Optimal Mechanisms for Repeated Communication (Job Market Paper)

Delong Meng

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Abstract

We study repeated communication between a long-run sender and a long-run receiver. In each period the sender observes the state of the world – which is i.i.d. across time – and reports the state to the receiver. The receiver takes an action based on the history of the sender's reports and public randomization signals. The receiver fully commits to her action at each point in history, and the sender commits to nothing. We allow arbitrary state space, action space, and preferences. We characterize the set of possible payoffs for the sender and the receiver when both are infinitely patient – i.e., as the discount factor goes to one. We also study the payoff set when the discount factor is less than (but close to) one. In particular we bound the rate of convergence to points on the frontier of the limit payoff set; the rate of convergence differs radically for discrete and continuous models, and we provide a unified view of the rate of convergence results based on the shape of the frontier of the limit payoff set. We discuss three applications of our results. First for dynamic CEO compensation we characterize the firm's revenue from the optimal contract as the interest rate goes to zero. Second we show that dynamic delegation – a common problem in agencies – is equivalent to our model. Third we study a reputation problem where the sender's preference is unknown, and we give a lower bound for the receiver's expected payoff as the discount factor goes to one.

Keywords: first-best, Folk Theorem, cheap talk, reputation

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1 Introduction

In many economic situations a decision maker constantly relies on the information given by an adviser who has a different incentive. Patients need prescriptions from their doctors. Journal editors consult the opinions of referees. Politicians hear from lobbyists on policy decisions. In these examples the sender (the one with the information) could have a different preference from the receiver (the decision maker): doctors want to prescribe expensive drugs; referees have different tastes from editors, and lobbyists have their own political agenda. If communication takes place only once, the sender has little incentive to truthfully reveal the information. We are interested in the case when communication occurs repeatedly, and the sender's behavior today could affect her future payoff. Our central economic question is as follows. What is the payoff set for the sender and the receiver when both are patient?

To state our question more precisely we consider a repeated communication model with a long-run sender and a long-run receiver. In each period the sender observes the state of the world – which is i.i.d. across time – and reports to the receiver. The receiver takes an action based on the history of the sender's reports and public randomization signals. Suppose the receiver fully commits to her action at each point in history, but the sender has no commitment power and simply reports the message that maximizes her expected utility. What are the payoffs for the sender and the receiver when the discount factor goes one? And what is the payoff set when the discount factor is less than (but close to) one?

A quick observation is that if the discount factor is zero (i.e. the static setting), then information transmission is limited. On the other hand even if the discount factor goes to one, in general the receiver cannot attain the first-best payoff (under complete information). For example if the preferences are zero-sum ($u_R = -u_S$), then the sender has no incentive to reveal any information, and the receiver cannot obtain any payoff above her babbling equilibrium payoff. Therefore the receiver's payoff from the optimal mechanism is generally between her static payoff and her first-best payoff. In this paper we characterize the payoff set as the discount factor goes to one, and we analyze the rate of convergence to points on the frontier of this limit payoff set.

We begin with a baseline model to illustrate our main results and proof techniques. In the baseline model the state of the world comes from a bounded interval of $\mathbb{R}$, and the action space is a compact subset of $\mathbb{R}$. Furthermore the sender's utility satisfies the single crossing condition. In this setting a necessary condition for any incentive compatible mechanism is that actions are non-decreasing in states of the world (i.e., the monotonicity condition). Thus the receiver's payoff from the optimal mechanism is bounded above by her payoff under complete information subject to the monotonicity condition. We prove that the optimal mechanism achieves this upper bound in the limit as the discount factor goes to one. We also characterize the frontier of the entire payoff set in the limit. We study this frontier for two reasons. First it is a byproduct of our analysis for the optimal mechanism; mathematically the frontier is the value of the Bellman equation. Second we are interested in the rate of convergence (i.e., what is the receiver's payoff when the discount factor is 0.99?), which critically depends on the shape of the frontier.

We next consider the rate of convergence – i.e., if the discount factor δ is equal to 0.99, how far is the payoff set from the limit payoff set? The answer differs radically for continuous and discrete models. For our baseline model with a continuous state of the world the frontier of the limit payoff set can be second-order Taylor approximated, and the rate of convergence to any point on the frontier is on the order $\Theta(1-\delta)$ (Theorem 4.1 and Theorem 4.2). For example suppose the receiver's payoff from the optimal mechanism converges to X as the discount factor goes to one. If the discount factor is 0.99, then we should expect the receiver's payoff from the optimal mechanism to be roughly $X - 0.01 \cdot C$ for some constant C.

However in a binary-state binary-action model that we construct in Section 5 the frontier is piece-wise linear. The rate of convergence to the kink point (the receiver's optimal point) is $\Theta(\sqrt{1-\delta})$ (Theorem 5.4). For example suppose the receiver's payoff from the optimal mechanism converges to X as discount factor goes to one. If the discount factor is 0.99, then the receiver's payoff from the optimal mechanism is around $X - 0.1 \cdot C$ for some C (note: in contrast to $X - 0.01 \cdot C$ from the continuous model). Hence the rate of convergence to the receiver's optimal payoff is slower than the continuous model ($\sqrt{1-\delta}$ versus $1-\delta$). However we also show that the $\Theta(\sqrt{1-\delta})$ bound is unique to the kink point. The rate of convergence to any other point on the frontier is $o(1-\delta)$ (Theorem 5.3). Therefore if we focus on any point slightly below the limit payoff of the optimal mechanism the binary model actually has a rate of convergence strictly faster than $1-\delta$. The difference between the kink point and non-kink points is best explained by Figure 1, which we discuss in detail in Section 5.

In Section 6 we present a general model that unifies the results on the limit payoff set and the rate of convergence. We allow states and and actions to be any compact subsets of $\mathbb{R}^N$, and we also allow arbitrary sender and receiver preferences. Now the receiver's actions must satisfy a cyclic monotonicity condition in Rochet (1987). We characterize the frontier of the limit payoff set and in particular the limit payoff from the optimal mechanism. We show that in the special case when the sender's preference is state-independent, the receiver could achieve the first-best payoff in the limit. We also derive the following upper bounds for the rate of convergence to the frontier of the limit payoff set: $O(\sqrt{1-\delta})$ if the point is a kink, $O(1-\delta)$ if the frontier around this point is smooth but non-linear, and $o(1-\delta)$ if the frontier around this point is locally linear. These bounds unify the different results on the rate of convergence for the baseline (continuous) model and the binary-state binary-action model.

In Section 7.4 we extend our model to a setting where the sender's utility function is unknown. In particular with probability p the sender has the same utility has the receiver, and with probability $1-p$ the sender has a different utility. For example the sender is a lobbyist who may or may not be a racist (Morris (2001)); the sender (a mechanic) either wants to fix the car or wants the most expensive repair (Ely and Välimäki (2003)), or the sender is either a loyal agent or a double agent (Sobel (1985)). We provide a lower bound for the receiver's payoff. Our lower bound is consistent with the observation in Ely and Välimäki (2003) that the bad reputation effect disappears if both the mechanic and the motorist are long-run players. We also construct an example in which unknown sender type significantly harms the receiver. In particular in a binary-state binary-action setting if the probability that the sender is a "bad type" exceeds 1/2, the receiver cannot do better than her payoff from the babbling equilibrium.

We discuss applications of our model in Section 7.

Applications First and foremost any example of long-term bilateral communication fits into our model. The sender is a think tank, and the receiver is a lawmaker. The sender is a mechanic, and the receiver is a motorist. The sender is a referee, and the receiver is the editor of a journal. In these examples the receiver's payoff approaches her first-best payoff as the discount factor goes to one. Beyond the examples of literal communication our model also applies to insurance with hidden income (e.g. Thomas and Worrall (1990)). We interpret the sender as a borrower and the receiver as a lender. If the borrower is risk-averse and the lender risk-neutral, then the efficient contract could be implemented, which is consistent with the result of Thomas and Worrall (1990). More generally our model is equivalent to dynamic delegation (see Section 7.3), where a principal repeatedly delegates tasks to an agent and specifies the set of permissible actions. Delegation is common in agency problems, and any application of repeated delegation (e.g. Lipnowski and Ramos (2017)) also fits into our model.

A less obvious application is dynamic CEO compensation (e.g. Garrett and Pavan (2012, 2015); Edmans and Gabaix (2011); Edmans et al. (2012)). The CEO privately observes a productivity shock before exerting her effort, and the principal could only contract on the output. Garrett and Pavan (2015) solved the optimal contract for the two-period model. Edmans and Gabaix (2011) analyzed a multi-period model, but assumed that the principal wants to implement a target effort level rather than solving for the optimal contract. We are interested in the performance of the optimal contract with an infinite time horizon. We interpret this dynamic contract problem in terms of repeated communication. In each period the principal first asks the CEO to report her productivity shock and then specifies an output level and a payment. We apply the results from our general model in Section 6. We show in Section 7.2 that if the CEO's cost of effort is strictly convex, then in the limit the principal achieves her first-best payoff. Moreover the rate of convergence to the first-best payoff is $O(\sqrt{1-\delta})$, but the rate of convergence to any point slightly below the first-best is $o(1-\delta)$.

In examples with transfers (e.g. insurance or CEO compensation) the rate of convergence captures the effect of the interest rate. Suppose the interest rate is small but non-zero. The rate of convergence gives the distance between the principal's expected profit (with interest rate) and her payoff in the limit (as interest rate goes to zero). In examples without transfers the rate of convergence captures the effect of impatience. Suppose the receiver is patient but not infinitely patient (i.e. $\delta = 0.99$). The rate of convergence gives the distance between the receiver's utility from repeated communication and her payoff in the limit.

Related literature The first strand of related literature is Folk Theorem with imperfect monitoring, such as Fudenberg et al. (1994), Fudenberg and Levine (1994), and Hörner et al. (2015). Closest to our model is Hörner et al. (2015). They study multi-player communication games; they established a Folk Theorem for the IPV case with persistent states and characterized a subset of truthful equilibria for the non-IPV case. The main difference between our work and theirs is that in our model the receiver could commit to her actions at each history, whereas they characterize the equilibria without commitment. Our proof technique is conceptually similar to the Fudenberg-Levine algorithm, but we use a Bellman equation instead of their linear programming approach. The Bellman equation allows us to find sharp bounds for the rate of convergence.

Hörner and Takahashi (2016) studied the rate of convergence in repeated games using the Fudenberg-Levine algorithm. For games with imperfect monitoring they derived a rate of $O(\sqrt{1-\delta})$ (Proposition 7 of their paper). Their bound relies on two features: discrete games and vertices. In contrast we derived three sets of results for the rate of convergence. For discrete models the rate of convergence to a vertex is indeed on the order of $O(\sqrt{1-\delta})$, but convergence is faster for non-vertex points as well as for continuous-state games. These differences could be seen in the second-order Taylor approximation. We elaborate on the connection between our work and Hörner and Takahashi (2016) in Sections 4 and 5.

Guo and Hörner (2018) studied a similar model as ours. They consider Markovian states, whereas we assume i.i.d. states, so their model is more general than ours in terms of state transitions across time. However within each period they assume binary states and rule out transfers. In contrast we allow arbitrary states, actions, and preferences, and we could interpret an action in our model as a transfer such as in the application to insurance. The bad reputation model of Ely and Välimäki (2003) is also relevant. Their main result assumes that the motorists are short-run players, but if both the mechanic and the motorist are long-run players, the motorist could attain the first-best as noted in Theorem 4 of their paper. In fact the "scoring rule" in their proof of Theorem 4 is closely related to our Bellman equation method. Other related papers include Lipnowski and Ramos (2017) who study repeated delegation where the agent incurs a cost of carrying out a project, and Chen (2017) who analyze repeated communication where states follow a Brownian motion. Kuvalekar et al. (2018) consider repeated cheap talk with no monitoring of the receiver's actions, whereas in our model the receiver commits to her actions. The literature on optimal taxation is also relevant (c.f. Golosov et al. (2003); Kapička (2013); Farhi and Werning (2013)), but those models typically assume that states (an agent's ability) are persistent instead of i.i.d. across time.

