambda}, Q^{L,T''}(\hat{\Lambda}))$ is a valid equilibrium corresponding to case (ii). Furthermore, because sales are strictly increasing in T with a fixed demand rate (by Lemma 1), we have that $S^{L,T''}(\hat{\Lambda}) > S^{F,T'}(\hat{\Lambda})$. And because purchased quality is the same, this implies that customer welfare is strictly larger, which in turn implies that the retailer's objective with (L,T'') is strictly larger than with (L,T') and therefore strictly larger than (F,T), which completes our argument.

Finally, because we proved that the objective is strictly larger with LIFO (and an optimal shelf life) for any exogenously fixed $p > -d$, the continuity of the optimal objective value implies that LIFO would be optimal even when the price p is optimally chosen.

EC.5.6. Proof of Proposition 11

(a) We compare the optimal policy with timestamps with the optimal policy without timestamps. The policy with timestamps cannot screen agents based on type so it will always use a shelf life of $q^{-1}(p - \theta_h)$, corresponding to the largest age at which a high-type customer would purchase an item knowing its age. Without timestamps, the retailer has a choice of strategy: selling to both customer types or selling only to the high type. Therefore, we consider each case separately and then derive the necessary and sufficient conditions under which the optimal policy with timestamps dominates each of the options.

Case 1. Assume that, without timestamps, the retailer *optimally sells to both customer types*. For the retailer to make any sales to low-type customers, it must be that a low-type customer is willing to purchase a new item. To that end, recall our standing assumption that $u(\theta_\ell, q(0)) > 0$, which in this case implies that:

$$0 < q_0 + \theta_\ell - p \quad \Leftrightarrow \quad q^{-1}(p - \theta_\ell) > 0. \tag{EC.5.8}$$

This also implies that a high-type customer would purchase the item, i.e., $q^{-1}(p - \theta_h) > 0$.

The expected purchased quality without timestamps and with LIFO and shelf life T is:

$$Q^{L,T} = \frac{\int_0^T \lambda e^{-\lambda t}(q_0 - bt)dt}{1 - e^{-\lambda T}} = q_0 - \frac{b}{\lambda} + \frac{bT}{e^{\lambda T} - 1}. \tag{EC.5.9}$$

It can be checked that:

$$\begin{aligned}
\frac{\partial Q^{L,T}}{\partial T} &= -\frac{b\big(1 + e^{\lambda T}(\lambda T - 1)\big)}{(e^{\lambda T} - 1)^2} < 0, \text{ for } \lambda T > 0 \\
\frac{\partial Q^{L,T}}{\partial \lambda} &= bT\Big(\frac{1}{(\lambda T)^2} + \frac{1}{2 - 2\cosh \lambda T}\Big) > 0, \text{ for } \lambda T > 0,
\end{aligned}$$

so quality is strictly decreasing in T and strictly increasing in λ. As the shelf life gets very large, the purchased quality converges to the finite limit $\lim_{T\to\infty} Q^{L,T} = q_0 - \frac{b}{\lambda}$, i.e., the quality derived from an expected customer arrival.

The shelf life without timestamps $\bar{T}^{\ell,h}$ is the largest T such that $Q^{L,T} + \theta_\ell \geq p$. Because the quality converges to a finite limit as $T \longrightarrow \infty$, the shelf life would actually be infinite if:

$$q_0 - \frac{b}{\lambda} + \theta_\ell \geq p. \tag{EC.5.10}$$

Our subsequent expression will allow for this (although we will also show that the shelf life cannot be infinite if timestamps are strictly optimal).

When the optimal shelf life is finite, it is the solution to the equation:

$$q_0 - \frac{b}{\lambda} + \frac{bT}{e^{\lambda T} - 1} + \theta_\ell = p \;\Leftrightarrow\; \frac{b - \lambda(q_0 + \theta_\ell - p)}{b} = \frac{\lambda T}{e^{\lambda T} - 1}. \tag{EC.5.11}$$

Consider the function $\frac{x}{e^x - 1}$ appearing in the right-hand-side of this equation. On the domain $[0, \infty)$, this takes values in the range $[0, 1]$, with $\lim_{x\to 0} \frac{x}{e^x-1} = 1$ and $\lim_{x\to\infty} \frac{x}{e^x-1} = 0$, and the function is continuous and strictly decreasing in x. Then, let us define the function $g : (-\infty, 1] \to \mathbb{R}_+$ such that $g(x)$ equals the inverse of the function $\frac{x}{e^x-1}$ for $x > 0$ and $g(x) = +\infty$ for $x \leq 0$. Note also that the limit of the inverse of $\frac{x}{e^x-1}$ as $x \to 0$ is $+\infty$, so $g(x)$ is continuous. Then, the optimal shelf life without timestamps is:

$$\bar{T}^{\ell,h} = \frac{1}{\lambda} g\Big(1 - \frac{\lambda(q_0 + \theta_\ell - p)}{b}\Big). \tag{EC.5.12}$$

Importantly, this expression is also valid when the condition in (EC.5.10) holds, because in that case the argument for g would be negative and we would obtain $\bar{T}^{\ell,h} = \infty$.

Case 2. Consider now the case that, without timestamps, the retailer *only sells to high type customers*. The relevant arrival process is a Poisson process with rate $\beta\lambda$, corresponding to the high types. The corresponding expected purchased quality with LIFO and shelf life T can be obtained by replacing λ with $\beta\lambda$ in (EC.5.9):

$$Q_h^{L,T} = q_0 - \frac{b}{\beta\lambda} + \frac{bT}{e^{\beta\lambda T} - 1}.$$

The shelf life without timestamps $\bar{T}^h$ is then generally expressed as:

$$\bar{T}^h = \frac{1}{\beta\lambda} g\Big(1 - \frac{\beta\lambda(q_0 + \theta_h - p)}{b}\Big). \tag{EC.5.13}$$

As before, we do *not* require the shelf life to be finite: if $\frac{\beta\lambda(q_0+\theta_h-p)}{b} \geq 1$, the shelf life would become infinite, $\bar{T}^h = +\infty$.

