Vagueness and Plural Predication*

Heather Burnett
University of California, Los Angeles
hburnett@ucla.edu

1 Introduction

This paper addresses the question of how to model the application of vague predicates like *tall* and *bald* to plural subjects like *John and Mary* and *the men*. In other words, we are interested in developing a logical analysis for natural language sentences like *Mary is tall* and *The men are bald*. In the past 30 years, much research has been devoted to finding the proper logical framework to model the application of non-vague predicates to pluralities (cf. [13], [16] among others). Additionally, there has been a lot of work on how to model the application of vague predicates to singular terms (cf. [6], [8], [4] among many others). However, the question of how to apply vague predicates to plural subjects and what complexities may arise in doing so has yet to be examined. This paper is a contribution to filling this gap. In particular, I argue that extending an analysis of predicate vagueness to incorporate pluralities is not immediately straightforward; that is, I show that sentences with vague predicates and certain kinds of plural subjects give rise to additional vague effects that are not present with singular subjects. Extending previous work on both plural predication and vague language, I propose a new logical system that models these effects.

In the remainder of this section, I present the data that an analysis of vagueness and plural predication aims to model. In section 2, I present a logical system based on [13] to treat (non-vague) plural predicates. In section 3, I present a recent prominent framework for modelling sentences with vague predicates and singular subjects: [4]’s similarity-based *Tolerant, Classical, Strict* (TCS) logic. Finally, in section 4, I present a system that incorporates the main proposals of these two frameworks and models both plural and singular predication with vague predicates.

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1 Link’s *Logic of Plurals and Mass Nouns* (LPM) is generally viewed as the standard approach to modelling the semantics of plurals in linguistics (see, for example, [2] and references within).
1.1 Vagueness in the AP Domain

In the literature, vagueness is most commonly discussed with reference to gradable adjectives and certain scalar nouns like *heap*. Following many authors (ex. [8], [5], [17], among others), I take vague language to be characterized by the presence of three (related) properties: *borderline cases* (objects for which it is difficult or even impossible to tell whether they satisfy the predicate), *fuzzy boundaries* (the observation that there appear to be no sharp boundaries between cases of a vague predicate and its negation), and *susceptibility to the Sorites paradox* (a paradox for classical logical systems that follows from the fuzzy boundaries property). Within the adjectival domain, we can distinguish between two classes of vague expressions: those that are vague with respect to a relative meaning, like open-scale adjectives (*tall, long, expensive, interesting* etc.) and those that are vague with respect to an absolute meaning, like closed-scale adjectives (*bald, empty, clean* etc.). Relative (open-scale) adjectives always display the three properties. For example, if we take the set of American males as the appropriate comparison class for *tallness*, we can easily identify the ones that are clearly tall: for example, anyone over 6 feet. Similarly, it is clear that anyone under 5ft9” (the average) is not tall. However, what about men who are 5ft11”? 5ft10.5”? It seems impossible to decide whether these borderline cases are tall or not tall. Furthermore, if we take a clearly tall person and we start subtracting millimetres from their height, it seems impossible to pinpoint the precise instance where subtracting a millimetre suddenly moves us from the height of a tall person to the height of a not tall person: the boundaries of *tall* appear fuzzy. The presence of a property similar to fuzzy boundaries also gives rise to a paradox (the Sorites) for a logical system with modus ponens and universal instantiation (cf. the works cited above for discussion).

As observed by [10] among others, absolute (closed-scale) adjectives exhibit contextual variation in whether they display the characterizing properties of vague language. For example, with *bald*, if we are in a context where the total absence of hair is extremely important (cf. [5] a description of such a context), adding a single hair to the head of a bald person might move them from being bald to not bald, and then *bald* would have a sharp meaning. However, in the majority of the contexts in which we use *bald*, it has borderline cases, fuzzy boundaries, and gives rise to the Sorites. For example, we normally allow for people to be considered bald even if they have some hair. In these contexts, there are people for whom it seems we can’t decide whether they are in the predicate’s extension or anti-extension, and it is not so clear at which exact number of hairs one goes from being bald to not bald. In summary, adjectival vague predicates come in two classes: vague with respect to a relative meaning

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2 In fact, this observation has led some authors (like Kennedy) to propose that the contextual effects observed with absolute adjectives are not due to vagueness, but a different pragmatic phenomenon called “imprecision”. However, in this paper, I follow the majority of work in the philosophical tradition that analyzes *bald* as vague, given that, when they are not used in strict contexts, absolute adjectives display exactly the characterizing properties of vague language.
and vague with respect to an absolute meaning, and these predicates contrast with non-vague predicates like to be prime and to be Canadian, which have a sharp meaning.