Several papers on repeated communication use a review strategy (c.f. Escobar and Toikka (2013), Jackson and Sonnenschein (2007), Renault et al. (2013)). Review strategy in general cannot attain the same limit payoff as the optimal mechanism. For example states and actions are both binary; the receiver wants to match the state, but the sender always prefers the lower action. The receiver could attain the first-best payoff (Proposition 5.2), but under a review strategy the sender would always report a sequence of low states followed by a sequence of high states. When the sender's preference is state-independent, Margaria and Smolin (2018) developed a variant of the review strategy with a reporting phase and an adjustment phase, but their mechanism cannot adapt to the general case of state-dependent sender preferences.

2 The Baseline Model

There is one long-run sender and one long-run receiver. The time horizon is infinite. In period t the sender observes the state $\theta_t \sim F[\underline{\theta}, \overline{\theta}]$, which is bounded and i.i.d. across time. After the sender observes θ_t a public randomization signal $x_t \in [\underline{x}, \overline{x}]$ is realized; without loss of generality assume that $x_t \sim U[0,1]$. Then the sender reports a message $m_t \in [\underline{\theta}, \overline{\theta}]$ to the receiver. The receiver takes an action $a_t \in A$; the action space is a compact subset of $\mathbb{R}$, and it could be either discrete or continuous. The receiver cannot observe previous states of the world, so her action depends only on the history of reports and public signals $\{m_0, \ldots, m_t; x_0, \ldots, x_t\}$. In summary the events in period t are as follows:

1. The sender sees the state θ_t.
2. The public signal x_t is realized.
3. The sender reports m_t.
4. The receiver takes action a_t based on history $\{m_0, \ldots, m_t; x_0, \ldots, x_t\}$.

The receiver has full commitment power. More precisely let $h_t = \{m_0, \ldots, m_t; x_0, \ldots, x_t\}$ denote the public history at time t. A mechanism is a deterministic function from the set of public history $\{h_t\}_{t=0}^{+\infty}$ to the action space A. For the ease of notation let $a(h_t)$ denote the action at history h_t. The receiver commits to taking action $a(h_t)$ at history t. By the revelation principle we could assume without loss of generality that the sender truthfully reveals the state of the world, which means $h_t = \{\theta_0, \ldots, \theta_t; x_0, \ldots, x_t\}$.

Let δ denote the discount factor. The sender's payoff is equal to

$$\sum_{t=0}^{+\infty}(1-\delta)\delta^t \cdot \mathbb{E}_{h_t} u_S(a(h_t), \theta_t),$$

and the receiver's payoff is equal to

$$\sum_{t=0}^{+\infty}(1-\delta)\delta^t \cdot \mathbb{E}_{h_t} u_R(a(h_t), \theta_t).$$

Assume that both $u_S(a,\theta)$ and $u_R(a,\theta)$ are bounded and continuous functions from $\mathbb{R}^2$ to $\mathbb{R}$. Assume that $\frac{\partial u_S(a,\theta)}{\partial \theta}$ exists, is bounded, and is strictly increasing in a for all θ. The last assumption on u_S is the standard single crossing condition in mechanism design.

Two examples The first example is delegation to a biased adviser. The sender is a biased adviser who prefers policy $\theta + b$, whereas the receiver wants to implement policy θ. The sender's utility is $u_S(a, \theta) = -(a - \theta - b)^2$, and the receiver's utility is $u_R(a, \theta) = -(a - \theta)^2$.

The second example is insurance with hidden income (Thomas and Worrall (1990)). The sender is a risk-averse borrower; the receiver is a risk-neutral lender. The state of the world is the sender's hidden income, and the action corresponds to the borrower's transfer to the lender. The borrower's utility is $u_S(a, \theta) = u(-a + \theta)$ where u is concave, and the lender's utility is $u_R(a, \theta) = a$.

The receiver's problem For the optimal mechanism the receiver chooses a function $a(h_t)$ that maximizes her expected payoff with respect to the sender's incentive constraint. We write the receiver's problem as follows:

$$\max_{a(h_t)} \sum_{t=0}^{+\infty} (1-\delta)\delta^t \cdot \mathbb{E}_{h_t} u_R(a(h_t), \theta_t)$$
$$s.t.$$
$$\theta_\tau \in \operatorname{argmax}_{\theta'_\tau} \sum_{t \geq \tau} (1-\delta)\delta^{t-\tau} \cdot \mathbb{E}_{h_t | h_{\tau-1}, x_\tau, \theta'_\tau} u_S(a(h_t), \theta_t) \qquad \forall h_{\tau-1}, x_\tau, \theta_\tau.$$

The second line is the sender's incentive constraint at history h_τ. Note that all subsequent history h_t (where $t \geq \tau$) starts with $h_{\tau-1}$, the public signal x_τ, and the sender's report θ'_τ.

We transform the receiver's problem into a recursive one. Suppose the receiver commits to actions $a(h_t)$ in some mechanism. Let u denote the sender's expected payoff from this mechanism. We have

$$u = \sum_{t=0}^{+\infty} (1-\delta)\delta^t \cdot \mathbb{E}_{h_t} u_S(a(h_t), \theta_t).$$

Let $u_{\theta,x}$ denote the sender's continuation payoff (starting from period 1) after seeing the public signal $x_0 = x$ and the state $\theta_0 = \theta$. We have

$$u_{\theta,x} = \sum_{t=1}^{+\infty} (1-\delta)\delta^{t-1} \cdot \mathbb{E}_{h_t | \theta_0 = \theta, x_0 = x} u_S(a(h_t), \theta_t).$$

Let $\mathtt{OPT}_\delta(u)$ denote the optimal mechanism with the constraint that the sender's expected payoff is u. In period 0 the receiver chooses the appropriate $u_{\theta,x}$. These $u_{\theta,x}$ must satisfy the sender's incentive constraint in period 0. They must also satisfy a promise-keeping constraint: the sender's total expected payoff is equal to u. Then starting from period 1 the receiver follows the actions specified by mechanism $\texttt{OPT}_\delta(u_{\theta,x})$. This recursive formulation is best captured in a Bellman equation below.

The Bellman equation Let $w_\delta(u)$ denote the receiver's maximal possible payoff from an incentive compatible mechanism such that the sender's expected payoff is equal to u. In other words $w_\delta(u)$ is the receiver's payoff from the mechanism $\texttt{OPT}_\delta(u)$. We write the maximization problem for $w_\delta(u)$ as a Bellman equation: choose a set of actions for today and a set of promised utilities for the future.

We first define the domain of the Bellman equation. Let

$$\begin{aligned} \underline{u} &= \min_a \int_{\underline{\theta}}^{\overline{\theta}} u_S(a,\theta)dF(\theta), \\ \overline{u} &= \int_{\underline{\theta}}^{\overline{\theta}} \max_{a_\theta} u_S(a_\theta,\theta)dF(\theta). \end{aligned}$$

The Bellman equation has domain on the interval $[\underline{u},\overline{u}]$. Because u_S is bounded, the domain is finite. The sender could secure a payoff of $\underline{u}$ by babbling, and the sender cannot attain more than $\overline{u}$ because the receiver is already taking the sender's most preferred action. The Bellman equation is as follows:

$$w_\delta(u) = \max\nolimits_{a_{\theta,x}\in A, u_{\theta,x}\in[\underline{u},\overline{u}]} \int_{\underline{x}}^{\overline{x}} \int_{\underline{\theta}}^{\overline{\theta}} [(1-\delta)\cdot u_R(a_{\theta,x},\theta) + \delta\cdot w_\delta(u_{\theta,x})]\cdot dF(\theta)\cdot dx$$

$$s.t.$$

$$\forall x,\theta \qquad \theta \in \operatorname{argmax}_{\theta'} (1-\delta)\cdot u_S(a_{\theta',x},\theta) + \delta\cdot u_{\theta',x}$$

$$\int_{\underline{x}}^{\overline{x}} \int_{\underline{\theta}}^{\overline{\theta}} [(1-\delta)\cdot u_S(a_{\theta,x},\theta) + \delta\cdot u_{\theta,x}]\cdot dF(\theta)\cdot dx = u$$

If the public randomization is x and the sender reports θ, the receiver takes action $a_{\theta,x}$ and promises the sender a continuation payoff of $u_{\theta,x}$. The receiver's payoff from the current period is $(1-\delta)\cdot u_R(a_{\theta,x},\theta)$, and her future payoff is $w_\delta(u_{\theta,x})$ because the sender gets an expected payoff of $u_{\theta,x}$. The choice of $a_{\theta,x}$ and $u_{\theta,x}$ must satisfy the incentive constraint and the promise-keeping constraint. We also require $u_{\theta,x}\in[\underline{u},\overline{u}]$, which is equivalent to a transversality condition: the receiver cannot use a Ponzi-scheme because the continuation payoffs will eventually get out of the boundary.

Public randomization ensures that $w_\delta(u)$ is well-defined and concave. Indeed if $u = \alpha u_1 + (1-\alpha)u_2$, then $w_\delta(u)$ must be at least the payoff from the following mechanism:

if the public signal is less than α, use the mechanism $\mathtt{OPT}_\delta(u_1)$, and if the public signal is more than α, use the mechanism $\mathtt{OPT}_\delta(u_2)$.

For a fixed δ the receiver's payoff from the optimal mechanism is $\max_u w_\delta(u)$, and receiver's payoff from the optimal mechanism in the limit is $\lim_{\delta\to 1} \max_u w_\delta(u)$.

3 The Receiver's Payoff as $\delta \to 1$

We begin with the following question: could the receiver attain her first-best payoff as $\delta \to 1$? By first-best we mean the receiver's maximal payoff under complete information. Mathematically this question is equivalent to

$$\lim_{\delta\to 1} \max_u w_\delta(u) \quad \stackrel{?}{=} \quad \max_{a_\theta} \int_{\underline{\theta}}^{\overline{\theta}} u_R(a_\theta, \theta) \cdot dF(\theta)$$

In general the left hand side is less than the right hand side. A simple example is the zero-sum case ($u_R = -u_S$). The sender has no incentive to reveal any information, so the receiver cannot achieve any payoff above her babbling equilibrium payoff.

We aim for a more modest objective: the receiver's maximal payoff under complete information subject to the constraint that actions $a_{\theta,x}$ are non-decreasing in θ. The receiver attains this payoff from the optimal mechanism as $\delta \to 1$.

Theorem 3.1. *We have*

$$\lim_{\delta\to 1} \max_u w_\delta(u) \quad = \quad \begin{array}{c} \max_{a_\theta} \int_{\underline{\theta}}^{\overline{\theta}} u_R(a_\theta, \theta) \cdot dF(\theta) \\ s.t. \\ a_{\theta,x} \textit{ is non-decreasing in } \theta \end{array}$$

We call the constraint that $a_{\theta,x}$ is non-decreasing in θ the monotonicity condition. It follows from Theorem 3.1 that the optimal mechanism gives the receiver her first-best payoff in the limit if and only if her first-best actions satisfy the monotonicity condition.

Corollary 3.2. *We have* $\lim_{\delta\to 1} \max_u w_\delta(u) = \max_{a_\theta} \int_{\underline{\theta}}^{\overline{\theta}} u_R(a_\theta, \theta) \cdot dF(\theta)$ *if and only if there exist actions* a_θ *such that (i)* a_θ *is non-decreasing in* θ*, and (ii)* $a_\theta \in \operatorname{argmax}_a u_R(a, \theta)$ *for all* θ*.*

The next example illustrates the role of the monotonicity condition.

Example 3.3. States come from $U[0,1]$. Actions come from $[0,1]$. Utilities are $u_S(a,\theta) = -(a-\theta)^2$, and

$$u_R(a,\theta) = \begin{cases} -(a-\theta-\frac{1}{2})^2 & \text{if } \theta < \frac{1}{2} \\ -(a-\theta+\frac{1}{2})^2 & \text{if } \theta > \frac{1}{2} \end{cases}.$$

The sender wants to match the state, but the receiver wants to match the state after switching the intervals $[0,\frac{1}{2}]$ and $[\frac{1}{2},1]$. The receiver's first-best payoff is 0, and her first-best actions are given by

$$a_\theta = \begin{cases} \theta+\frac{1}{2} & \text{if } \theta < \frac{1}{2} \\ \theta-\frac{1}{2} & \text{if } \theta > \frac{1}{2} \end{cases}.$$

Note that a_θ is not monotonic in θ, so according to Corollary 3.2 the receiver cannot attain payoff 0 in the limit of $\delta \to 1$.