(b) To express the gains in sales from using timestamps, it can be readily verified that $q^{-1}(p-\theta_\ell) := \frac{q_0+\theta-p}{b} < \bar{T}^{\ell,h}$ holds and that $q^{-1}(p-\theta_h) := \frac{q_0+\theta-p}{b} < \bar{T}^h$. The condition $\bar{T}^{\ell,h} < q^{-1}(p-\theta_h)$ is not automatically satisfied, but it will be implied by our subsequent requirement that timestamps strictly improve sales. It can also be readily checked

In Case 1, the difference in sales with timestamps and without timestamps is:

$$\begin{aligned}
\Delta_1 &:= S^{(L,q^{-1}(p-\theta_h),ts)} - S^{(L,\bar{T}^{\ell,h},nts)} \\
&= \underbrace{\Big(e^{-\lambda\bar{T}^{\ell,h}} - e^{-\lambda q^{-1}(p-\theta_h)}\Big)}_{\text{first arrival between } \bar{T}^{\ell,h} \text{ and } q^{-1}(p-\theta_h)} \cdot \underbrace{\beta}_{\text{type H}} - \underbrace{\Big(e^{-\lambda q^{-1}(p-\theta_\ell)} - e^{-\lambda\bar{T}^{\ell,h}}\Big)}_{\text{first arrival between } q^{-1}(p-\theta_\ell) \text{ and } \bar{T}^{\ell,h}} \cdot \underbrace{(1-\beta)}_{\text{type L}} \\
&= e^{-\lambda\bar{T}^{\ell,h}} - \beta \cdot e^{-\lambda q^{-1}(p-\theta_h)} - (1-\beta) \cdot e^{-\lambda q^{-1}(p-\theta_\ell)} \\
&= e^{-g\left(1-\frac{\lambda(q_0+\theta_\ell-p)}{b}\right)} - \beta \cdot e^{-\frac{\lambda(q_0+\theta_h-p)}{b}} - (1-\beta) \cdot e^{-\frac{\lambda(q_0+\theta_\ell-p)}{b}}.
\end{aligned}$$

In Case 2, the difference in sales with timestamps and without timestamps is:

$$\begin{aligned}
\Delta_2 &= S^{(L,q^{-1}(p-\theta_h),ts)} - S^{(L,\bar{T}^h,nts)} \\
&= \underbrace{\left(1 - e^{-\lambda q^{-1}(p-\theta_\ell)}\right)}_{\text{first arrival between 0 and } q^{-1}(p-\theta_\ell)} \cdot \underbrace{(1-\beta)}_{\text{type L}} - \underbrace{\left(e^{-\lambda q^{-1}(p-\theta_h)} - e^{-\lambda \bar{T}^h}\right)}_{\text{first arrival between } q^{-1}(p-\theta_h) \text{ and } \bar{T}^h} \cdot \underbrace{\beta}_{\text{type H}} \\
&= 1 - \beta - (1-\beta)e^{-\lambda q^{-1}(p-\theta_\ell)} - \beta e^{-\lambda q^{-1}(p-\theta_h)} + e^{-\lambda \bar{T}^h} \\
&= 1 - \beta - (1-\beta)e^{-\frac{\lambda(q_0+\theta_\ell-p)}{b}} - \beta e^{-\frac{\lambda(q_0+\theta_h-p)}{b}} + \beta e^{-\frac{1}{\beta} g\left(1 - \frac{\beta\lambda(q_0+\theta_h-p)}{b}\right)}.
\end{aligned}$$

For timestamps to result in strictly larger sales than the optimal policy without timestamps, it must be that timestamps yield strictly larger sales than the retailer's policy without timestamps *in each of the two cases*. Therefore, we can conclude that the gains in sales from using timestamps are strictly positive if $\Delta_1 > 0$ and $\Delta_2 > 0$, in which case the gains have the expression $\min(\Delta_1, \Delta_2)$.

(c) Consider again the necessary and sufficient conditions rewritten for convenience:

$$0 < \Delta_1 := e^{-\lambda \bar{T}^{\ell,h}} - \beta e^{-\lambda q^{-1}(p-\theta_h)} - (1-\beta)e^{-\lambda q^{-1}(p-\theta_\ell)} \qquad \text{(EC.5.14a)}$$

$$0 < \Delta_2 := 1 - \beta + \beta e^{-\lambda \bar{T}^h} - \beta e^{-\lambda q^{-1}(p-\theta_h)} - (1-\beta)e^{-\lambda q^{-1}(p-\theta_\ell)}. \qquad \text{(EC.5.14b)}$$

Because $\beta e^{-\lambda \bar{T}^h} \geq 0$, a set of sufficient conditions is obtained by imposing that:

$$\begin{aligned}
\min\left(e^{-\lambda \bar{T}^{\ell,h}}, 1-\beta\right) - \beta e^{-\lambda q^{-1}(p-\theta_h)} - (1-\beta)e^{-\lambda q^{-1}(p-\theta_\ell)} > 0 \;\Leftrightarrow \\
\min\left(e^{-g(1-\lambda q^{-1}(p-\theta_\ell))}, 1-\beta\right) - \beta e^{-\lambda q^{-1}(p-\theta_h)} - (1-\beta)e^{-\lambda q^{-1}(p-\theta_\ell)} > 0.
\end{aligned} \qquad \text{(EC.5.15)}$$

Consider the first term appearing in the expression and note that:

$$\begin{aligned}
1 - \beta < e^{-g(1-\lambda q^{-1}(p-\theta_\ell))} &\Leftrightarrow g(1 - \lambda q^{-1}(p-\theta_\ell)) < -\ln(1-\beta) \\
\left(\text{because } g^{-1}(x) = \frac{x}{e^x - 1}\right) &\Leftrightarrow 1 - \lambda q^{-1}(p-\theta_\ell) > \frac{-\ln(1-\beta)}{\frac{1}{1-\beta} - 1} \\
&\Leftrightarrow \lambda q^{-1}(p-\theta_\ell) < \frac{\beta + (1-\beta)\ln(1-\beta)}{\beta} \\
&\Leftrightarrow \theta_\ell < \frac{b}{\lambda}\,\frac{\beta + (1-\beta)\ln(1-\beta)}{\beta} + p - q_0.
\end{aligned} \qquad \text{(EC.5.16)}$$

The function $\frac{\beta+(1-\beta)\ln(1-\beta)}{\beta}$ is continuous and increasing and takes values in $[0,1]$ for $\beta \in [0,1]$ (with limit 0 as $\beta \to 0$ and limit 1 as $\beta \to 1$). Therefore, (EC.5.16) implies that $\theta_\ell < \frac{b}{\lambda} + p - q_0$ (and therefore $q^{-1}(p-\theta_\ell) < \infty$) and that we must require $\beta > 0$ to ensure that $\theta_\ell > p - q_0$ is feasible.