1.2 Vagueness in the DP Domain

The vague/non-vague distinction is preserved when these predicates apply to plural subjects. The truth conditions for sentences involving distributive predicates and some plural subjects are given by looking at how the singular predicate applies to the parts of the subject DP. For example, Both John and Peter are bald is true just in case each member of John and Peter is bald, and whether or not they count as bald will be determined in the same way as in the singular sentences John is bald and Peter is bald. Other sentences that show this pattern with vague predicates are those with subjects of the form all the girls, the four girls, and four girls.

The DPs discussed above force universal quantification over their parts with distributive predicates: sentences containing them are clearly true if the predicate holds of every atomic part of the subject, and they are clearly false otherwise. However, there is another class of DPs whose members give rise to the same effects as we find with vague predicates. As noticed by many authors (cf. [11], [1], a.o.), sentences like The girls have two eyes and These men are Canadian are clearly true when all of the parts of the subject are affected by the predicate and clearly false when none of them are; however, things are not so clear when we consider cases where all but a few parts are affected. As with absolute adjectives, if we are being as precise as possible, it may be inappropriate to use a sentence like These men are Canadian if a couple of irrelevant men have a different nationality. However, in many ‘looser’ contexts, it can be very natural to use such a sentence. In these situations, sentences with definite plural subjects display the three hallmark properties of vague language with respect to how many parts of the subject are affected by the predicate. For example, suppose we are speaking about a summer camp that has 100 girls, and the girls are required to perform a number of activities including going for a swim in the lake, although it is not extremely important that every single one participates in every activity. In this situation, it is very natural to say The girls jumped in the lake if only 99, 98 or 97 girls jumped in, and this sentence seems inappropriate if only a couple girls jump in; however, what about if 60 jump in? 61? 59? Just like with vague absolute adjectives, any particular cut-off point other than the entire group seems arbitrary. In contexts such as the one just discussed, definite plural DPs display borderline cases (cases where the predicate affects only some of the subject), fuzzy boundaries (the transition point between the borderline cases is not clear), and, provided the subject denotes a group that is sufficiently large, they are susceptible to a Sorites-type argument. Indeed, a unified treatment of this ‘part-structure’ vagueness and the vagueness associated with absolute adjectives is

\footnote{For space considerations, I will only discuss sentences with distributive predicates.}
already implicit in a number of works on the semantics and pragmatics of definite
noun phrases (for example in [11]).

In summary, examples like The men are Canadian involve pairing a vague
subject with a non-vague predicate. Furthermore, it is also possible to pair such
a subject with a vague predicate, as in the sentence The girls are tall. This
sentence is vague along two dimensions: the extent to which the members of
the girls are affected by the plural predicate (part-structure vagueness) and the
extent to which the affected members count as tall (predicate vagueness). This
paper provides a unified account of both these dimensions.

2 Plural predication: LPl

In this section, I present a simple logic for modelling non-vague plural predication
based on [13]'s system. The basic idea is that, instead of containing structure-
less individuals in an unordered domain like in classical first order logic (FOL),
the domain for the interpretation of plural individuals is (partially) ordered, and
pluralities denote sums/joins of singular individuals. The vocabulary and syntax
of the Logic of Plurals (LPl) are given below.

Definition 1. Vocabulary. The vocabulary of LPl is that of first order predi-
cate logic with the usual logical connectives and quantifiers (¬, ∧, ∨), singular
individual constants (A = \{a_1, a_2, \ldots\}), singular individual variables (V =
\{v_1, v_2, \ldots\}), and 1-place predicate symbols (Pr = \{P, Q, \ldots\}). For ease of
exposition, I restrict my attention to unary predicates. Additionally, there is
another series of plural individual constants (G = \{g_1, g_2, \ldots\}) and variables
(S = \{s_1, s_2, \ldots\}), a distinguished binary predicate: ≤, and an operator on 1-
place predicates: ∗.