For the constrained optimal actions with respect to the monotonicity condition the receiver solves the following problem:

$$\max_{a_\theta} \int_0^{1/2} -(a_\theta-\theta-\tfrac{1}{2})^2 \cdot d\theta + \int_0^{1/2} -(a_\theta-\theta+\tfrac{1}{2})^2 \cdot d\theta$$
$$s.t.$$
$$a_\theta \text{ is non-decreasing in } \theta$$

The solution to this constrained problem is $a_\theta = \frac{1}{2}$ for all θ, and the value of the objective function is $-\frac{1}{12}$. Theorem 3.1 implies that as $\delta \to 1$ the receiver's payoff from the optimal mechanism converges to $-\frac{1}{12}$. Notice that the receiver's action $a_\theta = \frac{1}{2}$ is independent of θ, so the receiver could attain $-\frac{1}{12}$ in a babbling equilibrium. Hence for this example the receiver cannot do better than the babbling payoff even if $\delta \to 1$.

One way to prove Theorem 3.1 is to use a review strategy. Every K periods the receiver checks whether the empirical frequency of the sender's reports is close to the actual frequency of θ. We use the Bellman equation approach instead of the review strategy for two reasons. First the Bellman equation approach gives us sharp bounds on the rate of convergence. Second we will drop the single crossing condition in Section 6, and the review strategy cannot handle general sender preferences.

We prove Theorem 3.1 as follows. First we prove the upper bound by weakening the incentive constraint of the Bellman equation. We then construct a greedy mechanism that attains the upper bound in the limit of $\delta \to 1$.

3.1 The upper bound

We first rewrite the incentive constraint in the Bellman equation. The incentive constraint says for all x and θ we have $\theta \in \text{argmax}_{\theta'}(1-\delta) \cdot u_S(a_{\theta',x}, \theta) + \delta \cdot u_{\theta',x}$. Under the single crossing condition this constraint is equivalent to the following two conditions: (i) $a_{\theta,x}$ is non-decreasing in θ, and (ii) the sender's utility can be expressed as an integral envelope formula.

Proposition 3.4. *Suppose for all x and θ we have $\theta \in \text{argmax}_{\theta'}(1-\delta) \cdot u_S(a_{\theta',x}, \theta) + \delta \cdot u_{\theta',x}$. Then the following two conditions hold:*

- *We have $a_{\theta,x}$ is non-decreasing in θ.*
- *Let $\underline{U}_x = (1-\delta)u_S(a_{\underline{\theta},x}, \underline{\theta}) + \delta u_{\underline{\theta},x}$. For all x and θ we have*

$$(1-\delta) \cdot u_S(a_{\theta,x}, \theta) + \delta \cdot u_{\theta,x} = \underline{U}_x + \int_{\underline{\theta}}^{\theta} (1-\delta) \frac{\partial u_S(a_{s,x}, s)}{\partial s} ds.$$

Conversely if these two conditions hold, we have $\theta \in \text{argmax}_{\theta'}(1-\delta) \cdot u_S(a_{\theta',x}, \theta) + \delta \cdot u_{\theta',x}$ for all x and θ.

Proposition 3.4 allows us to rewrite the Bellman equation as follows:

$$\begin{gathered} w_\delta(u) = \max_{a_{\theta,x} \in A, u_{\theta,x} \in [\underline{u}, \overline{u}]} \int_{\underline{x}}^{\overline{x}} \int_{\underline{\theta}}^{\overline{\theta}} [(1-\delta) \cdot u_R(a_{\theta,x}, \theta) + \delta \cdot w_\delta(u_{\theta,x})] \cdot dF(\theta) \cdot dx \\ s.t. \\ a_{\theta,x} \text{ is non-decreasing in } \theta \\ (1-\delta) \cdot u_S(a_{\theta,x}, \theta) + \delta \cdot u_{\theta,x} = \underline{U}_x + \int_{\underline{\theta}}^{\theta} (1-\delta) \frac{\partial u_S(a_{s,x}, s)}{\partial s} ds \\ \int_{\underline{x}}^{\overline{x}} \int_{\underline{\theta}}^{\overline{\theta}} [(1-\delta) \cdot u_S(a_{\theta,x}, \theta) + \delta \cdot u_{\theta,x}] \cdot dF(\theta) \cdot dx = u \end{gathered}$$

(We have $\underline{U}_x = (1-\delta)u_S(a_{\underline{\theta},x}, \underline{\theta}) + \delta u_{\underline{\theta},x}$ in the incentive constraint.)

Next define the function $w(u)$ as follows:

$$\begin{gathered} w(u) = \max_{a_{\theta,x}} \int_{\underline{x}}^{\overline{x}} \int_{\underline{\theta}}^{\overline{\theta}} u_R(a_{\theta,x}, \theta) dF(\theta) dx \\ s.t. \\ a_{\theta,x} \text{ is non-decreasing in } \theta \\ \int_{\underline{x}}^{\overline{x}} \int_{\underline{\theta}}^{\overline{\theta}} u_S(a_{\theta,x}, \theta) dF(\theta) dx = u \end{gathered}$$

The function $w(u)$ is the receiver's maximal payoff under complete information subject to the constraints that (i) actions are non-decreasing in states, and (ii) the sender gets an expected payoff of u. Since w is concave, it follows from Proposition 3.4 that $w_\delta(u) \leq w(u)$. Indeed $w(u)$ is equivalent to getting rid of the incentive constraint in the Bellman equation for $w_\delta(u)$. Therefore an upper bound for the receiver's payoff is the function $w(u)$.

Corollary 3.5. *For all $u \in [\underline{u}, \overline{u}]$ we have $w_\delta(u) \leq w(u)$. In particular $\limsup_{\delta \to 1} \max_u w_\delta(u) \leq \max_u w(u)$.*

3.2 Defining the mechanism $\texttt{GREEDY}_\delta(u)$

Recall that $\texttt{OPT}_\delta(u)$ is the optimal mechanism under the constraint that the sender's expected payoff is u. The receiver's payoff from mechanism $\texttt{OPT}_\delta(u)$ is equal to $w_\delta(u)$.

For all $u \in (\underline{u}, \overline{u})$ we define the mechanism $\texttt{GREEDY}_\delta(u)$ as follows. First let action $a^*_{\theta,x}$ denote a solution to the following problem:

$$\begin{aligned} a^*_{\theta,x} = \quad & \text{argmax}_{a_{\theta,x}} \int_{\underline{x}}^{\overline{x}} \int_{\underline{\theta}}^{\overline{\theta}} u_R(a_{\theta,x}, \theta) dF(\theta) dx \\ & s.t. \\ & a_{\theta,x} \text{ is non-decreasing in } \theta \\ & \int_{\underline{x}}^{\overline{x}} \int_{\underline{\theta}}^{\overline{\theta}} u_S(a_{\theta,x}, \theta) dF(\theta) dx = u \end{aligned}$$

In other words $a^*_{\theta,x}$ is the set of non-decreasing actions in θ that gives the receiver an expected payoff of $w(u)$ (defined in Section 3.1). Then define continuation payoff $u^*_{\theta,x}$ as follows:

$$u^*_{\theta,x} = \frac{u - (1-\delta) \cdot [\int_{\underline{x}}^{\overline{x}} \int_{\underline{\theta}}^{\overline{\theta}} \frac{\partial u_S(a^*_{s,x}, s)}{\partial s} \frac{1-F(s)}{f(s)} dF(s) dx - \int_{\underline{\theta}}^{\theta} \frac{\partial u_S(a^*_{s,x}, s)}{\partial s} ds + u_S(a^*_{\theta,x}, \theta)]}{\delta}. \quad (3.1)$$

Since $\frac{\partial u_S}{\partial s}$ and u_S are both bounded, the quantity in the bracket is bounded. Moreover since u is in the interior of $(\underline{u}, \overline{u})$, for δ close to 1 we have $u^*_{\theta,x} \in [\underline{u}, \overline{u}]$ for all θ and x.

The mechanism $\texttt{GREEDY}_\delta(u)$ works as follows:

- In period 0 the receiver takes action $a^*_{\theta,x}$ if the public signal is $x_0 = x$ and the sender reports $\theta_0 = \theta$.
- In period 1 and thereafter the receiver takes the actions specified in the mechanism $\texttt{OPT}_\delta(u^*_{\theta,x})$. Note: the receiver uses the mechanism $\texttt{OPT}_\delta(u^*_{\theta,x})$, not $\texttt{GREEDY}_\delta(u^*_{\theta,x})$.

The mechanism $\texttt{GREEDY}_\delta(u)$ is well-defined if $u^*_{\theta,x} \in [\underline{u}, \overline{u}]$ for all θ and x, which is true for δ close to 1. From now on when we say $\texttt{GREEDY}_\delta(u)$, we assume that δ is close to 1.

The receiver's expected payoff in period 0 is $w(u)$, and after period 0 the mechanism $\texttt{OPT}_\delta(u^*_{\theta,x})$ gives her an expected payoff of $w_\delta(u^*_{\theta,x})$. Therefore the receiver's total expected payoff from mechanism $\texttt{GREEDY}_\delta(u)$ is equal to

$$(1-\delta)\cdot w(u) + \delta \cdot \int_{\underline{x}}^{\overline{x}} \int_{\underline{\theta}}^{\overline{\theta}} w_\delta(u^*_{\theta,x}) \cdot dF(\theta)dx.$$

To check that the mechanism $\texttt{GREEDY}_\delta(u)$ is incentive compatible we derive the continuation payoff $u^*_{\theta,x}$ as follows. For the incentive constraints of the Bellman equation (given by Proposition 3.4) we set $\underline{U}_x$ the same for all x. In particular let $\underline{U}_x = u - \int_{\underline{x}}^{\overline{x}} \int_{\underline{\theta}}^{\overline{\theta}} (1-\delta) \frac{u_S(a^*_{\theta,x},\theta)}{\partial \theta} \frac{1-F(\theta)}{f(\theta)} dF(\theta)dx$ for all x. This construction fulfills the promise-keeping constraint. Then the incentive constraint implies that

$$(1-\delta)\cdot u_S(a^*_{\theta,x},\theta) + \delta\cdot u^*_{\theta,x} = u - \int_{\underline{x}}^{\overline{x}} \int_{\underline{\theta}}^{\overline{\theta}} (1-\delta) \frac{u_S(a^*_{s,x},s)}{\partial s} \frac{1-F(s)}{f(s)} dF(s)dx + \int_{\underline{\theta}}^{\theta} (1-\delta) \frac{\partial u_S(a^*_{s,x},s)}{\partial s} ds,$$

which gives us the expression (3.1). Therefore the $u^*_{\theta,x}$ given by (3.1) satisfies the incentive constraint.

Notice that the mechanisms $\texttt{GREEDY}_\delta(u)$ and $\texttt{OPT}_\delta(u)$ differ in two ways. First the greedy mechanism implements actions $a^*_{\theta,x}$ in period 0, and this choice may not be optimal. Second the greedy mechanism requires $\underline{U}_x$ the same for all x, and this restriction may not be optimal either.

3.3 The receiver's payoff as $\delta \to 1$

Recall that $w(u)$ is the receiver's maximal payoff under complete information subject to the constraints that $a_{\theta,x}$ is non-decreasing in θ and the sender gets an expected payoff of u. Corollary 3.5 shows that $w_\delta(u)$ is bounded above by $w(u)$. The next theorem states that the receiver could attain this upper bound as $\delta \to 1$.

Theorem 3.6. *For all $u \in (\underline{u}, \overline{u})$ we have $w_\delta(u) \to w(u)$ as $\delta \to 1$.*

Theorem 3.6 implies Theorem 3.1. Indeed since $w(u)$ is continuous, it suffices to show that as $\delta \to 1$ the receiver could attain a payoff of $w(u)$ for all u in the interior of $(\underline{u}, \overline{u})$. The receiver's payoff from the optimal mechanism converges to $\max_u w(u)$.