Then, for any $\beta > 0$ and θ_ℓ satisfying (EC.5.16), condition (EC.5.15) is equivalent to:

$$e^{-\lambda q^{-1}(p-\theta_h)} < \frac{(1-\beta)\left(1-e^{-\lambda q^{-1}(p-\theta_\ell)}\right)}{\beta} \Leftrightarrow \theta_h > -\frac{1}{\lambda}\ln\frac{(1-\beta)\left(1-e^{-\frac{\lambda(q_0+\theta_\ell-p)}{b}}\right)}{\beta}. \quad \text{(EC.5.17)}$$

Importantly, the latter condition cannot hold if $\beta = 1$, so we must require $\beta \in (0,1)$. With this requirement, a sufficiently high θ_h that satisfies (EC.5.17) can always be found.

To summarize, the sufficient conditions that must be imposed are:

$$\begin{aligned} &\beta \in (0,1) \\ &\theta_\ell < p - q_0 + \frac{b}{\lambda}\frac{\beta + (1-\beta)\ln(1-\beta)}{\beta} \\ &\theta_h > -\frac{1}{\lambda}\ln\frac{(1-\beta)\left(1-e^{-\frac{\lambda(q_0+\theta_\ell-p)}{b}}\right)}{\beta}, \end{aligned}$$

which exactly correspond to the conditions in the statement.

To conclude, we verify one additional premise. Recall that all the instances we consider require that the interarrival time of batches should satisfy $\ell_n \geq \frac{1}{\beta\lambda}g\left(1-\frac{\beta\lambda(q_0+\theta_h-p)}{b}\right)$. To allow a *finite* interarrival time, it is necessary and sufficient to ensure that $\beta\lambda q^{-1}(p-\theta_h) < 1$. The concern is that the upper bound on θ_h from (EC.5.17) may make it infeasible to satisfy this additional constraint (if we chose to). We show that this is not the case by proving that, under any feasible β and any feasible θ_ℓ in the system above, we can find θ_h satisfying the requirement. To that end, note that (EC.5.17) and (EC.5.16) imply:

$$\begin{aligned} -\beta\,\lambda\,q^{-1}(p-\theta_h) &< \beta\ln\frac{(1-\beta)\left(1-e^{-\lambda q^{-1}(p-\theta_\ell)}\right)}{\beta} \\ \text{(from (EC.5.16), for any feasible } \theta_\ell) &< \beta\ln\frac{(1-\beta)\left(1-e^{-\frac{\beta+(1-\beta)\ln(1-\beta)}{\beta}}\right)}{\beta} \Leftrightarrow \\ \beta\,\lambda\,q^{-1}(p-\theta_h) &> -\beta\ln\frac{(1-\beta)\left(1-e^{-\frac{\beta+(1-\beta)\ln(1-\beta)}{\beta}}\right)}{\beta}. \end{aligned}$$

The function appearing in the right-hand-side is increasing on the interval $\beta \in [0,1]$ and takes values in $[0,4]$. Therefore, we readily have that this does not impose any constraint on the left-hand-side, which is positive. In particular, this shows that if we want to explicitly impose a requirement that ℓ_n be finite, this can be done by just adding the requirement $\beta\lambda q^{-1}(p-\theta_h) < 1$, and we will still recover feasible instances.

EC.6. Numerical Illustration of Results for Risk Aversion Extension

In the running example, this section illustrates the results for the extension with customer risk-aversion in §5.4. The running example has price $p = 1$, customer arrival rate $\lambda = 1$, Poisson replenishment with rate $\mu = 1$ and batch size $B = 1$, and quality $q(\tau) = 2 - 0.5\tau^{\alpha}$. As illustrated in Figure EC.5, higher risk-aversion among customers favors FIFO because customers purchase the freshest item with FIFO, hence they have lower probability of getting old items that have significantly lower quality.

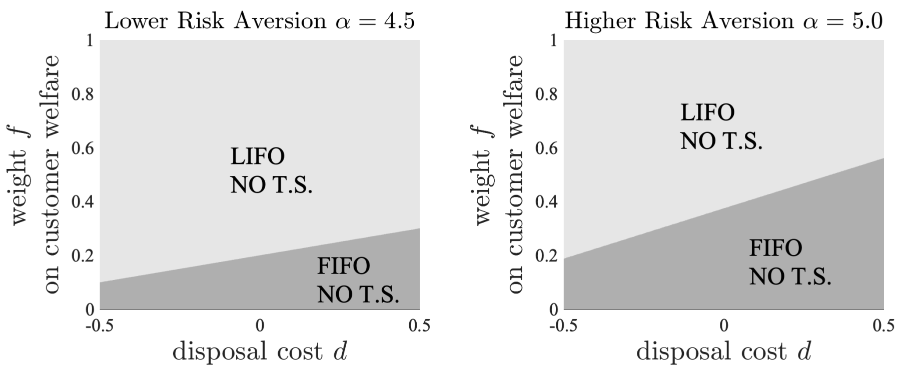


Figure EC.5 Effect of risk aversion on optimal issuance and timestamp policy with $u(\tau) = 2 - 0.5\tau^{\alpha}$.