Definition 2. Syntax. The syntax of LPl is given as follows:
1. i) If \(x \in V \cup A\), then \(x \in \text{I-term}\). ii) If \(x \in G \cup S\), then \(x \in \text{P-term}\).
2. If \(P \in Pr\), then \(P^* \in \text{PlurP}\).
3. Atomic Formula: i) If \(x \in \text{I-term}\) and \(P \in Pr\), then \(P(x)\) is an atomic
   formula, ii) If \(x \in \text{P-term}\) and \(P^* \in \text{PlurP}\), then \(P^*(x)\) is an atomic formula,
   iii If \(x, y \in \text{P-term}\), then \(x \leq y\) is an atomic formula.
4. Well-Formed Formula (wff): Defined as in FOL.

With respect to the semantics of LPl, we first define the structure into which
pluralities are interpreted.

Definition 3. Plural Model Structure. A plural model structure \(M\) is a tuple
\(<D, \vee>\), where \(D\) is a finite set of singular/plural individuals, \(\vee\) is a binary op-
eration that is associative, commutative and idempotent.

In other words, \(<D, \vee>\) is a finite join semi-lattice⁴.

⁴ Link’s 1983 LPM had plural individuals being interpreted in a complete atomic
Boolean Algebra instead of a join semi-lattice; however, I follow the majority of
Definition 4. Atom. $g_1 \in D$ is an atom iff there is no $g_2 \in D$ such that $g_2 < g_1$.

We write $AT(D)$ for the set of atoms of $\langle D, \vee \rangle$.

Definition 5. Plural Model. A plural model $M$ is a tuple $\langle D, \vee, m \rangle$, where $\langle D, \vee \rangle$ is a plural model structure and $m$ is an interpretation function of the usual sort for the non-logical vocabulary: If $a_1 \in A$, then $m(a_1) \in AT(D)$; If $g_1 \in G$, then $m(g_1) \in D$; and if $P \in Pr$, then $m(P) \in P(AT(D))$.

Note that I will often write $a_1$ for $m(a_1)$ and $g_1$ for $m(g_1)$.

A major insight of Link’s paper is to propose that there exists a non-arbitrary link between the interpretation of a singular predicate and its plural counterpart; in particular, he proposes that plural distributive predicates are derived from singular predicates through a * operator which generates all the individual sums of members of the extensions of $P$.

Definition 6. Interpretation of * . $m(P^*)$ is the closure of $P$ under $\vee$.

Truth in a plural model is defined as follows:

Definition 7. Truth in a plural model. For $M$ be a plural model,

1. $M \models P(a_1)$ iff $m(a_1) \in m(P)$
2. $M \models P^*(g_1)$ iff $m(g_1) \in m(P^*)$
3. $M \models g_1 \leq g_2$ iff $m(g_1) \vee m(g_2) = m(g_2)$
4. $M \models \neg \phi$ iff $M \not \models \phi$
5. $M \models \phi \land \psi$ iff $M \models \phi$ and $M \models \psi$
6. $M \models \forall v_1 \phi$ iff for every $a_2 \in AT(D), M \models \phi[a_2/v_1]$
7. $M \models \exists v_1 \phi$ iff for every $g_2 \in D, M \models \phi[g_2/v_1]$

We can prove that all predicates in this system are distributive\(^5\); that is, when a plural property $P^*$ holds of a plurality $g_1$, the corresponding singular property $P$ holds of all the singular individuals that make up $g_1$.

Theorem 1. Distributivity. For $M$, a plural model, $g_1 \in D$, and $P \in P(AT(D))$, $M \models P^*(g_1)$ iff for all atoms $a_1 \leq g_1$, $M \models P(a_1)$

Proof. $\Rightarrow$ Suppose $M \models P^*(g_1)$ and let $a_1$ be an atom such that $a_1 \leq g_1$ to show that $M \models P(a_1)$. Since $\langle D, \vee \rangle$ is atomic, $g_1$ is the join of the set atoms below it (viz. [7] (p. 118)), and, since $a_1$ is an atom below $g_1$, by definition 6, $M \models P(a_1)$. $\Leftarrow$ Suppose that, for all atoms $a_1 \leq g_1$, $M \models P(a_1)$ to show $M \models P^*(g_1)$. Immediate from definition 6. \hfill \Box

the more recent interpretations of his work in having plural model structures lack a bottom element (cf. [12] (pp. 302-303) for arguments that semi-lattices are more appropriate for modelling the semantics of plurals than full BAs.).