We illustrate the proof of Theorem 3.6 through an example. The key idea is to show that the mechanism $\texttt{GREEDY}_\delta(u)$ gives the receiver a payoff $w(u)$ in the limit.

Example 3.7. Suppose $\theta \sim U[0,1]$ and $a \in [0, 1+b]$ for some $b > 0$. The sender's utility is $u_S(a,\theta) = -(a-\theta-b)^2$, and the receiver's utility is $u_R(a,\theta) = -(a-\theta)^2$. We have $\underline{u} = -\frac{1}{3} - b - b^2$ and $\overline{u} = 0$. For simplicity we focus on the interval $u \in [-b^2, 0)$. For this interval we have $w(u) = -(b-\sqrt{-u})^2$; we also have $a^*_{\theta,x} = \theta + b - \sqrt{-u}$ for all x, so we can ignore public randomization and write a^*_θ instead of $a^*_{\theta,x}$. The promised utility u^*_θ from (3.1) simplifies to

$$u^*_\theta = \frac{u - (1-\delta)\cdot(\sqrt{-u}(2\theta-1)+u)}{\delta},$$

which means

$$u^*_\theta - u = -\frac{1-\delta}{\delta}\cdot\sqrt{-u}(2\theta-1).$$

The mechanism $\texttt{GREEDY}_\delta(u)$ implements a^*_θ in period 0 and uses u^*_θ as the promised utility. The receiver's payoff from period 0 is equal to $w(u)$, and her future payoff is $w_\delta(u^*_\theta)$. Hence the receiver's payoff from $\texttt{GREEDY}_\delta(u)$ is equal to $(1-\delta)\cdot w(u) + \delta\int_0^1 w_\delta(u^*_\theta)d\theta$, which is at most her payoff from $\texttt{OPT}_\delta(u)$. We obtain that

$$\text{payoff from } \texttt{OPT}_\delta(u) = w_\delta(u) \;\geq\; \text{payoff from } \texttt{GREEDY}_\delta(u) = (1-\delta)\cdot w(u) + \delta\int_0^1 w_\delta(u^*_\theta)d\theta.$$

Divide both sides by $1-\delta$ and rearrange this inequality:

$$w(u) - w_\delta(u) \leq \frac{\delta}{1-\delta}\cdot\int_0^1 (w_\delta(u) - w_\delta(u^*_\theta))d\theta. \tag{3.2}$$

We are left to show that the right hand side goes to 0 as $\delta \to 1$. First-order Taylor approximation implies that the right hand side is approximately

$$w'_\delta(u)\cdot\int_0^1 \frac{\delta}{1-\delta}(u-u^*_\theta)d\theta = w'_\delta(u)\cdot\int_0^1 \sqrt{-u}(2\theta-1)d\theta = 0.$$

The key idea is that $u^*_\theta - u$ is on the order of $\frac{1-\delta}{\delta}$ and has mean 0; i.e., $\frac{\delta}{1-\delta}(u-u^*_\theta) = \sqrt{-u}(2\theta-1)$ is independent of δ. Therefore the difference $w(u) - w_\delta(u)$ converges to 0. The above approximation has a few loose ends: in the appendix we formally prove that w'_δ converges and that the Taylor approximation is valid.

The receiver's first-best payoff is 0. The receiver could achieve the first-best because the actions $a_\theta = \theta$ are non-decreasing in θ. The mechanism $\texttt{GREEDY}_\delta(-b^2)$ attains the first-best in the limit because $w(-b^2) = -(b - \sqrt{b^2})^2 = 0$. In this mechanism the period-0 actions are $a^*_\theta = \theta$, and promised utilities are $u^*_\theta = -b^2 - \frac{1-\delta}{\delta} b(2\theta - 1)$.

4 The Rate of Convergence

Theorem 3.6 says $w_\delta(u)$ converges to $w(u)$ for all $u \in (\underline{u}, \overline{u})$ as $\delta \to 1$. More specifically Theorem 3.1 says the receiver's payoff from the optimal mechanism converges to $\max_u w(u)$ as $\delta \to 1$. What if $\delta = 0.99$? How far is the receiver's payoff from the limit? In this section we argue that the answer is generally on the order of $1 - \delta$.

We show that under certain conditions $w_\delta(u)$ converges to $w(u)$ at a rate of $\Theta(1 - \delta)$. More precisely for each $u \in (\underline{u}, \overline{u})$ there exist constants C_1 and C_2 such that for all δ sufficiently close to 1 we have $C_1 < \frac{w(u) - w_\delta(u)}{1-\delta} < C_2$. It follows from this result that the receiver's payoff from the optimal mechanism converges to $\max_u w(u)$ at rate of $\Theta(1 - \delta)$ as long as the maximizer of $w(u)$ is in the interior of $(\underline{u}, \overline{u})$.

For the upper bound we assume that $w'(u)$ exists and is locally Lipschitz continuous for all $u \in (\underline{u}, \overline{u})$. Then the rate of convergence is $O(1 - \delta)$.

Theorem 4.1. *For all $u \in (\underline{u}, \overline{u})$ we have $\limsup_{\delta \to 1} \frac{w(u) - w_\delta(u)}{1-\delta}$ is finite.*

We prove the upper bound by showing that the mechanism $\texttt{GREEDY}_\delta(u)$ already gives a payoff that is $O(1 - \delta)$ away from $w(u)$. In fact this distance is generally on the order of $\Theta(1 - \delta)$. We next establish a corresponding lower bound of $\Omega(1 - \delta)$. We show that as $\delta \to 1$ the mechanisms $\texttt{OPT}_\delta(u)$ and $\texttt{GREEDY}_\delta(u)$ are very similar.

For the lower bound we assume that $w''(u)$ exists, and $w''(u)$ is negative and continuous. The fact that $w''(u) < 0$ is critical. It rules out cases like $u_S = u_R$, in which $w(u)$ is a line, and the rate of convergence is 0. More generally Theorem 6.4 shows that when $w''(u) = 0$, the rate of convergence is $o(1 - \delta)$, which is strictly faster than $1 - \delta$.

For the lower bound we also impose two additional assumptions. Fix $u \in (\underline{u}, \overline{u})$. First let $a^*_{\theta,x}$ denote the period-0 actions implemented by the mechanism $\texttt{GREEDY}_\delta(u)$. Let

$$z^*_{\theta,x} = \int_{\underline{\theta}}^{\overline{\theta}} u_S(a^*_{s,x}, s) dF(s) - \int_{\underline{\theta}}^{\overline{\theta}} \frac{\partial u_S(a^*_{s,x}, s)}{\partial s} \frac{1 - F(s)}{f(s)} dF(s) + \int_{\underline{\theta}}^{\theta} \frac{\partial u_S(a^*_{s,x}, s)}{\partial s} ds - u_S(a^*_{\theta,x}, \theta).$$

Assume that $\mathbb{E}_{\theta,x}[z^*_{\theta,x} | z^*_{\theta,x} > 0] > 0$ for all x. Since $\mathbb{E}_{\theta,x} z^*_{\theta,x} = 0$, this assumption ensures that z^* has a non-trivial spread.

Second let $a_{\theta,x}(k)$ denote a family of actions indexed by $k \in \mathbb{N}$. Let

$$z_{\theta,x}(k) = \int_{\underline{\theta}}^{\overline{\theta}} u_S(a_{s,x}(k), s) dF(s) - \int_{\underline{\theta}}^{\overline{\theta}} \frac{\partial u_S(a_{s,x}(k), s)}{\partial s} \frac{1 - F(s)}{f(s)} dF(s) + \int_{\underline{\theta}}^{\theta} \frac{\partial u_S(a_{s,x}(k), s)}{\partial s} ds - u_S(a_{\theta,x}(k), \theta).$$

Assume that if $\int_{\underline{x}}^{\overline{x}} \int_{\underline{\theta}}^{\overline{\theta}} u_R(a_{\theta,x}(k), \theta) dF(\theta) dx \to w(u)$ and $\int_{\underline{x}}^{\overline{x}} \int_{\underline{\theta}}^{\overline{\theta}} u_S(a_{\theta,x}(k), \theta) dF(\theta) dx \to u$ as $k \to +\infty$, then $z_{\theta,x}(k) \to z^*_{\theta,x}$ in distribution for almost all x. More precisely for a measure-one set of x we have $z_{\theta,x}(k) \to z^*_{\theta,x}$ in distribution (as random variables in terms of θ). This assumption allows us to bound the distance between the payoff from $\texttt{OPT}_\delta(u)$ and the payoff from $\texttt{GREEDY}_\delta(u)$.

Theorem 4.2. *For all $u \in (\underline{u}, \overline{u})$ we have* $\liminf_{\delta \to 1} \frac{w(u) - w_\delta(u)}{1-\delta} > 0$.

Theorem 4.1 and Theorem 4.2 together imply that the receiver's payoff from the optimal mechanism converges to $\max_u w(u)$ at a rate of $\Theta(1-\delta)$ as long as the maximizer of $w(u)$ is in the interior of $(\underline{u}, \overline{u})$. The upper bound follows directly from Theorem 4.1. The lower bound requires slightly more work. The maximizer of $w_\delta(u)$ could be different from the maximizer of $w(u)$, but they are the same in the limit.

Corollary 4.3. *Suppose there exists a $u^* \in (\underline{u}, \overline{u})$ such that $w(u^*) = \max_u w(u)$. Then $w(u^*) - \max_u w_\delta(u) = \Theta(1 - \delta)$.*

Assumptions for Theorem 4.1 are easy to check. Assumptions for Theorem 4.2 look messy, but the purpose is straightforward: the continuation payoffs $u_{\theta,x}$ in mechanism $\texttt{OPT}_\delta(u)$ must have a non-trivial spread, so that the sender could extract a positive information rent. We don't know what conditions on the primitives could ensure that the agent extracts a positive information rent. See Section 6 for further discussion.

Example 4.4. Consider the example in Section 3.3 where both the sender and the receiver have quadratic utility. Again for simplicity we focus on the interval $u \in [-b^2, 0)$. We have shown that $w_\delta(u) \to w(u)$ as $\delta \to 1$. We now check that this example satisfies the assumptions in Theorem 4.1 and Theorem 4.2. For the upper bound we have $w(u) = -(b - \sqrt{-u})^2$, so $w'(u) = 1 - \frac{b}{\sqrt{-u}}$ is locally Lipschitz continuous. Thus Theorem 4.1 applies.

Now let's check the lower bound. First off we have $w''(u) = -\frac{b}{2(-u)^{3/2}}$ is continuous, and $w''(u) < 0$ for all $u < 0$. Next we have $z^*_\theta = \sqrt{-u}(1 - 2\theta)$, so $\mathbb{E}_\theta[z^*_\theta | z^*_\theta > 0] = \sqrt{-u} \cdot \frac{1}{2}$, which is positive for all $u < 0$. Thus z_θ^* has a non-trivial spread, and the first assumption for Theorem 4.2 is satisfied. For the second assumption suppose actions $a_{\theta,x}(k)$ satisfy $\int_0^1 \int_0^1 -(a_{\theta,x}(k) - \theta - b)^2 \cdot d\theta dx \to u$ and $\int_0^1 \int_0^1 -(a_{\theta,x}(k) - \theta)^2 \cdot d\theta dx \to -(b - \sqrt{-u})^2$. Let $\epsilon_{\theta,x}(k) = a_{\theta,x}(k) - a_\theta^* = a_{\theta,x}(k) - (\theta + b - \sqrt{-u})$. We have $\int_0^1 \int_0^1 -(-\sqrt{-u} + \epsilon_{\theta,x}(k))^2 \cdot d\theta dx \to u$ and $\int_0^1 \int_0^1 -(b - \sqrt{-u} + \epsilon_{\theta,x}(k))^2 \cdot d\theta dx \to -(b - \sqrt{-u})^2$. We deduce that both $\int_0^1 \int_0^1 [2\sqrt{-u} \cdot \epsilon_{\theta,x}(k) - \epsilon_{\theta,x}(k)^2] \cdot d\theta dx$ and $\int_0^1 \int_0^1 [-2(b - \sqrt{-u}) \cdot \epsilon_{\theta,x}(k) - \epsilon_{\theta,x}(k)^2] \cdot d\theta dx$ go to 0. The first term times $b - \sqrt{-u}$ plus the second term times $\sqrt{-u}$ is equal to $\int_0^1 \int_0^1 -b \cdot \epsilon_{\theta,x}(k)^2 \cdot d\theta dx$. We obtain that $\int_0^1 \int_0^1 \epsilon_{\theta,x}(k) \cdot d\theta dx \to 0$. As a result we have $z_{\theta,x}(k) \to z_\theta^*$ in distribution. Thus Theorem 4.2 applies.