EC.7. Numerical Experiments with Heterogeneous Customers

In all our numerical experiments with heterogeneous customers, we observe that LIFO is optimal, even in parameter regions where timestamps become strictly optimal. We numerically calculated the optimal policy (issuance, shelf life and whether to timestamp items) in the following settings. We considered two forms of heterogeneity in customer utility. In the first, $u(\theta, q(\tau)) = \theta + q(\tau)$ and θ has a uniform distribution on $[0, \epsilon]$. In the second, $u(\theta, q(\tau)) = \theta q(\tau)$ and θ has a uniform distribution on $[1, 1+\epsilon]$. To favor FIFO, we consider linear quality degradation schedules $q(\tau) = q_0 - b\tau$ (recall that strict convexity in $q(\tau)$ would favor LIFO). To further favor FIFO, we consider a range of disposal cost values d that includes extremely high disposal cost, up to 50% greater than the initial quality $q(0)$ of an item. We consider Poisson replenishment with rate $\mu = 1$ and batch size $B = 1$. We consider all possible combinations of the parameter levels characterized by Table EC.2. For each parameter, we consider the number of levels specified in Table EC.2, ranging in *equal increments* from the minimum to the maximum in the table. This amounts to $101,088$ experiments. In each problem instance, we compare the performance of three policies: LIFO issuance without timestamps, LIFO issuance with timestamps, and FIFO issuance with timestamps. (We do not include FIFO issuance without timestamps because it is always strictly dominated by LIFO without timestamps). The results are summarized in Figure EC.6 and Figure EC.7.

Table EC.2 Table of Parameters

Parameter	Number of Levels	Minimum	Maximum	Increment Size
weight on customer welfare f	3	0	1	0.5
customer heterogeneity ϵ	7	0	3	0.5
initial quality q_0	3	1	3	1
quality degradation rate b	6	0.25	1.5	0.25
customer arrival rate λ	5	0.5	1	0.1
disposal cost d	5	-0.5	1.5	0.5
price p	2 to 7	$\max(-d, 0)$	q_0	0.5

Figure EC.6, which depicts histograms that compare the three policies, shows that LIFO issuance is optimal in all our experiments. The first row shows that with *required* timestamps, either LIFO or FIFO issuance could be dominant, which is consistent with Proposition 4. With required timestamps, the gains from using the correct issuance can be as high as 40% with additive utility and 40% with multiplicative utility. The second row of Figure EC.6 shows to *not* timestamp is optimal in the majority of problem instances and that losses from requiring timestamps could be very high (535% with additive utility and 565% with multiplicative utility). The third row considers only those instances where timestamps are optimal (46 in a model with additive utility and 3 in a model with multiplicative utility) and shows that LIFO issuance (with timestamps) is preferred to FIFO issuance (with timestamps) in every such instance. Thus, LIFO issuance is optimal in all our tests.

Figure EC.7 allows examining how the gains from using the optimal issuance or from using timestamps depend on the retailer's objective and on the degree of customer heterogeneity. The first row shows that with required timestamps, LIFO issuance leads to larger gains than FIFO issuance when the retailer places more weight on customer welfare (i.e., under a larger f). This is consistent with Proposition 4(b), which states that LIFO dominates FIFO at sufficiently large f. With required timestamps, the gains from using the optimal issuance are more pronounced if customers have homogeneous preferences. The second row of the figure shows that losses from timestamps are larger when the retailer prioritizes direct profit/sales; losses decrease as the retailer places more weight on customer welfare (i.e., higher f), which is consistent with our findings in Proposition 3(b) that timestamps improve customer welfare. Interestingly, although timestamps may become optimal in the presence of customer heterogeneity (which is consistent with the results in Section 5.5),