\(^5\) Note that in this paper the term *distributivity* is used as it is commonly used in linguistics to mean “reducible to individual predication.” It should not be interpreted as the property of binary relations that bears the same name in abstract algebra. In fact, the semi-lattice described above is not distributive, in this second sense, because there no bottom element.
In summary, with LPl, we can analyze sentences with non-vague predicates that have both singular and non-vague plural subjects. However, the system does not provide a way for modelling the properties of vague language (borderline cases etc.) with either singulars or plurals. Furthermore, Theorem 1 shows that LPl is not equipped to treat part-structure vagueness: all its predicates are fully distributive.

3 Vague Predication: TCS

In this section, I outline [4]'s Tolerant, Classical, Strict framework. This system was originally developed as a way to preserve the intuition that vague predicates are tolerant (i.e. satisfy $\forall x\forall y[P(x) \land x \sim_P y \rightarrow P(y)]$, where $\sim_P$ is an indifference relation for a predicate $P$), without running into the Sorites paradox. [4] adopt a non-classical logical framework with three notions of satisfaction: classical truth, tolerant truth, and its dual, strict truth. Formulas are tolerant/strictly satisfied based on classical truth and predicate-relative, possibly non-transitive indifference relations. For a given predicate $P$, an indifference relation, $\sim_P$, relates those individuals that are viewed as sufficiently similar with respect to $P$. For example, for the predicate tall, $\sim_{tall}$ would be something like the relation “not looking to have distinct heights”. In this framework, we say that John is tall is tolerantly true just in case John has a very similar height to someone who is classically tall (i.e. has a height greater than or equal to the contextually given ‘tallness’ threshold). The framework is defined (using the notation adopted in this paper) as follows:

Definition 8. Language. The language of TCS is that of first order predicate logic with neither identity nor function symbols.

For the semantics, we define three notions of truth: one that corresponds to truth in classical FOL ($c$-truth), and two that are novel: $t$-truth and its dual $s$-truth.

Definition 9. C(lassical) Model. A c-model is a tuple $\langle D, m \rangle$ where $D$ is a non-empty domain of individuals and $m$ is an interpretation function for the non-logical vocabulary: for a constant $a_1$, $m(a_1) \in D$; for a predicate $P$, $m(P) \in \mathcal{P}(D)$.

Definition 10. T(olerant) Model. A t-model is a tuple $\langle D, m, \sim \rangle$, where $\langle D, m \rangle$ is a c-model and $\sim$ is a function that takes any predicate $P$ to a binary relation $\sim_P$ on $D$. For any $P$, $\sim_P$ is reflexive and symmetric (but possibly not transitive).

A non-empty set with a reflexive, symmetric relation on it is often called a tolerance space (ex. [15]). Thus, for any $P$, the structure $\langle D, \sim_P \rangle$ is a tolerance space.

C-truth is defined as classical truth in either a c-model or a t-model.

Definition 11. c-truth in a model. Let $M$ be either a c-model such that $M = \langle D, m \rangle$ or a t-model such that $M = \langle D, m, \sim \rangle$. 
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1. \( M \models^c P(a_1) \) iff \( m(a_1) \in m(P) \)
2. \( M \models^c \neg \phi \) iff \( M \not\models^c \phi \)
3. \( M \models^c \phi \land \psi \) iff \( M \models^c \phi \) and \( M \models^c \psi \)
4. \( M \models^c \forall v_1 \phi v_1 \) iff for every \( a_2 \) in \( D \), \( M \models^c \phi[a_2/v_1] \)

Definition 12. t-truth and s-truth. Let \( M \) be a t-model.

1. \( M \models^t P(a_1) \) iff \( \exists a_2 \sim P a_1 : M \models^c P(a_2) \)
2. \( M \models^t \neg \phi \) iff \( M \not\models^s \phi \)
3. \( M \models^t \phi \land \psi \) iff \( M \models^t \phi \) and \( M \models^t \psi \)
4. \( M \models^t \forall x \phi x \) iff for all \( a_2 \in D : M \models^t \phi[a_2/x] \)
5. \( M \models^s P(a_1) \) iff \( \forall a_2 \sim P a_1 : M \models^c P(a_2) \)
6. \( M \models^s \neg \phi \) iff \( M \not\models^t \phi \)
7. \( M \models^s \phi \land \psi \) iff \( M \models^s \phi \) and \( M \models^s \psi \)
8. \( M \models^s \forall v_1 \phi v_1 \) iff for all \( a_2 \in D : M \models^s \phi[a_2/v_1] \)

In summary, TCS models sentences with singular subjects and both vague and non-vague predicates (i.e. Mary is tall and Mary is Canadian).