We now explain why the rate of convergence is $\Theta(1 - \delta)$. We first examine the rate of convergence from the mechanism $\texttt{GREEDY}_\delta(u)$. We showed in Section 3.3 (inequality (3.2)) that

$$w(u) - w_\delta(u) \leq \frac{\delta}{1 - \delta} \cdot \int_0^1 (w_\delta(u) - w_\delta(u_\theta^*)) d\theta.$$

We claim that the right hand side is on the order of $1 - \delta$. Second-order Taylor approximation implies that

$$w_\delta(u) - w_\delta(u_\theta^*) \approx -w_\delta'(u) \cdot (u - u_\theta^*) - \frac{w_\delta''(u)}{2} \cdot (u - u_\theta^*)^2.$$

We know from Section 3.3 that $u - u_\theta^*$ is on the order of $\frac{1-\delta}{\delta}$ and has mean 0. Hence the w_δ' term disappears after integration, and only the w_δ'' term remains. Since $u - u_\theta^*$ is on the order of $1 - \delta$, we have $(u - u_\theta^*)^2$ is on the order of $(1 - \delta)^2$. We deduce that

$$w(u) - w_\delta(u) \lesssim \frac{\delta}{1 - \delta} \cdot \int_0^1 \frac{|w_\delta''(u)|}{2} \cdot (u - u_\theta^*)^2 d\theta = O(1 - \delta). \tag{4.1}$$

Thus the upper bound is indeed on the order of $1 - \delta$. For the lower bound we have to show that as $\delta \to 1$ the mechanism $\texttt{OPT}_\delta(u)$ looks similar to the mechanism $\texttt{GREEDY}_\delta(u)$. In particular the actions implemented by $\texttt{OPT}_\delta(u)$ converge to a_θ^*, and the continuation payoffs converge to u_θ^*.

Remark. Hörner and Takahashi (2016) find that in repeated games with imperfect monitoring the equilibrium payoff sets converge to a particular payoff vector at a rate of $\sqrt{1 - \delta}$ (Proposition 4 and Proposition 7). In contrast we obtained a rate of convergence of $1 - \delta$. The key difference is that they consider discrete games, and the $\sqrt{1 - \delta}$ bound is tight at the vertices.

Notice from (4.1) that the rate of convergence is proportional to $\frac{w''(u)}{2}(1 - \delta)$. If states and actions are discrete, then $w(u)$ is piece-wise linear, which means $w''(u) = 0$ if $(u, w(u))$ lies on a line, and $w''(u) = -\infty$ if $(u, w(u))$ is a vertex. Hörner and Takahashi (2016) captured the case when $w''(u) = -\infty$, which means the rate of convergence must be slower than $1 - \delta$.

5 A Binary-State Binary-Action Model

In this section we consider a model with binary states and binary actions. The receiver wants the action to match the state, and the sender always prefers the lower action. We use this model to demonstrate that the rate of convergence in a discrete model[1] radically differs from $\Theta(1 - \delta)$, and we explain the connection between our results and those of Hörner and Takahashi (2016).

In Section 4 we derived a rate of convergence of $\Theta(1 - \delta)$. In contrast Hörner and Takahashi (2016) derived a rate of convergence $O(\sqrt{1 - \delta})$ for repeated games with imperfect monitoring (Proposition 7), and they constructed an example (Proposition 4) in which the rate of convergence is $\Omega(\sqrt{1 - \delta})$. Their model differs from ours in two ways. First they consider discrete games, so the Pareto-frontier is piece-wise linear, whereas in our baseline model the limit $w(u)$ is smooth and non-linear. Second their construction of the lower bound focuses on a vertex, and the rate of convergence could be faster for non-vertex points. We illustrate these differences through a binary-state binary-action model. We show that the rate of convergence is $\Theta(\sqrt{1 - \delta})$ for the vertex (Theorem 5.4), and is strictly faster than $1 - \delta$ for all other points (Theorem 5.3).

5.1 The model and the main idea

We define the following binary-state binary-action model (a variant of Guo and Hörner (2018)). The state of the world is either $l = 0$ or $h = 1$ with equal probability. An action is also $l = 0$ or $h = 1$. The receiver wants to match the state; the sender always wants the lower action. The receiver's utility is $-(a - \theta)^2$, and the sender's utility is $-a$. The receiver's maximal payoff under the constraint that the sender gets an expected payoff of u is equal to

$$w(u) = \min\{u + \frac{1}{2}, -u - \frac{1}{2}\}.$$

[1]Another difference between this section and the baseline model in Section 3 is that we now assume the sender's preference is state-independent, so the single crossing condition is no longer satisfied. We show in Example 6.6 that state-independent preference under a continuous state of the world still yields a rate of convergence of $\Theta(1 - \delta)$.

Notice that $w(u)$ is piece-wise linear with a peak at $u = -\frac{1}{2}$. We have $w''(u) = 0$ for all $u \neq -\frac{1}{2}$ and $w''(u) = -\infty$ for $u = -\frac{1}{2}$. Our argument in Section 4 suggests that the rate of convergence is proportional to $\frac{w''(u)}{2}(1-\delta)$, so we should expect the rate of convergence to be strictly faster than $1-\delta$ for all $u \neq -\frac{1}{2}$ and strictly slower than $1-\delta$ for $u = -\frac{1}{2}$.

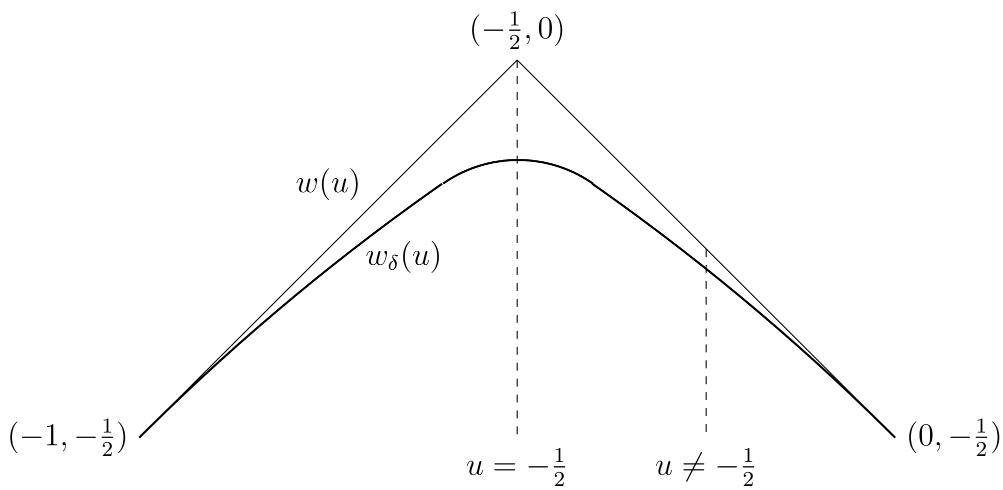

Figure 1: convergence is fast for $u \neq -\frac{1}{2}$ and slow for $u = -\frac{1}{2}$

Figure 1 illustrates why the rate of convergence is fast for $u \neq -\frac{1}{2}$ but slow for $u = -\frac{1}{2}$. In Figure 1 the piece-wise linear function is $w(u)$, and the curve below it represents $w_\delta(u)$. For $u > -\frac{1}{2}$ we have $w_\delta(u)$ has a slope close to -1, and for $u < -\frac{1}{2}$ the slope of $w_\delta(u)$ is close to 1. However because $w(u)$ has a kink at $(-\frac{1}{2}, 0)$, the slope of $w_\delta(u)$ changes sharply as u approaches $-\frac{1}{2}$ (from ± 1 to 0). Hence the distance between $w_\delta(-\frac{1}{2})$ and $w(-\frac{1}{2})$ is much larger than the distance between $w_\delta(u)$ and $w(u)$ for $u \neq -\frac{1}{2}$.

5.2 The Bellman equation

For consistency with the baseline model we focus on deterministic mechanisms and assume that public randomization occurs before the sender makes a report. Recall that $w_\delta(u)$ is the receiver's maximal payoff when the discount factor is δ and the sender's expected payoff is u.

Without writing down the Bellman equation we know that $w_\delta(u)$ is symmetric around $-\frac{1}{2}$. Indeed consider the mechanism that gives the sender an expected utility u; let $a(h_t)$ denote the receiver's action at history h_t. We construct another mechanism as follows: if $\theta_t = l$, then take the opposite action as $a(h_{t-1}, h)$, and if $\theta_t = h$, then take the opposite action as $a(h_{t-1}, l)$. This mechanism is incentive compatible and gives the sender a payoff

$-1-u$, and the receiver gets the same payoff as in the mechanism for u. Hence $w_\delta(u)$ is symmetric about $-\frac{1}{2}$.

We now write down the Bellman equation as follows:

$$w_\delta(u) = \max_{a_{l,x},a_{h,x},u_{l,x},u_{h,x}} \int_{\underline{x}}^{\overline{x}} [(1-\delta) \cdot \tfrac{-(a_{l,x}-0)^2-(a_{h,x}-1)^2}{2} + \delta \cdot \tfrac{w_\delta(u_{l,x})+w_\delta(u_{h,x})}{2}] \cdot dx$$

$$s.t.$$

$$a_{l,x}, a_{h,x} \in \{0,1\}; u_{l,x}, u_{h,x} \in [-1,0]$$

$$\text{IC}h \qquad (1-\delta)(-a_{h,x}) + \delta u_{h,x} \geq (1-\delta)(-a_{l,x}) + \delta u_{l,x}$$

$$\text{IC}l \qquad (1-\delta)(-a_{l,x}) + \delta u_{l,x} \geq (1-\delta)(-a_{h,x}) + \delta u_{h,x}$$

$$\int_{\underline{x}}^{\overline{x}} [(1-\delta)\tfrac{-a_{l,x}-a_{h,x}}{2} + \delta \tfrac{u_{l,x}+u_{h,x}}{2}] \cdot dx = u$$

Since $w_\delta(u)$ is concave and is symmetric about $-\frac{1}{2}$, for all δ the optimal mechanism gives the sender an expected payoff of $-\frac{1}{2}$.

Proposition 5.1. *For all $u \in [-\delta, \delta-1]$ the optimal choice of actions are $a_{l,x} = 0$ and $a_{h,x} = 1$ for all x. Consequently for all $u \in [-\delta, \delta-1]$ we have*

$$w_\delta(u) = \frac{\delta}{2} \cdot w_\delta(\frac{u}{2}) + \frac{\delta}{2} \cdot w_\delta(\frac{u+1-\delta}{\delta}).$$

We briefly explain how we got the above equation. Since the sender's preference is state-independent, the sender must be indifferent between reporting l or h. Indeed the ICh and ICl are the same inequality with opposite signs. We deduce that $(1-\delta)(-a_{h,x}) + \delta u_{h,x} = (1-\delta)(-a_{l,x}) + \delta u_{l,x}$. As a result for both $\theta = l$ and $\theta = h$ we have $\int_{\underline{x}}^{\overline{x}} [(1-\delta)(1-a_{\theta,x}) + \delta u_{\theta,x}] dx = u$. Thus if we set $a_{\theta,x} = \theta$ for all x, then we have $u_{l,x} = \frac{u}{\delta}$ and $u_{h,x} = \frac{u+1-\delta}{\delta}$ for all x.

The optimal mechanism starts with $u = -\frac{1}{2}$. As long as $u \in [-\delta, \delta-1]$ the receiver takes the action that matches the sender's report. If the sender reports l, the receiver takes action l, and the sender's continuation payoff becomes $\frac{u}{\delta}$. If the sender reports h, the receiver takes action h, and the sender's continuation payoff becomes $\frac{u+1-\delta}{\delta}$. The receiver then follows the optimal mechanism that gives the sender a payoff $\frac{u}{\delta}$ or $\frac{u+1-\delta}{\delta}$. We skip the depiction of the mechanism for $u < -\delta$ or $u > \delta - 1$ because these fringe cases are irrelevant to our discussion of the limit of w_δ and the rate of convergence.