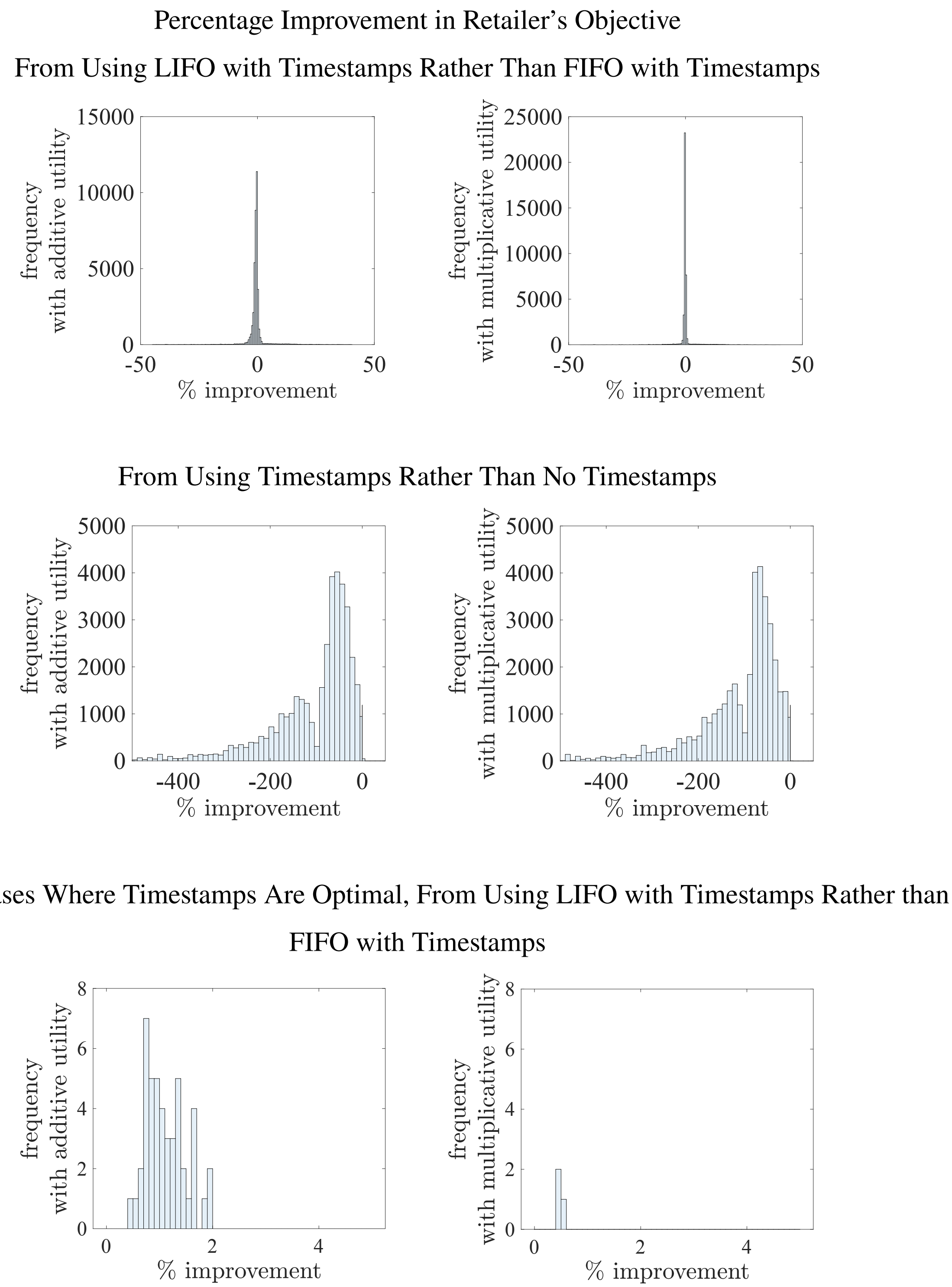


Figure EC.6 **In all cases, the retailer optimizes the shelf life. As is evident from the last row, LIFO is optimal in all cases where timestamps are optimal.**

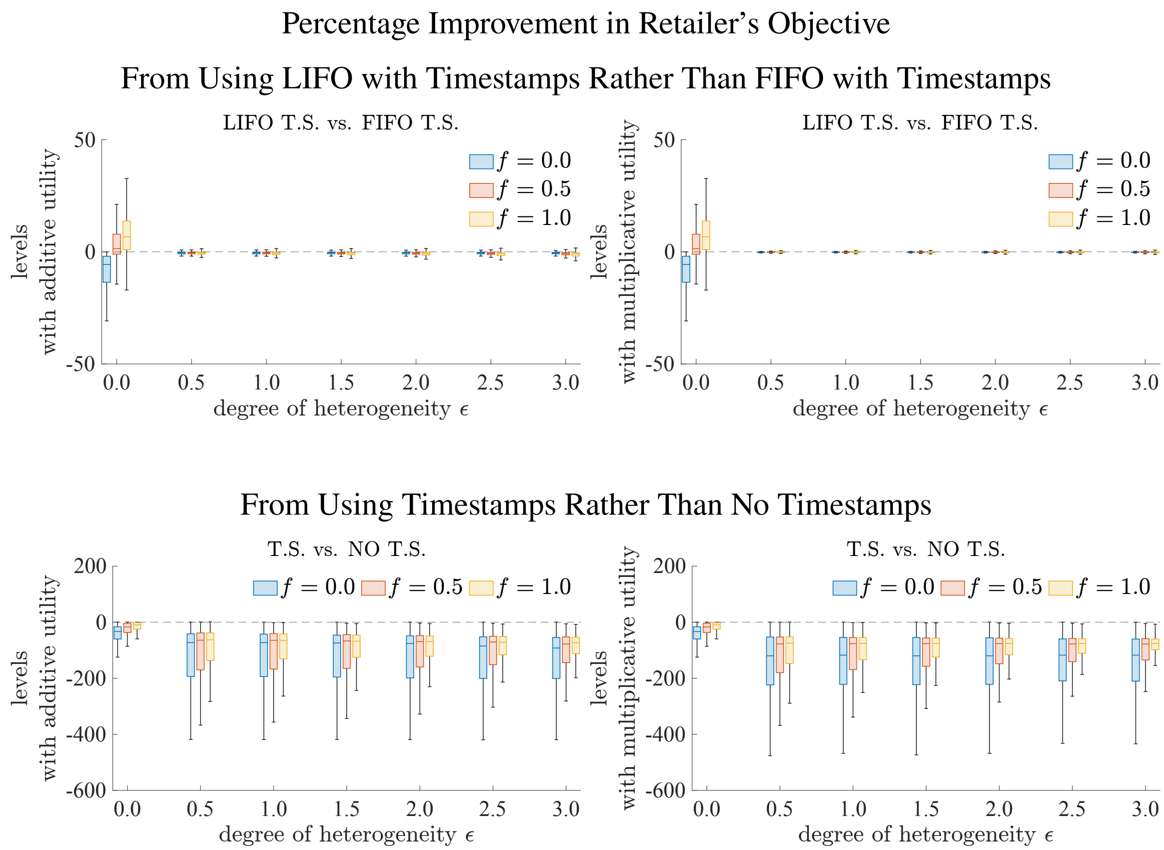


Figure EC.7 **Blue, red, and yellow indicate cases with $f = 0$, $f = 0.5$, and $f = 1$. In all cases the retailer optimizes the shelf life.**

larger heterogeneity does not necessarily lead to increased gains from using timestamps; in fact, the largest gains occur with a small degree of heterogeneity ($\epsilon = 0.5$), whereas with large heterogeneity, timestamps only create gains in few problem instances and can lead to substantial losses.

EC.8. Pure vs. Mixed Strategy Equilibrium

This section analyzes mixed strategy equilibria and shows that the pure-strategy equilibrium discussed in the main body of the paper is Pareto-dominant: the retailer achieves a strictly larger objective and customers achieve a strictly larger welfare than in any other mixed-strategy equilibrium. Subsequently, we also discuss practical ways in which retailers could induce this (preferred) Pareto-dominant equilibrium.

With no timestamps, a customer cannot observe the quality of the items currently on the shelf. Hence, we focus on mixed strategy equilibria wherein the long-run average purchased quality $Q \geq p$, an arriving customer purchases the item on offer with probability $\eta \in (0, 1]$, and if $Q > p$ then $\eta = 1$. This equilibrium concept is adopted from the literature on unobserved queues with strategic customers (Hassin and Haviv 2003).