4 Vague Language and Plural Predication: PTCS

In this section, I enrich the framework above with the structure of LPl in order to treat sentences with plural subjects. The language of this new system, Plural Tolerant Classical Strict (PTCS), is that of LPl, and c-truth is defined in the same way as truth in LPl as well.

Definition 13. Plural c-model. A plural c-model is a tuple \( \langle D, \lor, m \rangle \) where \( \langle D, \lor \rangle \) is a finite (atomic) join semi-lattice and \( m \) is an interpretation function for the non-logical vocabulary: \( m(a_1) \in AT(D) \); \( m(g_1) \in D \), \( m(P) \in \mathcal{P}(AT(D)) \). Also, \( m(P^*) \) is the closure of \( P \) under \( \lor \).

Definition 14. C-truth in a plural model for formulas involving mereological relations (c-truth for logical connectives is the same as truth in FOL/LPl). For \( M \) be a plural model,

1. \( M \models^c P(a_1) \) iff \( m(a_1) \in m(P) \)
2. \( M \models^c P^*(g_1) \) iff \( m(g_1) \in m(P^*) \)
3. \( M \models^c g_1 \leq g_2 \) iff \( m(g_1) \lor m(g_2) = m(g_2) \)

First of all, since classical truth and c-truth coincide, (Classical) Distributivity is also a theorem of PTCS:

Theorem 2. Classical Distributivity. For \( M \), a plural c-model or a plural t-model (to be defined below), \( g_1 \in D \), and \( P \in \mathcal{P}(AT(D)) \), \( M \models^c P^*(g_1) \) iff for all atoms \( a_1 \leq g_1 \), \( M \models^c P(a_1) \) (Proof is immediate from Theorem 1.)

T-models are defined as follows:
Definition 15. Plural t-model. A plural t-model $M$ is a tuple $\langle D, \lor, m, \sim \rangle$ such that $\langle D, \lor, m \rangle$ is a plural c-model and $\sim$ is a function that takes any predicate $P$ to a binary relation $\sim_P$ on $\text{AT}(D)$ that is reflexive and symmetric, but possibly not transitive.

With respect to defining indifference relations with plural predicates, a first option might be to have $\sim_P$ be given as part of the model (i.e. have them given entirely based on context), like $\sim_P$.

Definition 16. $\sim$. (First try). For all $P^*$, $\sim_P$ is a binary relation on $D$ that is reflexive and symmetric (but possibly not transitive).

T/S-truth for the formulas involving mereological relations is defined below (t/s-truth for logical connectives is the same as in TCS).

Definition 17. T-truth and s-truth. Let $M$ be a plural t-model.

1. $M \vDash t^P(a_1)$ iff $\exists a_2 \sim_P a_1 : M \vDash c^P(a_2)$
2. $M \vDash t^{P^*}(g_1)$ iff $\exists g_2 \sim_P g_1 : M \vDash c^{P^*}(g_2)$
3. $M \vDash s^P(a_1)$ iff $\forall a_2 \sim_P a_1 : M \vDash c^P(a_2)$
4. $M \vDash s^{P^*}(g_1)$ iff $\forall g_2 \sim_P g_1 : M \vDash c^{P^*}(g_2)$
5. $M \vDash s^g(a_1) \leq g_2$ iff $M \vDash c^g(a_1) \leq g_2$

Firstly, note that the interpretation of formulas involving $\leq$ is the same regardless of which type of satisfaction we are considering. This reflects the fact that mereological relations are part of the model structure.\footnote{See [4] for the same ‘crisp’ approach to indifference predicates: predicates in the language that express the $\sim$ relations in the model. This is not a necessary feature of the system; in fact, allowing for part structure relations to be vague may have some empirical explanatory potential (cf. [3]’s analysis of the non-countability of mass nouns). However, exploring this possibility is out of the scope of this work.}

Note secondly that the definition of $\sim_P$ is very weak. In fact, I argue that the definition 16 is insufficient to account for how the tolerant truth of sentences with non-vague plural subjects is calculated. In particular, it seems that we want a link between the tolerant truth of a sentence like The three girls are tall, where the three girls refers to the group of Mary, Sarah and Isabelle, and the tolerant truth of the sentences Mary is tall; Sarah is tall and Isabelle is tall. In other words, we want the tolerant truth of sentences with vague distributive predicates and non-vague plural subjects to be calculated on the basis of whether the predicate tolerantly applies to the subjects’ atoms. However, if we allow $\sim_P$ to be just any reflexive and symmetric relation between pluralities, this dependency is not there, even though, as shown by theorem 2, the predicates of PTCS are all classically distributive. The fact that, with no further restrictions on plural indifference relations, classical distributivity does not imply tolerant distributivity is shown by theorem 3.\footnote{This result is somewhat surprising, given that it is a theorem of singular TCS that classical validity implies tolerant validity ([4]’s Corollary 1).}
Theorem 3. c-distributivity $\nRightarrow$ t-distributivity. It is not the case that, for all t-models $M$, $g_1 \in D$, and $P \in \mathcal{P}(AT(D))$,

