Do the following exercises. Most of it should be straightforward for you. If you don’t find these to be straightforward review, please talk to me privately.

Some review is included on pages 2-6, and you may also find it helpful to consult the Partee et al. readings linked just below (where you’ll also find the first batch of exercises).

- **Partee, ter Meulen & Wall 1990:** §1 [PDF here], page 23ff, problems 1, 4, 5, 6, and 7.
- **Partee et al. 1990:** §2 [PDF here], page 36-37, problems 1 and 2.
- **Propositional logic exercises:**
  - Define $\rightarrow$ using (only) $\neg$ and $\land$.
  - Define $\rightarrow$ using (only) $\neg$ and $\lor$.
  - Show that $\phi \lor \psi$ and $\neg(\neg\phi \land \neg\psi)$ are logically equivalent.
  - Let $D = \{p, q, r\}$ and let $\alpha = p \lor (q \rightarrow r)$. Show that the definitions in (1) and the truth-table method agree about how the truth value of $\alpha$ depends on those of $p$, $q$, $r$ in all possible models.
  - Explain why each of the following entailments holds, or describe a counter-model if it does not.

    * $\phi \equiv \phi$
    * $\phi \lor \psi \equiv \phi$
    * $\phi \land \psi \equiv \phi$
    * $\phi \equiv \phi \lor \psi$
    * $\phi \equiv \phi \land \psi$
    * $(\phi \land \psi) \rightarrow \chi \equiv \psi \rightarrow \chi$
    * $(\phi \lor \psi) \rightarrow \chi \equiv \psi \rightarrow \chi$

- Explain why each of the following statements follows from our definitions.

    * $\phi$ and $\psi$ are logically equivalent iff $\phi \leftrightarrow \psi$ is a tautology.
    * $\phi$ and $\psi$ are logically equivalent iff $\phi \equiv \psi$ and $\neg \phi \equiv \psi$.
    * $\phi$ is a contradiction iff $\neg \phi$ is a tautology.
    * A contradiction entails anything: e.g. ‘It’s raining and it’s not raining’ entails ‘Dan is God’.
1 Set theory review

Definition 1. (Set) An abstract collection of distinct (!) objects, which are called the members or elements of the set. Order does not matter, and repetitions are not meaningful: \( \{a, b\} = \{b, a\} = \{a, b, b\} \).

Some examples:

- \( \{\text{Bill Clinton, Nelson Mandela, 76}\} \)
- \( \{x \mid x^2 = 9\} = \{-3, 3\} \)
- Singleton set: e.g. \( \{\text{Dan}\} \)
- Empty set: \( \emptyset = \{\} \)
- Domain of discourse \( D \) ("for all" and "for some" are implicitly relativized to \( D \))

Some important concepts:

- Membership: \( x \in A \) iff \( x \) is an element of \( A \).
- Non-membership: \( x \notin A \) iff it is not the case that \( x \in A \).
- Extensionality: \( A = B \) iff, for all \( x, x \in A \) iff \( x \in B \).
- Venn diagrams
- Intersection: \( A \cap B = \{x \mid x \in A \text{ and } x \in B\} \)
- Union: \( A \cup B = \{x \mid x \in A \text{ or } x \in B\} \)
- Complement: \( -A = \{x \mid x \notin A\} \) (also written \( \overline{A} \), sometimes \( A' \))
- Relative complement/set difference: \( A - B = \{x \mid x \in A \text{ and } x \notin B\} \) (also written \( A \backslash B \))
- Subset: \( A \subseteq B \) iff, for all \( x \in A, x \in B \).
- Strict subset: \( A \subset B \) iff \( A \subseteq B \) and \( A \neq B \).
- Superset: \( A \supseteq B \) iff \( B \subseteq A \).
- Strict superset: \( A \supset B \) iff \( A \supseteq B \) and \( A \neq B \).
- Powerset: \( \mathcal{P}(A) = \{B \mid B \subseteq A\} \). (often written \( \wp(A) \))
- Cardinality: \( |A| \) is the number of elements in \( A \).

Generalizations: If \( \mathcal{A} = \{A_1, A_2, \ldots\} \) is a collection of sets, then

- \( \bigcap \mathcal{A} = \{x \mid \text{for all } A \in \mathcal{A}, x \in A\} \).
- \( \bigcup \mathcal{A} = \{x \mid \text{for some } A \in \mathcal{A}, x \in A\} \).
Some important facts:

- \( A \cup A = A \) (idempotency)
- \( A \cap A = A \) (idempotency)
- \( A \cup B = B \cup A \) (commutativity)
- \( A \cap B = B \cap A \) (commutativity)
- \( (A \cup B) \cup C = A \cup (B \cup C) \) (associativity)
- \( (A \cap B) \cap C = A \cap (B \cap C) \) (associativity)
- \( A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \) (distributivity)
- \( A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \) (distributivity)
- \( A \cup -A = D \)
- \( A \cap -A = \emptyset \)
- \( -(A) = A \)
- \( A - B = A \cap -B \)
- \( -(A \cup B) = -A \cap -B \) (de Morgan’s Law 1)
- \( -(A \cap B) = -A \cup -B \) (de Morgan’s Law 2)
- \( A \subseteq B \) iff \( A \cup B = B \)
- \( A \subseteq B \) iff \( A \cap B = A \)

2 Relations

Definition 2. (Tuple) A \((n-)\)tuple is an ordered list of length \( n \), where repetitions are permitted. For example, \( \{a, b\} \neq \{b, a\} \). \( \{a, a\} \) is a perfectly good tuple, while \( \{a, a\} \) is just a stupid way to write \( \{a\} \).

