Divergence and Gambling

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Divergence

**Definition**

If $X$ is a random variable with probability densities $p(x)$, then the **surprisal** associated with the value $X = x$ is

$$s(x) = \log \frac{1}{p(x)}.$$ 

The **entropy** is the average surprisal, $H = E[p(X)]$. 
## Divergence

<table>
<thead>
<tr>
<th></th>
<th>x</th>
<th>r</th>
<th>a</th>
<th>b</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p(x)$</td>
<td>$\frac{1}{4}$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{4}$</td>
<td></td>
</tr>
<tr>
<td>$p$-code</td>
<td>00</td>
<td>1</td>
<td>01</td>
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<tr>
<td>$q(x)$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{4}$</td>
<td>$\frac{1}{4}$</td>
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<tr>
<td>$q$-code</td>
<td>0</td>
<td>10</td>
<td>11</td>
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rabarbararabaa ...
The **Kullback-Leibler divergence** from $p$ to $q$ is

$$D(p \parallel q) = E \left[ \log \frac{1}{q(X)} \right] - E \left[ \log \frac{1}{p(X)} \right] = \sum_x p(x) \log \frac{p(x)}{q(x)},$$

where $E[\cdot]$ is the expectation with respect to $p$.

**Properties of the KL divergence**

1. **Nonnegativity:** $D(p \parallel q) \geq 0$;
2. **Coincidence:** $D(p \parallel q) = 0$ if and only if $p = q$;
3. **Asymmetry:** Often, $D(p \parallel q) \neq D(q \parallel p)$.

Divergence

<table>
<thead>
<tr>
<th>x</th>
<th>1</th>
<th>2</th>
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<tbody>
<tr>
<td>p(x)</td>
<td>.5</td>
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</table>

<table>
<thead>
<tr>
<th>x</th>
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<tr>
<td>p(x)</td>
<td>.3</td>
<td>.7</td>
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</table>
Divergence

\[ p(x) \]

\[ \begin{array}{c|ccc}
  x & 1 & 2 & 3 \\
  \hline
  p(x) & \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\
\end{array} \]
Divergence

\[ p(\text{abaab} \ldots) = p(a) \cdot p(b|a) \cdot p(a|b) \cdot p(a|a) \ldots \]
\[ q(\text{abaab} \ldots) = q(a) \cdot q(b) \cdot q(a) \cdot q(a) \cdot q(b) \ldots \]
Problem

A horse race has three horses:

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<tr>
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<th>3</th>
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<tr>
<td>$p(x)$</td>
<td></td>
<td>.2</td>
<td>.3</td>
<td>.5</td>
</tr>
<tr>
<td>$o(x)$</td>
<td></td>
<td>4</td>
<td>4</td>
<td>2</td>
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<tr>
<td>$b(x)$</td>
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<td>?</td>
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Which capital distribution $b$ maximizes the expected rate of return, $E[R] = E[o(X)b(X)]$?
Gambling

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<td>4</td>
<td>2</td>
</tr>
<tr>
<td>$b(x)$</td>
<td>0</td>
<td>1</td>
<td>0</td>
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</table>
Gambling


Gambling

Definition

The **doubling rate** is the logarithm of the rate of return,

\[ W(X) = \log R(X) = \log o(X)b(X). \]

Proportional Gambling

The doubling rate attains its maximum at \( b^* = p \), regardless of the odds.

\[
\begin{array}{c|ccc}
  x & 1 & 2 & 3 \\
  \hline
  p(x) & .2 & .3 & .5 \\
  o(x) & 4 & 4 & 2 \\
\end{array}
\]
Gambling

![Graph showing capital growth over rounds](image-url)
Problem: Dependent Roulette

A bookmaker draws the 52 cards in a deck one by one:

\[ \text{\textbullet, \textbullet, \textbullet, \textbullet, \textbullet, \textbullet, \textbullet, \ldots, \textbullet, \textbullet} \]

Before each draw, you can place bets on \( \text{\textbullet} \) and \( \text{\textbullet} \). Is this a favorable game, and what is its doubling rate?

Cover and Thomas (1991), Example 6.3.1.