- If $M \models c\ P^*(g_1)$ iff for all atoms $a_1 \leq g_1$, $M \models c\ P(a_1)$, then $M \not\models t\ P^*(g_1)$ iff for all atoms $a_1 \leq g_1$, $M \not\models t\ P(a_1)$.

Proof. Let $M$ be a plural t-model $\langle D, \lor, m, \sim \rangle$, where $D$ is the join semi-lattice generated by the atoms $\{a_1, a_2, a_3\}$ and let $m(P) = \{a_1, a_2\}$. Therefore, by definition 13, $m(P^*) = \{a_1, a_2, a_1 \lor a_2\}$. Furthermore, let $\sim_p = \{(a_1, a_2), (a_2, a_1)\}$ + reflexivity, and let $\sim_p = \{(a_1 \lor a_3, a_1 \lor a_2), (a_1 \lor a_2, a_1 \lor a_2), (a_1, a_2), (a_2, a_1)\}$ + reflexivity. Finally, let $m(g_1) = a_1 \lor a_3$.

Clearly, $P^*$ is classically distributive. However, $M \models t\ P^*(g_1)$, but $M \not\models t\ P(a_3)$.

Therefore, $P^*$ is not tolerantly distributive. Therefore, in order to reflect the relationship between plural tolerant distributive predication and singular tolerant predication, I propose that plural indifference relations are constructed out of singular ones through closure under pointwise join, the binary operation over pairs defined below.

Definition 18. Pointwise join. $(\lor)$ For $\langle w, x \rangle$ and $\langle y, z \rangle$, $\langle w, x \rangle \lor \langle y, z \rangle = \langle w \lor y, x \lor z \rangle$

Definition 19. $\sim_*$ (final). For all $P^*$, $\sim_p$ is the closure of $\sim_p$ under $\lor$.

We first verify that $\sim_p$ has the required properties to be an indifference relation: it is reflexive and symmetric.

Theorem 4. For all $P^*$, $\langle D, \sim_p \rangle$ is a tolerance space.

Proof. Since, by assumption, $D$ is non-empty, we must show that $\sim_p$ is reflexive and symmetric. Let $P$ be a singular predicate. Reflexivity. Let $g_1 \in D$ to show $\langle g_1, g_1 \rangle \in \sim_p$. Let $A$ be the set of atoms under $g_1$. By the reflexivity of $\sim_p$, for all $a_1 \in A$, $\langle a_1, a_1 \rangle \in \sim_p$. By definition 19, the pointwise join of all the pairs $\langle a_1, a_1 \rangle$, for $a_1 \in A$, is in $\sim_p$, i.e. $\langle \lor A, \lor A \rangle \in \sim_p$. Since, by assumption and the atomicity of $\langle D, \lor \rangle$, $\lor A = g_1$, $\langle g_1, g_1 \rangle \in \sim_p$. Symmetry. Let $\langle g_1, g_2 \rangle \in \sim_p$ to show $\langle g_2, g_1 \rangle \in \sim_p$. Call the set of atoms under $g_1$, $A$ and the set of atoms under $g_2$, $B$. Because $\langle D, \lor \rangle$ is atomic, $\langle g_1, g_2 \rangle = \langle \lor A, \lor B \rangle$. Since $\langle g_1, g_2 \rangle \in \sim_p$, it is the pointwise join of some subset $R$ of $\sim_p$. Since $\sim_p$ is symmetric, the inverse of $R$, $R^{-1}$ is also a subset of $\sim_p$. Consider the pointwise join of $R^{-1}$: $\langle \lor B, \lor A \rangle$, a.k.a $\langle g_2, g_1 \rangle$. By definition 19, $\langle g_2, g_1 \rangle \in \sim_p$.