Lemma EC.7 establishes an upper bound on the optimal shelf life that a retailer should use. The rationale developed in Part (a) is that purchased quality decreases with the shelf life, for any fixed probability π that an arriving customer that finds an item on offer makes a purchase. Part (b) establishes that an equilibrium with $\pi = 1$ exists if and only if the retailer sets the shelf life at or below the threshold specified in (EC.8.1). Furthermore, sales and customer welfare, and hence the retailer's optimal objective value, are higher in an equilibrium with $\pi = 1$ than in any equilibrium with $\pi < 1$.

Our notation $Q^{I,T}(\eta\lambda, \mu)$ serves to emphasize the dependence of purchased quality Q on the issuance I, shelf life T, arrival rate λ and purchase probability η.

LEMMA EC.7. *Under either FIFO ($I = F$) or LIFO ($I = L$) issuance:*

(a) Purchased quality $Q^{I,T}(\eta\lambda, \mu)$ strictly decreases with the shelf life T for any $\eta \in [0, 1]$.

(b) At each shelf life $T \in (0, \bar{T}_I]$ where

$$\bar{T}_I := \sup\{T \geq 0 : Q^{I,T}(\lambda, \mu) \geq p\}, \tag{EC.8.1}$$

an equilibrium exists wherein every customer that finds an item on the shelf makes a purchase ($\eta = 1$). In that equilibrium with $\eta = 1$, both sales and customer welfare are higher than in any equilibrium with $\eta \in [0, 1)$ at the same shelf life T. At each shelf life $T > \bar{T}_I$, only equilibria with $\eta < 1$ may exist wherein both sales and customer welfare are lower than in the equilibrium with $\eta = 1$ with shelf life $\bar{T}_I$.

The lemma implies that the retailer may only consider pure strategies because the retailer can achieve higher sales and customer welfare than mixed strategies for a given shelf life. In practice, to induce this preferred equilibrium, a retailer could offer free samples or discounts to maximize the demand rate and corresponding quality that customers experience and come to expect. Many grocery retailers, such as Walmart, Costco, and Whole Foods Market offer free samples, temporary price discounts, or other promotions (e.g., "buy one get one free") as a way to attract customers, make them familiar with the product, and eventually lock them in. Once customers buy the product at the maximum demand rate and become acquainted with the purchased quality, the preferred equilibrium would be sustained even after the promotions end.

Proof of Lemma EC.7. Let us fix a sample path of batch and customer arrivals, and let T_1 and T_2 be two shelf lives with $T_1 < T_2$.

(a) Let us consider FIFO issuance first. According to Lemma EC.1(a), for an item that exists on the shelf under both policies at the same time, there are more older items on the shelf under (F, T_2)

than under (F,T_1). Thus, for each item that is sold under (F,T_2), we have that either: (i) the item was sold under (F,T_1) at an earlier time or (ii) the item was disposed of under (F,T_1), and it was only sold under (F,T_2). In the latter event, which occurs with positive probability, the item had an age in $[T_1,T_2)$ when it was purchased under (F,T_2), so its quality was strictly lower than $q(T_1)$ and thus strictly lower than any purchased quality under (F,T_1). These observations together imply that the long-run average purchased quality is strictly lower under (F,T_2) than under (F,T_1).

Next, we consider LIFO issuance. According to Lemma EC.3(a), the retailer has greater sales under (L,T_2) than under (L,T_1). Lemma EC.1(b) implies that the incremental sales under (L,T_2) come from the sales of items of age in $[T_1,T_2)$, whose quality is strictly lower than that of any items of age in $[0,T_1)$. Hence, it follows that the long-run average purchased quality is strictly lower under (L,T_2) than under (L,T_1).

(b) For convenience, let us refer to the equilibrium where each customer purchases with probability η as "equilibrium η".

Suppose that we have $T \in (0,\bar{T}^I]$ for $I \in \{F,L\}$. We first establish that a pure strategy equilibrium $\eta = 1$ (wherein all customers buy the product when it is in stock) exists. Suppose all customers buy the product when it is offered, i.e., $\eta = 1$. By the definition of $\bar{T}^I$ and Part (a), we have $Q^{I,T}(\lambda,\mu) \geq p$, which implies that the expected utility from a purchase is non-negative. This proves the existence of a pure strategy equilibrium.

We next treat the case where $T > \bar{T}_I$ for $I \in \{F,L\}$. Suppose that there exists a mixed strategy $\eta \in (0,1)$ in which customers purchase the product with probability η when it is offered and the average quality of purchased items is equal to the price p (so that customers are indifferent between a purchase and no purchase). Note that the expected utility from a purchase is lower with this mixed strategy than with the pure strategy ($\eta = 1$) because $Q^{I,T}(\eta\lambda,\mu) - p = 0 \leq Q^{I,T}(\lambda,\mu) - p$ where the inequality follows because $T \leq \bar{T}^I$. In addition, Lemma EC.3(c) implies that sales are lower with the mixed strategy than with the pure strategy. Combining these observations, we obtain that both the sales and the customer welfare $(Q^{I,T}(\eta\lambda,\mu) - p)S^{I,T}(\eta\lambda,\mu)$ are higher with the pure strategy equilibrium.