As in the continuous-state model w_δ converges to w.

Proposition 5.2. *For all $u \in [-1,0]$ we have $\lim_{\delta\to1} w_\delta(u) = w(u) = \min\{u+\frac{1}{2}, -u-\frac{1}{2}\}$.*

5.3 Rate of convergence

Since the optimal mechanism gives the sender an expected payoff $-\frac{1}{2}$, we should focus on how fast $w_\delta(-\frac{1}{2})$ converges to $w(-\frac{1}{2}) = 0$. However as a technical point of interest we first examine the rate of convergence for $u \neq -\frac{1}{2}$.

Theorem 5.3. *For all $u \neq -\frac{1}{2}$ we have*

$$\lim_{\delta \to 1} \frac{w(u) - w_\delta(u)}{1 - \delta} = 0.$$

The proof of Theorem 5.3 follows a similar argument given in Section 4. Taylor-approximation suggests that the rate of convergence is on the order of $\frac{|w''(u)|}{2}(1-\delta)$. In this case $w(u)$ is linear for $u \neq -\frac{1}{2}$, which means $w''(u) = 0$, so we should expect a rate of convergence strictly faster than $1 - \delta$.

We next study the rate of convergence for $u = -\frac{1}{2}$. Since $w(u)$ has a kink at this point, we have $w''(-\frac{1}{2}) = -\infty$, so we should expect the rate of convergence to be strictly slower than $1 - \delta$. We show that the rate of convergence is in fact $\Theta(\sqrt{1-\delta})$, which is consistent with the finding of Hörner and Takahashi (2016). The next theorem establishes this result and gives a bound for the constants.

Theorem 5.4. *For all $\delta > 0.695$ we have $w_\delta(-\frac{1}{2})$ is between $-1.21582\sqrt{1-\delta}$ and $-0.107653\sqrt{1-\delta}$.*

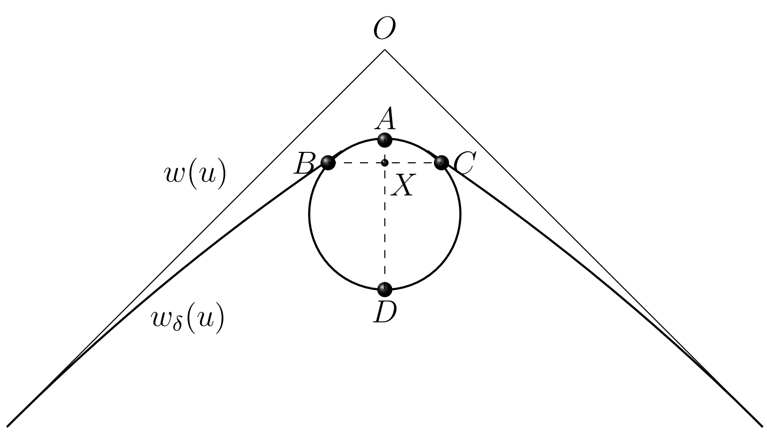

Figure 2: the power of a point argument

Figure 2 gives an intuitive argument for why the rate of convergence is $\Theta(\sqrt{1-\delta})$. The piece-wise linear function is $w(u)$, and the solid curve below it is $w_\delta(u)$. Let O denote the point $(-\frac{1}{2}, 0)$. We label the following three points on $w_\delta(u)$. Let A denote the point $(-\frac{1}{2}, w_\delta(-\frac{1}{2}))$. Let B denote the point $(-\frac{1}{2} - \frac{1-\delta}{2\delta}, w_\delta(-\frac{1}{2} - \frac{1-\delta}{2\delta}))$. Let C denote the point $(-\frac{1}{2} + \frac{1-\delta}{2\delta}, w_\delta(-\frac{1}{2} + \frac{1-\delta}{2\delta}))$. Let X denote the midpoint of BC. The Bellman equation from Proposition 5.1 implies that $AX = w_\delta(-\frac{1}{2}) - \frac{w_\delta(-\frac{1}{2} - \frac{1-\delta}{2\delta}) + w_\delta(-\frac{1}{2} + \frac{1-\delta}{2\delta})}{2} = \frac{1-\delta}{\delta} \cdot |w_\delta(-\frac{1}{2})| = \frac{1-\delta}{\delta} \cdot AO$ (for $u = -\frac{1}{2}$ we have $\frac{u}{\delta} = -\frac{1}{2} - \frac{1-\delta}{2\delta}$ and $\frac{u+1-\delta}{\delta} = -\frac{1}{2} + \frac{1-\delta}{2\delta}$). We deduce that

$$AX = \frac{1-\delta}{\delta} AO.$$

Now consider the circumcircle of triangle ABC. Let D denote the intersection between AX and the circumcircle. The Power of a Point Theorem states that

$$AX \cdot DX = BX \cdot CX.$$

We know that BX and CX are both on the order of $1 - \delta$ because $BX = CX = \frac{1-\delta}{2\delta}$. We also know that AX is equal to $(1 - \delta) \cdot AO$, while DX is on the order of AO. Hence $BX \cdot CX$ is on the order of $(1 - \delta)^2$, and $AX \cdot DX$ is on the order of $(1 - \delta) \cdot AO^2$. We deduce that AO is on the order of $\sqrt{1 - \delta}$.

Note that this circle method gives a rate of convergence of $\sqrt{1 - \delta}$ only if point O is a kink. If point O is on a line, then we no longer have XD on the order of AO because points B and C (on average) converge to $w(u)$ slower than point A to point O. In fact it follows from Theorem 5.3 that if O is on a line, then the ratio of XD to AO is unbounded because the rate of convergence is $o(1 - \delta)$.

6 Arbitrary states, actions, preferences

We now allow the state space Θ to be any compact subset of $\mathbb{R}^N$ and the action space A any compact subset of $\mathbb{R}^M$ for some positive integers N and M. Preferences u_R and u_S are bounded and continuous functions from $\mathbb{R}^N \times \mathbb{R}^M$ to $\mathbb{R}$. This general model includes both the continuous and discrete models we considered in previous sections. We focus on deterministic mechanisms as before. We characterize the receiver's payoff in the limit and provide a unified view of the rate of convergence results.

We have to rewrite the Bellman equation to account for multi-dimensional states. As before the domain of the Bellman equation is between $\underline{u} = \min_a \mathbb{E}_\theta u_S(a, \theta)$ and $\overline{u} = \mathbb{E}_\theta \max_a u_S(a, \theta)$. The sender could always secure a payoff of $\underline{u}$ by babbling and cannot achieve more than $\overline{u}$ because the receiver is already taking the sender's most preferred actions. The Bellman equation is as follows:

$$w_\delta(u) = \max_{a_{\theta,x} \in A, u_{\theta,x} \in [\underline{u}, \overline{u}]} \mathbb{E}_{\theta,x}[(1-\delta) \cdot u_R(a_{\theta,x}, \theta) + \delta \cdot w_\delta(u_{\theta,x})]$$
$$s.t.$$
$$\forall x, \theta \qquad \theta \in \text{argmax}_{\theta'} (1-\delta) \cdot u_S(a_{\theta',x}, \theta) + \delta \cdot u_{\theta',x}$$
$$\mathbb{E}_{\theta,x}[(1-\delta) \cdot u_S(a_{\theta,x}, \theta) + \delta \cdot u_{\theta,x}] = u$$

We also have to redefine the upper bound $w(u)$. For multi-dimensional states and actions the implementability condition is characterized by the cyclic monotonicity condition in Rochet (1987). In particular we define

$$w(u) = \max_{a_{\theta,x}} \mathbb{E}_{\theta,x} u_R(a_{\theta,x}, \theta)$$
$$s.t.$$
$$a \text{ satisfies CMON}$$
$$\mathbb{E}_{\theta,x} u_S(a_{\theta,x}, \theta) = u$$

We have CMON stands for cyclic monotonicity: CMON holds if whenever $\theta_l = \theta_0$ we have

$$\sum_{k=0}^{l-1} d(\theta_k, \theta_{k+1}) \leq 0,$$

where

$$d(\theta', \theta) = u(a_{\theta',x}, \theta) - u(a_{\theta',x}, \theta').$$

By Rochet (1987) CMON is the necessary and sufficient condition for implementability. In other words there exists $t_{\theta,x}$ such that $\theta \in \text{argmax}_{\theta'} u(a_{\theta',x}, \theta) + t_{\theta',x}$ if and only if $a_{\theta,x}$ satisfies CMON. If θ is one-dimensional and u_S satisfies the single crossing condition, then CMON reduces to the monotonicity condition that $a_{\theta,x}$ is non-decreasing in θ.

Theorem 6.1. *For all $u \in (\underline{u}, \overline{u})$ we have* $\lim_{\delta \to 1} w_\delta(u) = w(u)$.

The proof of Theorem 6.1 follows the same strategy as the argument we gave in Section 3.3. We construct a mechanism $\texttt{GREEDY}_\delta(u)$ that implements the actions $a^*_{\theta,x}$ (solution to the optimization problem for $w(u)$) in period 0. For the continuation values we look at the $t_{\theta,x}$ such that $\theta \in \text{argmax}_{\theta'} u(a^*_{\theta',x}, \theta) + t_{\theta',x}$. Normalize the $t_{\theta,x}$ such that $\mathbb{E}_\theta t_{\theta,x} = 0$ for all x. Then set

$$u^*_{\theta,x} = u + \frac{1-\delta}{\delta} t_{\theta,x}. \tag{6.1}$$

In the next period use the mechanism $\texttt{OPT}_\delta(u^*_{\theta,x})$; i.e., the mechanism that gives the sender a payoff of $u^*_{\theta,x}$ and the receiver a payoff of $w_\delta(u^*_{\theta,x})$. The mechanism $\texttt{GREEDY}_\delta(u)$ gives the receiver an expected payoff of $(1-\delta)\cdot w(u) + \delta \cdot \mathbb{E}_{\theta,x} w_\delta(u^*_{\theta,x})$, and in the limit this payoff converges to $w(u)$.

Theorem 6.1 implies that the receiver could achieve her first-best payoff if and only if the first-best actions satisfy the cyclic monotonicity condition.

Corollary 6.2. *The receiver's payoff from the optimal mechanism converges to her first-best payoff if and only if there exist actions a_θ such that (i) a_θ satisfies CMON, and (ii) $a_\theta \in \operatorname{argmax}_a u_R(a,\theta)$ for all θ.*

A special case of Corollary 6.2 is when sender's preference is state-independent. In this case any set of actions $a_{\theta,x}$ would satisfy the cyclic monotonicity condition, and the receiver could attain her first-best payoff in the limit. Indeed the receiver could pick $u_{\theta,x}$ such that $(1-\delta)\cdot u_S(a_{\theta,x}) + \delta \cdot u_{\theta,x} = u$, where $u_S(a)$ is a short-hand for $u_S(a,\theta)$ when sender's preference is state-independent. The sender is indifferent between reporting any θ, so for any set of actions $a_{\theta,x}$ the receiver could find continuation payoffs $u_{\theta,x}$ such that it is incentive compatible for the sender to report the truth.

Corollary 6.3. *If $u_S(a,\theta)$ is independent of θ, then the receiver's payoff in the optimal mechanism approaches her first-best payoff as $\delta \to 1$.*

We now discuss the rate of convergence. We have three sets of results for the upper bound.

Theorem 6.4. *Fix $u \in (\underline{u}, \overline{u})$.*

- *We have $\limsup_{\delta\to 1} \frac{w(u)-w_\delta(u)}{\sqrt{1-\delta}}$ is finite.*
- *If there is a neighborhood of u for which w' exists for all points and is locally Lipschitz continuous, then $\limsup_{\delta\to 1} \frac{w(u)-w_\delta(u)}{1-\delta}$ is finite.*
- *If $w''(u) = 0$, then $\lim_{\delta\to 1} \frac{w(u)-w_\delta(u)}{1-\delta} = 0$.*

Roughly speaking if $w(u)$ is locally linear, the rate of convergence is $o(1-\delta)$. If $w(u)$ is smooth but non-linear, then the rate is $O(1-\delta)$. If $w(u)$ has a kink at u, the rate is $O(\sqrt{1-\delta})$. These results provide a unified view of the upper bounds in Theorem 4.1, Theorem 5.3, and Theorem 5.4.