With this new definition of $\sim_p$, we can prove that tolerant distributivity holds in PTCS:

Theorem 5. Tolerant Distributivity. Let $M$ be a plural t-model, let $P$ be a predicate, and let $g_1 \in D$. $M \models t\ P^*(g_1)$ iff for all atoms $a_1 \leq g_1$, $M \models t\ P(a_1)$. 

Proof. ⇒ Suppose \( M \models^t P^*(g_1) \) and let \( a_1 \) be an atom such that \( a_1 \leq g_1 \). Since \( M \models^t P^*(g_1) \), by definition 17, there is some group \( g_2 \) such that \( g_2 \sim P^* g_1 \) and \( M \models^c P^*(g_2) \). By definition 19, there is some atom \( a_2 \leq g_2 \) such that \( a_2 \sim_P a_1 \). Furthermore, by classical distributivity (Theorem 2), \( M \models^c P(a_2) \). Therefore, by definition 17, \( M \models t P^*(a_1) \).

⇐ Suppose for all atoms \( a_1 \leq g_1 \), \( M \models t P^*(a_1) \) to show \( M \models^t P^*(g_1) \). Call the set of atoms under \( g_1 \) \( A \). Since \( P \) tolerantly holds on all the members of \( A \), by definition 17, they are all related to some other atom for which \( P \) classically holds. Let \( B = \{ a : x \sim_P a \mid x \in A \} \). Now consider the group \( \bigvee B \), call it \( g_2 \). By Theorem 2, \( M \models^c P^*(g_2) \). Furthermore, by definition 19, \( g_1 \sim_P g_2 \). So, \( M \models^t P^*(g_1) \).

In summary, the mereological extension of TCS that I have presented correctly assigns interpretations to sentences with singular subjects and both vague and non-vague predicates (Mary is tall/Canadian), and non-vague plural subjects with both vague and non-vague predicates (The four girls are tall/Canadian). However, we still cannot model sentences like The girls are tall/Canadian: although we need it to create the proper interpretation for sentences with vague predicates and non-vague plural subjects, definition 19 enforces universal (tolerant) distributive quantification over the atoms of the subject (this fact is reflected in Theorem 5). Thus, we have not yet accounted for the vague effects created by subjects like the girls.

4.1 Vague Subjects

To account for vagueness associated with the subject DP, I add to the language a generalized quantifier lifter: \( I \).

Definition 20. Syntax. 1) If \( g_1 \in P\text{-Term} \), then \( I g_1 \in GQ\text{-term} \). 2) If \( \overline{I g_1} \in GQ\text{-term} \) and \( P^* \in \text{PlurP} \), then \( \overline{I g_1}(P^*) \) is an atomic formula.

For the classical semantics, the proposal is essentially that of [9] (p.48), based on a proposal by [14]: rather than denoting in \( D \), subject DPs can be viewed as denoting generalized quantifiers; that is, they denote second order properties as defined below.

Definition 21. Semantics. For all \( g_1 \in D \), I maps \( g_1 \) to the family of properties containing it: 1) \( m(I g_1) = \{ P^* : M \models^c P^*(g_1) \} \) 2) C-truth in a t/c-model is defined as: \( M \models^c I g_1(P^*) \) iff \( P^* \in \{ Q^* : M \models^c Q^*(g_1) \} \)

It is easily proven from the definitions above that \( M \models^c I g_1(P^*) \) iff \( M \models^c P^*(g_1) \).

For the tolerant/strict semantics: just like how properties of individuals are associated with indifference relations by \( \sim \), I propose that properties of properties are also associated with indifference relations that express how similar they are with respect to an individual. For example, in the same way that \( \sim_{\text{tall}} \) relates elements of \( AT(D) \) that have an irrelevant difference in height, \( \sim_{I g_1} \) relates elements of \( \star P(D) \) that map the relevant parts of \( g_1 \) to true.
Definition 22. T/S-truth in a plural t-model. Add to definition 17:

1. $M \vDash_{I_{g_1}} (P^*)$ iff $\exists Q^* \sim_{I_{g_1}} P^* : M \vDash_t Q^*(g_1)$
2. $M \vDash_{I_{g_2}} (P^*)$ iff $\forall Q^* \sim_{I_{g_1}} P^* : M \vDash_{I_{g_2}} Q^*(g_1)$

To see how this new system works, consider the following example.