Finally, we show that a pure strategy equilibrium cannot exist in this case. We first consider FIFO issuance. Suppose that there exists a mixed or pure strategy equilibrium $\eta \in (0,1]$. Then, Lemma EC.4(d) gives that $Q^{F,T}(\eta\lambda,\mu) \leq Q^{F,T}(\lambda,\mu) < p$ where the last inequality follows because $T > \bar{T}_F$. This implies that the expected utility from a purchase is negative and customers would not buy the product, and thus no such equilibrium can exist. We next examine LIFO issuance. It is straightforward to see that the pure strategy $\eta = 1$ cannot be an equilibrium because $Q^{L,T}(\lambda, \mu) < p$ for $T > \bar{T}_L$. Suppose there exists an equilibrium $\eta < 1$. Because it is a mixed strategy equilibrium, customers expect zero utility from a purchase (and hence zero customer welfare). Hence, it remains to show that the retailer can achieve strictly higher sales while keeping the customers' expected utility from a purchase equal to zero by reducing the shelf life from T to $\bar{T}_L$. Consider the equilibrium wherein the retailer uses shelf life $\bar{T}_L$ and customers play the pure strategy $\eta = 1$. Using parts (b-c) of Lemma EC.4 and the fact that $R^L(0; \eta\lambda, \mu) = R^L(0; \lambda, \mu) = 0$, we obtain that the (unique) solution $S > 0$ to $R^L(S) - pS = 0$ (at which customers expect zero utility from a purchase) is greater in the pure strategy with shelf life $\bar{T}_L$ than in the mixed strategy η with shelf life T. This proves that the retailer achieves higher sales in the pure strategy equilibrium than in the mixed strategy equilibrium η.

EC.9. Extension: Endogenous Production

In this extension, we consider a setting where the retailer chooses the production rate μ and the integer batch size B, in addition to the issuance rule, shelf life, and whether or not to timestamp, to maximize

$$pS^\pi - dD^\pi + fW^\pi - c\mu B = \left[(1-f)p + d\right]S^\pi + fS^\pi Q^\pi - (c+d)B\mu,$$

where S^π, D^π, and W^π are the corresponding sales, disposal, and customer welfare, respectively, and c is the unit production cost. Notice that we may regard that the retailer solves the optimization problem in two stages: first, she decides the production rate μ and the batch size B (outer optimization); second, she chooses the issuance rule, shelf life, and whether or not to timestamp items (inner optimization). We address the following four claims.

EC.9.1. LIFO is optimal

It is sufficient to show that LIFO dominates FIFO when the retailer does not provide a timestamp. (With a timestamp customers to pick the newest item first, i.e., LIFO.)

Let us denote the optimal objective under issuance I in the absence of a timestamp when the production rate is μ and the batch size is B by $V^I(\mu, B)$. We establish in the proof of Theorem 2 that $V^F(\hat{\mu}, \hat{B}) \le V^L(\hat{\mu}, \hat{B})$ for any $\hat{\mu}$ and $\hat{B}$. Hence, it follows that $V^F(\hat{\mu}, \hat{B}) \le \sup_{\mu,B} V^L(\mu, B)$ for any $\hat{\mu}$ and $\hat{B}$, which in turn implies that $\sup_{\mu,B} V^F(\mu, B) \le \sup_{\mu,B} V^L(\mu, B)$. This proves the claim.

EC.9.2. The retailer does not timestamp items with homogeneous customers

When customers are homogeneous, Proposition 4 suggests that the retailer's optimal objective strictly decreases with the introduction of a timestamp for any given (μ, B). That is, if we denote the optimal objective with and without a timestamp when the production rate is μ and the batch size is B by $V_{\mathsf{ts}}(\mu, B)$ and $V(\mu, B)$, respectively, we have $V_{\mathsf{ts}}(\hat{\mu}, \hat{B}) < V(\hat{\mu}, \hat{B})$ for any $\hat{\mu}$ and $\hat{B}$. This gives that $V_{\mathsf{ts}}(\hat{\mu}, \hat{B}) < \sup_{\mu,B} V(\mu, B)$ for any $\hat{\mu}$ and $\hat{B}$, and hence $\sup_{\mu,B} V_{\mathsf{ts}}(\mu, B) \le \sup_{\mu,B} V(\mu, B)$. Therefore, the retailer will choose not to timestamp items.

EC.9.3. The retailer timestamps items when the disposal cost d is sufficiently low with heterogeneous customers

When customers are heterogeneous, we establish in the proof of Theorem 2 that in the limit where d tends to $-p$ the retailer maximizes customer welfare and is strictly better off with a timestamp than without a timestamp for each choice of (μ, B). That is, if we denote the optimal objective with and without a timestamp when the disposal cost is d, the production rate is μ and the batch size is B by $V_{\mathsf{ts}}(\mu, B; d)$ and $V(\mu, B; d)$, respectively, we have $V(\hat{\mu}, \hat{B}; -p) < V_{\mathsf{ts}}(\hat{\mu}, \hat{B}; -p)$ for any $\hat{\mu}$ and $\hat{B}$. This holds true in a non-strict sense after taking supremum over (μ, B):

$$\sup_{\mu,B} V(\mu, B; -p) \le \sup_{\mu,B} V_{\mathsf{ts}}(\mu, B; -p).$$

Recall from the proof of Theorem 2 that $V(\mu, B; d)$ is convex and decreasing in d and $V_{\mathsf{ts}}(\mu, B; d)$ is linear and decreasing in d for each (μ, B). Because the supremum of convex functions is convex and a convex function defined on $\mathbb{R}$ is continuous, both $\sup_{\mu,B} V(\mu, B; d)$ and $\sup_{\mu,B} V_{\mathsf{ts}}(\mu, B; d)$ are continuous in d. Thus, by the Intermediate Value Theorem, there exists a threshold $\bar{d}$ such that

$$\sup_{\mu,B} V(\mu, B; d) \le \sup_{\mu,B} V_{\mathsf{ts}}(\mu, B; d)$$

for $d \le \tilde{d}$. That is, the retailer uses a timestamp for $d \le \tilde{d}$.

EC.10. Extension: Mixed Issuance

This section extends our base model by relaxing the assumption that, without timestamps, customers perfectly adhere to the retailer's issuance. We consider two more general settings where some fraction of customers deviate from the retailer's issuance, and confirm that LIFO issuance is remains optimal.

Cherry-Pickers. When the retailer uses a shelf design that induces FIFO issuance, some customers might learn that they could get fresher items if they exerted some effort to reach for lower (or deeper) on the shelf. To formalize this behavior, we assume that when the retailer imposes FIFO, a fraction of customers deviate and search through items of several ages looking for a fresher item, instead of taking the oldest item. We take the maximum number of age layers through which customers search as an exogenous parameter, which could be random and independently given among customers. We also assume that customers would *not* deviate from LIFO because doing so would be irrational, as LIFO already maximizes their expected utility.