For the lower bound we don't even know when the rate of convergence is 0. For example if the sender and the receiver have exactly the same preference, or if their preferences are affine transformations of each other, then the rate of convergence is 0. The main hurdle for generalizing Theorem 4.2 is Lemma B.1, which ensures that $u_{\theta,x} - u$ are non-zero. In general for the $\Omega(1-\delta)$ bound we need the following condition:

Condition X: Fix $u \in (\underline{u}, \overline{u})$. There exists a $C > 0$ and a $\Delta > 0$ such that if

$$\mathbb{E}_{\theta,x} u_R(a_{\theta,x}, \theta) \in [w(u) - \Delta, w(u) + \Delta]$$
$$\mathbb{E}_{\theta,x} u_S(a_{\theta,x}, \theta) \in [u - \Delta, u + \Delta]$$
$$\theta \in \text{argmax}_{\theta'} u_S(a_{\theta',x}, \theta) + t_{\theta',x} \ \forall x, \theta, \text{ where } \mathbb{E}_{\theta,x} t_{\theta,x} = 0,$$

then

$$\mathbb{E}_{\theta,x}[t_{\theta,x} | t_{\theta,x} > 0] > C \text{ and } \mathbb{E}_{\theta,x}[t_{\theta,x} | t_{\theta,x} < 0] < -C.$$

For Theorem 4.2 we made two cumbersome assumptions: (i) the statement is true for $\Delta = 0$, and (ii) the $t_{\theta,x}$ converges as $\Delta \to 0$. We don't know how to reduce Condition X to assumptions on the primitives.

Proposition 6.5. *Fix $u \in (\underline{u}, \overline{u})$. Suppose Condition X holds for u. Suppose there is a neighborhood of u for which w'' exists, is negative, and is continuous for all points in this neighborhood. Then* $\liminf_{\delta \to 1} \frac{w(u) - w_\delta(u)}{1-\delta} > 0$.

Example 6.6. The state of the world is $\theta \sim U[0,1]$. The action space is $[-\frac{1}{2}, \frac{3}{2}]$. We have $u_R(a,\theta) = -(a-\theta)^2$ and $u_S(a,\theta) = -a$. In other words the receiver wants the action to match the state, but the sender prefers lower actions. We have $w(u) = -(u + \frac{1}{2})^2$ for all $u \in [-1, 0]$. In particular we have $w(-\frac{1}{2}) = 0$ is the receiver's limit payoff from the optimal mechanism.

Theorem 6.4 implies that rate of convergence for $u = -\frac{1}{2}$ is $O(1-\delta)$. In this example we also have $w(-\frac{1}{2}) - w_\delta(-\frac{1}{2}) = \Omega(1-\delta)$. We check that Proposition 6.5 applies to this setting. First notice that $w''(u) = -2$ for all $u \in [-1, 0]$. Next we check Condition X. Since the sender's preference is state-independent, we need $u_S(a_{\theta,x}, \theta) + t_{\theta,x} = -a_{\theta,x} + t_{\theta,x}$ the same for all θ. Let μ denote $\mathbb{E}[a_{\theta,x}]$. We deduce that $t_{\theta,x} = a_{\theta,x} - \mu$. Suppose $\mathbb{E}[-a_{\theta,x}] \in [-\frac{1}{2} - \Delta, -\frac{1}{2} + \Delta]$ and $\mathbb{E}[-(a_{\theta,x} - \theta)^2] \in [-\Delta, \Delta]$. If Δ is close to 0, we have $\mathbb{E}[(a_{\theta,x} - \mu) | a_\theta - \mu > 0]$ is close to $\mathbb{E}[\theta - \frac{1}{2} | \theta \geq \frac{1}{2}] = \frac{1}{4}$, which is positive. As a result $t_{\theta,x}$ has a non-trivial spread, and we indeed have $w(-\frac{1}{2}) - w_\delta(-\frac{1}{2}) = \Theta(1-\delta)$.

7 Applications and Extensions

We begin with applications that directly follow from our model. We then discuss applications that are variants or extensions of our model.

7.1 Direct applications

The first example is a literal communication problem. The sender is a biased adviser, and the receiver is a lawmaker. The state θ is the optimal policy. The sender prefers policy $\theta + b$, and the receiver prefers policy θ. The sender's utility is $u_S(a, \theta) = -(a - \theta - b)^2$, and the receiver's utility is $u_R(a, \theta) = -(a - \theta)^2$. Since u_S satisfies the single crossing condition, and the receiver's first-best actions $a^*_\theta = \theta$ are increasing in θ, the lawmaker could achieve her first-best payoff as $\delta \to 1$.

The second example is a communication problem where sender's preference is state-independent. For example the sender is a salesman who always wants the consumer to purchase the good. Or the sender is a doctor who always wants to prescribe the most expensive medicine. Or the sender is a car mechanic who always wants the most expensive repair. Corollary 6.3 implies that in these cases the consumer, the patient, or the motorist could attain their first-best payoff as $\delta \to 1$. For more applications of a state-independent sender preference see Lipnowski and Ravid (2017) and Chen (2017).

Another example is insurance with hidden income (Thomas and Worrall (1990)). The sender is a borrower; the receiver is a lender. The state θ is the sender's hidden income in each period, and the action a corresponds to the borrower's transfer to the lender. The borrower's utility is $u_S(a, \theta) = u(-a + \theta)$, and the lender's utility is $u_R(a, \theta) = a$. Suppose u is differentiable and strictly concave. Then $\frac{\partial u_S}{\partial \theta} = u'(-a + \theta)$ is strictly increasing in a. The efficient contract gives a constant consumption, so $-a + \theta = \mathbb{E}[\theta]$. Since the actions $a_\theta = \theta + \mathbb{E}[\theta]$ are non-decreasing in θ, the efficient contract can be implemented in the limit as $\delta \to 1$, as noted in Thomas and Worrall (1990).

Note that the first two examples do not involve transfers, while the third example treats the receiver's action as a transfer. For the setting without transfers the i.i.d. case of Guo and Hörner (2018) is a special case of our model, and we refer to their paper for more applications of dynamic allocation without money.

7.2 Dynamic CEO compensation

Consider a CEO compensation model studied in Edmans and Gabaix (2011) and Garrett and Pavan (2015). In each period the agent observes a shock θ_t before exerting an effort e_t at cost $c(e_t)$. Assume that $\theta_t \in F[\underline{\theta}, \overline{\theta}]$ with $\mathbb{E}[\theta_t] = 0$. Assume that $e_t \in \mathbb{R}$;[2] also assume that c is differentiable and strictly convex with $c(0) = 0$ and $c'(0) = 0$. The principal then observes an output $y_t = \theta_t + e_t$ and rewards the agent w_t. The principal has limited liability, which means $w_t \geq 0$. Both the principal and the agent are risk-neutral. In summary the time-line in period t is as follows:

1. The agent observes θ_t.

2. The agent exerts effort e_t.

3. The principal observes $y_t = \theta_t + e_t$ and pays the agent w_t.

4. The agent gets payoff $w_t - c(e_t)$, and the principal gets payoff $y_t - w_t$.

Garrett and Pavan (2015) derived the optimal contract for a two-period model. Edmans and Gabaix (2011) studied the multi-period model, but assumed that the principal wants to implement a specific effort level rather than solving for the optimal contract. We are interested in the performance of the optimal contract with infinitely many periods.

We formulate this optimal contract problem in terms of repeated communication. The state of the world is the shock θ. An action is a two-dimensional vector (y, w). In period t the sender (agent) observes θ_t. Then the receiver (principal) specifies an output y_t and pays the agent w_t. The sender exerts effort $y_t - \theta_t$ and gets payoff $w_t - c(y_t - \theta_t)$, and the receiver gets payoff $y_t - w_t$. This formulation fits into our general model in Section 6.

Note that we now assume the receiver chooses output y_t rather than the sender chooses an effort level. The assumption is innocuous. Indeed in each period the receiver specifies an output y_t, which forces the sender to choose effort $y_t - \theta_t$. If the sender chooses a different effort level, then the receiver observes an output different from y_t, and the receiver could punish the sender by setting $w = 0$ forever.

Let e^* denote the maximizer of $e - c(e)$; in other words $e^* = c'^{-1}(1)$. Assume that the domain of y includes the interval $[e^* + \underline{\theta}, e^* + \overline{\theta}]$. Assume the domain of w is $[0, \overline{w}]$ where $\overline{w} > e^*$. These assumptions ensure that the receiver could implement the first-best effort level e^*.

[2] We allow efforts to be negative, which simplifies our analysis. For the case of non-negative efforts see the analysis in Carroll and Meng (2016) who studied a static version of this problem.

From Theorem 6.1 we need to figure out which actions are implementable in a static mechanism design problem with transfers. The incentive constraint now becomes $\theta \in \text{argmax}_{\theta'}(1-\delta)\cdot(w_{\theta'} - c(y_{\theta'}-\theta)) + \delta \cdot u_{\theta'}$. Since c is convex, the necessary and sufficient condition for implementability is that y_θ is non-decreasing in θ. Therefore the receiver's payoff in the limit is

$$\begin{aligned} w(u) = \quad & \max_{y_\theta, w_\theta} \mathbb{E}_\theta[y_\theta - w_\theta] \\ & s.t. \\ & y_\theta \text{ is non-decreasing in } \theta \\ & \mathbb{E}_\theta[w_\theta - c(y_\theta - \theta)] = u \end{aligned}$$

The solution to this problem is $y_\theta^* = \theta + e^*$ and $w_\theta^* = u + c(e^*)$. Consequently we have

$$w(u) = e^* - c(e^*) - u.$$

From Theorem 6.1 for any $u \in (\underline{u}, \overline{u})$ the receiver could attain $w(u)$ in the limit of $\delta \to 1$. Notice that $e^* - c(e^*)$ is the full surplus, so the receiver wants to set $u = 0$. However in this case we have $\underline{u} = 0$ because of limited liability. For arbitrarily small $\epsilon > 0$ there is a mechanism $\texttt{GREEDY}_\delta(\epsilon)$ for which the receiver could attain $e^* - c(e^*) - \epsilon$ in the limit. Taking $\epsilon \to 0$ we know that the receiver could extract the full surplus $e^* - c(e^*)$ as $\delta \to 1$.

Proposition 7.1. *The principal's payoff from the optimal contract converges to $e^* - c(e^*)$ as $\delta \to 1$.*

We do not have a closed-formed solution for the optimal contract. However we know that in the mechanism $\texttt{GREEDY}_\delta(u)$ the first period outputs are $y_\theta^* = e^* + \theta$, so the first period effort e^* is independent of θ. Edmans and Gabaix (2011) assume that in every period the principal wants to implement a target effort level independent of θ, and such a contract is typically not optimal, although the first period does resemble the greedy mechanism we constructed.

For the rate of convergence we know from Theorem 6.4 that for arbitrarily small $\epsilon > 0$ the rate of convergence to $e^* - c(e^*) - \epsilon$ is $o(1-\delta)$. However for the limit $e^* - c(e^*)$ we cannot immediately apply Theorem 6.4 because $u = \underline{u} = 0$ (Theorem 6.4 requires that u is in the interior of $(\underline{u}, \overline{u})$). The next proposition shows that the rate of convergence to the first-best $e^* - c(e^*)$ is $O(\sqrt{1-\delta})$.

Proposition 7.2. *We have $w(\underline{u}) - (e^* - c(e^*)) = O(\sqrt{1-\delta})$.*

7.3 Dynamic delegation

In our communication model the sender makes a report, and the receiver takes an action. In a delegation problem a principal chooses a set of permissible actions, and an agent chooses an action from this set. Alonso and Matouschek (2008) shows that this problem is equivalent to the sender-receiver game where the receiver commits to her actions. We show that this equivalence also holds for our setting.