A 2-tuple is called a (ordered) pair. A 3-tuple is called a (ordered) triple.

Definition 3. (Cartesian product)

- \( A \times B = \{(x, y) \mid x \in A \text{ and } y \in B\} \).
- \( A \times B \times C = \{(x, y, z) \mid x \in A \text{ and } y \in B \text{ and } z \in C\} \).
- etc.
Note that $A \times A$ is usually written $A^2$ for any $A$; $A \times A \times A$ is written $A^3$; and so on.

**Definition 4. (Relation)** An $n$-ary relation is a set of $n$-tuples.

For example, a *binary* (2-ary) relation is any subset of $D^2$. A *binary relation on* $X$ is a subset of $X^2$.

Convention: instead of $(a,b) \in R$ we usually write $aRb$ of $R(a,b)$.

### 3 Functions

**Definition 5. (Function)** A function is a relation $R$ with the property that, for any $x$, there is a unique $y$ such that $R(x,y)$. (For functions, we usually use $f$ or $F$ as a variable, rather than $R$.)

**Definition 6. (Domain and range)** The *domain* of $f$ is $\{ x \mid$ for some $y, f(x) = y \}$. The *range* of $f$ is $\{ y \mid$ for some $x, f(x) = y \}$.

When $f$ has domain $A$ and range $B$, we frequently specify $f : A \to B$. 
4 Propositional logic review

4.1 Syntax

Fixing some \((\text{non-empty, possibly infinite})\) set of propositional atoms \(\mathcal{A} \subseteq \{p_i \mid i \in \mathbb{N}\}\):

**Definition 7. (Syntax of \(L_{PL}\))** The language \(L_{PL}\) is defined recursively as follows:

- all elements of \(\mathcal{A}\) are in \(L_{PL}\).
- if \(\phi \in L_{PL}\) and \(\psi \in L_{PL}\), then \(\neg \phi, \phi \land \psi, \phi \lor \psi, \phi \rightarrow \psi, \phi \leftrightarrow \psi \in L_{PL}\).
- Nothing else is in \(L_{PL}\).

Obviously \(L_{PL}\) is a different language depending on which atomic propositions are in \(\mathcal{A}\). We’ll usually concentrate on variants where \(\mathcal{A}\) is either very small or infinite. Note that \(L_{PL}\) is infinite no matter what \(|\mathcal{A}|\) is.

4.2 Semantics

A model for \(L_{PL}\) is a pair \(M = \langle \{0, 1\}, [\cdot]^M \rangle\), where \([\cdot]^M\) maps every expression in \(L_{PL}\) to \(\{0, 1\}\) (i.e., assigns every proposition a truth-value).

The truth-values of sentences containing the connectives \(\neg, \land, \lor, \rightarrow, \leftrightarrow\) can be defined from the truth-values of their component propositions in either of two ways. The simplest way is to enumerate all possible cases:

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Each line of a truth table defines a class of models: the models in which the propositions in question have the truth-values in question. (Not all combinations will be possible, e.g., if one proposition entails the other.) These determine the truth-values of all possible ways of combining simple or derived propositions into larger propositions using \(\neg, \land, \lor, \rightarrow, \leftrightarrow\).

- Since all connectives are either unary or binary, we only need these four lines to enumerate all the cases — and we could if we like go through and enumerate all possible connectives.
- (Aside: if you like, you could think of the lines of a truth-table as an extensional representation of a set of “possible worlds”, where the number of possible worlds is \(2^n\) for \(n\) atomic propositions. This doesn’t mean that there can’t \textbf{be} more possible worlds, but rather that a language with \(n\) atomic propositions is only capable of \textbf{expressing} the differences between \(2^n\) classes of possible worlds.)
A more elegant way to proceed is to use the following inductive definition:

(1) a. $\neg \phi^M = 1$ iff $\phi^M = 0$.
   
b. $(\phi \land \psi)^M = 1$ iff both $\phi^M = 1$ and $\psi^M = 1$.
   
c. $(\phi \lor \psi)^M = 1$ iff $\phi^M = 1$ or $\psi^M = 1$ (or both!).
   
d. $(\phi \rightarrow \psi)^M = 1$ unless both $\phi^M = 1$ and $\psi^M = 0$.
   
e. $(\phi \leftrightarrow \psi)^M = 1$ iff $\phi^M = \psi^M$.

4.3 Logical relations

$\phi$ entails $\psi$ iff, for every model $M$ for which $\phi^M$ is true, $\psi^M$ is also true. In symbols: $\phi \models \psi$.

- Thinking in terms of truth-tables: $\phi \models \psi$ iff: every possible way of permuting the truth-values of the atomic propositions which makes $\phi$ true also makes $\psi$ true.

- Equivalently: if the set of models $M = \{ M \mid (\phi)^M = 1 \}$ is a subset of the set of models $M' = \{ M \mid (\psi)^M = 1 \}$.

$\phi$ and $\psi$ are LOGICALLY EQUIVALENT iff, for all models $M$, $(\phi)^M = (\psi)^M$.

$\phi$ is a TAUTOLOGY (or VALID) iff $\phi$ is true in every model.

- Equivalently: if $\psi \models \phi$ for every $\psi$.

$\phi$ is a CONTRADICTION iff $\phi$ is true in no model — i.e. if there’s no $\psi \models \phi$ such that $\psi \models \phi$.

References