Example 1. Let $M$ be a group t-model $(D, \lor, m, \sim)$ such that $D$ is the join semi-lattice generated by the atoms $\{a_1, a_2, a_3\}$. Let $m(P) = \{a_1, a_2\}$ and $m(Q) = \{a_1, a_2, a_3\}$. Therefore, by the definition of $\sim$, $m(P^*) = \{a_1, a_2, a_1 \lor a_2\}$ and $m(Q^*) = \{a_1, a_2, a_3, a_1 \lor a_2, a_2 \lor a_3, a_1 \lor a_3, a_1 \lor a_2 \lor a_3\}$. Let $m(g_1) = a_1 \lor a_2$ and $m(g_2) = a_1 \lor a_2 \lor a_3$. By definition 21, $m(I_{g_1}) = \langle m(P^*), m(Q^*), \{a_1 \lor a_2\}, \{a_1 \lor a_2, a_1\}, \{a_1 \lor a_2\} \rangle$ and $m(I_{g_2}) = \langle m(P^*), m(Q^*), \{a_1 \lor a_2 \lor a_3\}, \{a_1 \lor a_2 \lor a_3, a_1\}, \{a_1 \lor a_2 \lor a_3, a_2\} \rangle$ (note $m(I_{g_2}) \subset m(I_{g_1})$). So $M \vDash_{I_{g_1}} (P^*)$ but $M \not\vDash_{I_{g_2}} (P^*)$.

Now, let $\sim_P = \{\langle a_1, a_1\rangle, \langle a_2, a_2\rangle, \langle a_3, a_3\rangle\}$ (i.e. $P$ is a non-vague predicate), so by definition 19, $\sim_P$ is the point-wise join of $\sim_P$. Therefore, $\sim_P$ has no members beyond is required for reflexivity. Finally, let $\sim_{I_{g_2}} = \{\langle m(P^*), m(Q^*)\rangle, \langle m(Q^*), m(P^*)\rangle\}$ + reflexivity. By definitions 17 and 22, $M \vDash_{I_{g_2}} (P^*)$.

The example can be summarized as follows: Suppose that the girls refers to the group $a_1 \lor a_2 \lor a_3$. Then the girls does not classically map $P^*$ to true in the model because $P^*$ does not affect $a_3$. However, since $P^*$ is indifferent from $Q^*$, and $Q^*$ tolerantly maps $a_1 \lor a_2 \lor a_3$ to true, the girls will tolerantly hold of $P^*$. In other words, the girls tolerantly maps $P^*$ to true even though $P$ does not tolerantly hold of $a_3$ because cases where the predicate holds of two members of the group are ‘just as good’ as cases where the predicate holds of all three members, i.e. $a_3$ is an irrelevant member. Thus, this example illustrates how a sentence with a vague subject and a non-vague predicate could be tolerantly true even if the predicate does not hold of the entire subject.

Furthermore, the example above can serve as the required case to prove that tolerant distributivity does not hold between $I_{g_1}$ and their atoms.

Theorem 6. Non-maximality. Let $M$ be a plural t-model, let $P$ be a predicate, and let $g_1 \in D$. $M \vDash_{I_{g_1}} (P^*) \not\rightarrow$ for all atoms $a_1 \leq g_1$, $M \vDash P(a_1)$.

Proof. In example 1, $M \vDash_{I_{g_2}} (P^*)$, but $a_3$ is an atom under $g_2$, and it is not the case that $M \vDash P(a_3)$. $\square$

Theorem 6 is the result that we need to reflect the observation that bare definite plural (i.e. vague) subjects tolerate exceptions. An interesting corollary of this fact is that, at the tolerant level, the equivalence between a group and the corresponding Montagovian individual breaks down: in particular, in some models, $I_{g_1}$ will tolerantly map more properties to true than will hold of $g_1$.

Corollary 1. $M \vDash_{I_{g_1}} (P^*) \not\rightarrow M \vDash P^*(g_1)$
Proof. Immediately from theorems 5 and 6.

Although, for ease of exposition, the example features a non-vague predicate, the recursion in definition 22 allows for sentences with borderline subjects to be tolerantly true even if they only tolerantly satisfy the predicate. Thus, we capture ‘double vagueness’ cases like *The girls are tall*.

In conclusion, I presented a new system, PTCS, that combines the insights from Link’s LPM and Cobreros et al.’s TCS to model distributive predication with singular and plural vague and non-vague predicates. The empirical scope of this paper was limited to certain kinds of definite plurals combined with distributive predicates; however, the analysis outlined above provides a basis for extending this approach to vague DPs with other kinds of predication and opens a new line of research into the distribution and properties of vague constituents outside the adjectival domain.

References