It can be readily argued that Lemmas 1 and 2 hold in this setting. Specifically, increasing the shelf life T strictly reduces purchased quality $Q^{I,T}$ and strictly increases sales $S^{I,T}$ under any issuance I, so customers purchase items if and only if the shelf life does not exceed an issuance-specific threshold $\bar{T}_I$. The threshold is again larger with LIFO than FIFO, i.e., $\bar{T}_L \geq \bar{T}_F$, and for shelf lives below the threshold, FIFO guarantees strictly lower quality, but strictly larger sales, than LIFO, which immediately implies a result analogous to Proposition 1: for any fixed $T \leq \bar{T}_F$, FIFO dominates LIFO if and only if the retailer does not value customer welfare much relative to immediate profits, i.e., $f \leq \bar{f}$.

We claim that an analogous result to Lemma 3 also holds, as formalized next.

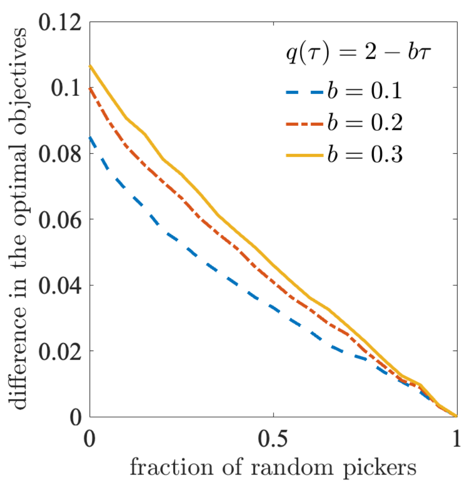
LEMMA EC.8. *For the setting with cherry-pickers and any shelf lives $T_F \leq \bar{T}_F$ and $T_L \leq \bar{T}_L$ satisfying $S^{F,T_F} = S^{L,T_L} = S$, LIFO issuance with shelf life T_L achieves a higher total quality of sold items than FIFO issuance with shelf life T_F, i.e., $R^F(S) \leq R^L(S)$. Moreover, the maximal sales possible under LIFO exceed those under FIFO, i.e., $S^{F,\bar{T}_F} \leq S^{L,\bar{T}_L}$.*

Proof sketch. Following the same steps in the proof of Lemma 3, suppose that we obtain the same sales S under (F, T_F) and (L, T_L) with $0 < T_F < T_L$, i.e., $S^{F,T_F} = S^{L,T_L} = S$. We define a hidden-FIFO policy analogously: the retailer offers items of age at most T_F to customers and only discards items of age strictly above T_L. The results presented in **Steps 1-3** in the proof of Lemma 3 follow from analogous arguments. Therefore, it suffices to argue that Equation (EC.3.12) holds. As in the proof of Lemma 3, we can prove this by induction, showing that the inequality holds after every arrival (of a batch or a customer). Obviously, the inequality holds at $t_0 = 0$. Suppose that it holds until time t_n and consider any $x \in [0, T_L]$. If a batch arrived at t_{n+1}, we have (EC.3.13). If a customer arrived at t_{n+1}, (EC.3.14) still holds because customers only have access to items below age T_F under the hidden-FIFO policy (regardless of how many layers they dig through), whereas they have access to *all* items on the shelf under the forced-LIFO policy. The argument then follows from the same steps as in the proof of Lemma 3.

This result immediately implies Theorem 1, which means that LIFO dominates FIFO even in this setting with cherry-pickers. At an intuitive level, this finding should not be too surprising given our original result on LIFO's dominance, because the FIFO issuance with cherry-picking behavior would still preserve a consistent dynamic of age groups on the shelf, which is sufficient for the results.

Random Pickers. Here, we assume that a fixed fraction of customers deviate from the retailer's issuance and instead pick an item randomly among all the items on the shelf. In reality, such behavior could occur without timestamps, when all items are displayed with the same level of accessibility, e.g., on a single flat shelf.

Unfortunately, this setting is more difficult to analyze because the evolution of the inventory of items with distinct age groups does not follow a consistent pattern under either issuance, which make it challenging to compare LIFO and FIFO analytically. However, we conduct a detailed set of numerical experiments and show some representative results in Figure EC.8. The figure plots the difference in the retailer's optimal objectives with LIFO and FIFO as a function of the fraction of random pickers, for a linear and a logarithmic quality depreciation schedule $q(\tau)$. Note that LIFO continues to be optimal and, as expected, the gains from implementing LIFO versus FIFO increase when the quality of items drops faster with age, but decrease as more customers pick randomly.



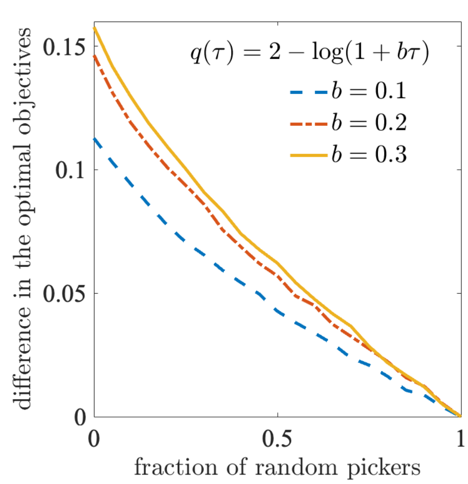


Figure EC.8 The difference in the retailer's optimal objective under LIFO and FIFO as a function of the fraction of customers who pick items randomly, under a linear schedule $q(\tau) = 2 - b\tau$ (left) and under a logarithmic schedule $q(\tau) = 2 - \log(1 + b\tau)$ (right). In this example, the price is $p = 1$, customer arrival rate is $\lambda = 1.0$, replenishment is Poisson with rate $\mu = 1.1$ and batch size $B = 1$, the disposal cost is $d = 0$, and the weight on customer welfare is $f = 0.7$.

References

Ross SM (1996) *Stochastic Processes*, 2nd ed. (Wiley, New York).