In a delegation problem events in period t are as follows:

1. The agent observes the state θ_t.
2. The public randomization x_t is realized.
3. The principal delegates a set of actions $D_t \subseteq A$.
4. The agent chooses an action $a_t \in D_t$.

The principal commits to the choice of D_t, which is a function of $x_0, x_1, \ldots, x_t, a_0, a_1, \ldots, a_{t-1}$. The question is whether the principal in this delegation model could get the same payoff as the receiver in the communication model. More precisely let $w_\delta^D(u)$ denote the principal's maximal possible payoff in the delegation model when the agent gets an expected utility of u. Recall that $w_\delta(u)$ is the receiver's maximal possible payoff in the communication model when the sender gets an expected utility of u. Is it true that $w_\delta^D(u) = w_\delta(u)$? The answer is yes.

Proposition 7.3. *We have $w_\delta^D(u) = w_\delta(u)$ for all δ and u. In other words the delegation model and the communication model have the same payoff sets for all $\delta < 1$.*

Proof. Consider any incentive compatible mechanism in the communication model. Suppose at history h_t the public signal is x; the sender reports θ, and the receiver takes action $a_{\theta,x}$ and promises the sender a continuation payoff $u_{\theta,x}$. We claim that there exists a well-defined mapping from $a_{\theta,x}$ to $u_{\theta,x}$. In particular if $a_{\theta,x} = a_{\theta',x}$, then we must have $u_{\theta,x} = u_{\theta',x}$. Indeed suppose on the contrary $u_{\theta,x} < u_{\theta',x}$, then the mechanism is not incentive compatible because when the state is θ, the sender would rather report θ'. Hence there exists a function $f : A \to \mathbb{R}$ such that $f(a_{\theta,x}) = u_{\theta,x}$ for all θ.

The principal in the delegation model could delegate to the agent the set of actions $D_x = \{a_{\theta,x} : \theta \in \Theta\}$. If the agent picks action a, the principal gives the agent a continuation payoff $f(a)$. Then the agent who observes θ would choose action $a_{\theta,x}$ and gets a continuation payoff $u_{\theta,x}$. Thus the agent's actions in the two models are exactly the same. □

Example 7.4. Suppose in the communication model the receiver has utility $u_R = -(a-\theta)^2$, and sender has utility $u_S = -(a-\theta-b)^2$, where $\theta \sim U[0,1]$ and $a \in [0, 1+b]$. For all $u \in [-b^2, 0)$ the mechanism $\texttt{GREEDY}_\delta(u)$ in the first period implements actions $a^*_\theta = \theta + b - \sqrt{-u}$ and gives continuation payoffs $u^*_\theta = u - \frac{1-\delta}{\delta} \cdot \sqrt{-u}(2\theta - 1)$ (see Section 3.3).

In the delegation model in the first period the principal delegates the set $[b - \sqrt{-u}, b + 1 - \sqrt{-u}]$. If the agent takes action a, the continuation payoff is $u - \frac{1-\delta}{\delta} \cdot \sqrt{-u} \cdot (2(a - b + \sqrt{-u}) - 1)$. Now the agent's problem is

$$\max_a -(1-\delta) \cdot (a - \theta - b)^2 - (1-\delta) \cdot \sqrt{-u} \cdot (2(a - b + \sqrt{-u}) - 1).$$

The first-order condition yields that $a_\theta = \theta + b - \sqrt{-u}$, which is consistent with the actions implemented by the mechanism $\texttt{GREEDY}_\delta(u)$. By Theorem 3.6 the receiver's payoff from mechanism $\texttt{GREEDY}_\delta(-b^2)$ converges to 0. Therefore in the delegation model the principal could attain the first-best payoff of 0 as $\delta \to 1$.

We next extend Example 7.4 to a setting of delegation to an adviser with unknown bias. The principal knows that the agent has utility $-(a-\theta-b)^2$ for some $b \in [0, \bar{b}]$, but doesn't know the exact value of b. For example a CEO knows that a division manager prefers more ambitious projects, does not know the precise ambition of the manager. We show that the CEO could still achieve her first-best in the limit.

Example 7.5. Suppose the principal has utility $-(a-\theta)^2$, and the agent has utility $-(a - \theta - b)^2$. The principal knows that $b \in [0, \bar{b}]$ for some finite $\bar{b} > 0$, but doesn't know the actual value of b. Example 7.4 shows that if the principal knows b, she could attain her first-best. Could the principal still attain her first-best when delegating to an agent with unknown bias?

We use "type b" to denote the agent with utility indexed by b. We know from Theorem 3.6 that there exists a greedy mechanism for which the principal could attain payoff 0 from type $\bar{b}$ in the limit of $\delta \to 1$; i.e., mechanism $\texttt{GREEDY}_\delta(-\bar{b}^2)$ for type $\bar{b}$. This mechanism promises type $\bar{b}$ agent a payoff of $-\bar{b}^2$ and gives the principal a payoff of $-\epsilon$, where ϵ goes to 0 as δ goes to 1. We claim that the same mechanism gives the principal at least $-\epsilon$ from all other types.

Proposition 7.6. *Suppose the principal gets $-\epsilon$ when type $\bar{b}$ participates in the mechanism $\texttt{GREEDY}_\delta(-\bar{b}^2)$ for type $\bar{b}$. Then the principal gets at least $-\epsilon$ when type $b \in [0, \bar{b}]$ participates in this mechanism.*

It follows from Proposition 7.6 that the principal attains the first-best no matter what the distribution of b is. Furthermore the rate of convergence is still $\Theta(1-\delta)$ because the principal's payoff from type b is at least her payoff from type $\bar{b}$, and the greedy mechanism has a rate of convergence of $O(1-\delta)$.

We defer the proof to the appendix. Intuitively type $\bar{b}$ prefers action $\theta + \bar{b}$; type b prefers action $\theta + b$, and the principal prefers action θ. Type $\bar{b}$ has the incentive to report higher states, and the mechanism for $\bar{b}$ guards against this incentive to misreport. Type b also wants to report higher states, but not as much as type $\bar{b}$, and if the mechanism for $\bar{b}$ is stringent enough to make $\bar{b}$ report truthfully, it could also guard against a less biased agent of type b.

7.4 Reputation with long-run players

In many applications the sender's preference is unknown to the receiver. Examples include Sobel (1985) where the sender is either a government spy or a foreign spy. In Morris (2001) the sender (adviser) is either a racist or a non-racist. In Ely and Välimäki (2003) the sender (mechanic) either wants to fix the real problem of the car or always wants the most expensive repair.

In this section we assume that the sender has two possible types: with probability p the sender has utility $u_S = u_R$, and with probability $1-p$ the sender has utility $u_S = u'_S \neq u_R$. We first prove a lower bound for the receiver's payoff. We then show in a binary-state binary-action example for which the lower bound is tight.

For the lower bound we consider the general model in Section 6. The receiver's payoff from the sender with unknown type is at least her payoff from the sender with type $u'_S \neq u_R$.

Proposition 7.7. *Let M denote any mechanism. Let w denote the receiver's payoff when type $u'_S \neq u_R$ participates in the mechanism M. If type $u_S = u_R$ participates in the mechanism M, the receiver's payoff is at least w.*

Proof. Suppose type $u_S = u_R$ follows exactly the same reporting strategy as type $u'_S \neq u_R$. The receiver's payoff is w, so the sender's payoff is also w because $u_S = u_R$. Hence if type $u_S = u_R$ uses a different reporting strategy from type $u'_S \neq u_R$, then type $u_S = u_R$ must get at least w, which means the receiver also gets at least w. □

It follows from Proposition 7.7 that if the receiver could attain the first-best payoff from type $u'_S \neq u_R$, then the receiver could also attain the first-best payoff from this sender with unknown preference. Moreover the rate of convergence is as fast as the rate for the $u_S \neq u_R$ type sender.

Example 7.8. Consider the binary-state binary-action model of Section 5. The receiver wants to match the state. Suppose with probability p the sender also wants to match the state, but with probability $1-p$ the sender always wants the lower action. Proposition 5.2 implies that receiver attains her first-best payoff in the limit from the sender who wants the lower action. We deduce from Proposition 7.7 that the same mechanism gives the receiver her first-best payoff in the limit from this sender whose type is unknown. This observation is consistent with Ely and Välimäki (2003) Theorem 4 where the bad reputation problem vanishes when both the mechanic and the motorist are long-run players.

We next provide an example that attains the lower bound in Proposition 7.7. States and actions are both binary. The receiver wants to match the state: $u_R(a, \theta) = \mathbb{1}(a = \theta)$. With probability p the sender is a good type who has the same utility as the receiver. With probability $1 - p$ the sender is a bad type who wants to mismatch: $u_S(a, \theta) = \mathbb{1}(a \neq \theta)$. What is the receiver's optimal payoff?

The answer is not obvious. Proposition 7.7 implies that the receiver's payoff is at least $\frac{1}{2}$ (if the sender is a bad type, the receiver cannot do better than the babbling equilibrium payoff). However the receiver's expected payoff, under complete information about the sender's type, is $p + \frac{1}{2}(1 - p)$ because the receiver gets payoff 1 from the good type. Therefore the receiver's payoff from this unknown sender is between $\frac{1}{2}$ and $p + \frac{1}{2}(1 - p)$. The question is whether the receiver could do better than $\frac{1}{2}$, and if so, could she achieve $p + \frac{1}{2}(1 - p)$?

Proposition 7.9. *The receiver's optimal payoff is* $\max\{p, 1/2\}$.

- *If* $p \leq \frac{1}{2}$, *the optimal mechanism is to ignore the sender's reports and take random actions, and the receiver's payoff is* $\frac{1}{2}$.
- *If* $p > \frac{1}{2}$, *the optimal mechanism is to always take the action that corresponds to the sender's report, and the receiver's payoff is* p.

The receiver either listens to the sender all the time (if $p > \frac{1}{2}$) or takes random actions all the time (if $p < \frac{1}{2}$). The reason is that the good type and the bad type could imitate each other in two ways. They could pretend to be the other type and follow exactly the same reporting strategy as the other type. Or they could pretend to be the other type and follow the opposite reporting strategy (if the other type reports m in state 1, I report

m in state 0). These two imitations restrict the set of payoffs for the good type and the bad type.

Proof of Proposition 7.9. Let u_g and u_b denote the expected payoffs of the good type and the bad type. Since the good type has the same utility has the receiver, and bad type wants to mismatch, the receiver's expected utility is equal to

$$p \cdot u_g + (1-p) \cdot (1-u_b).$$

We claim that $u_g + u_b \geq 1$. Indeed if $u_g < 1 - u_b$, then the good type would pretend to be the bad type and follow the same reporting strategy as the bad type. Similarly we also have $u_b < 1 - u_g$, so the bad type would pretend to be the good type and follow the same reporting strategy as the good type.

Next we need $u_g \geq u_b$. Otherwise the good type would pretend to be the bad type, but uses the bad type's report in the other state: if $\theta_t = 0$, report the bad type's message in $\theta_t = 1$, and vice versa.

We also need $u_b \geq u_g$. Otherwise the bad type would pretend to be the good type, but uses the good type's report in the other state: if $\theta_t = 0$, report the good type's message for $\theta_t = 1$, and vice versa.

We deduce that $u_g = u_b \geq \frac{1}{2}$. Consequently if $p < \frac{1}{2}$, the optimal solution is $u_g = u_b = \frac{1}{2}$, which corresponds to babbling. If $p > \frac{1}{2}$, the optimal solution is $u_g = u_b = 1$, which corresponds to the receiver always follows the sender's recommendation. □

8 Conclusion

Throughout the paper we assume that the receiver commits to her mechanism. We could potentially drop the commitment assumption as follows. The sender and the receiver write a "non-binding" contract in period 0, which specifies the receiver's action for each history of reports and public signals. If the receiver ever deviates from an action specified in this contract, they switch to the babbling equilibrium forever. A caveat is that babbling must give the sender and the receiver a low enough payoff. For example suppose states and actions are both binary. The sender always wants the high action. The receiver wants the action to match the state, but the cost of mismatch is higher in the high state than in the low state. In a babbling equilibrium the receiver would always take the high action, which is exactly what the sender prefers. However if the receiver has commitment, she could punish the sender by taking low actions. That's why the setting of Guo and Hörner (2018) requires commitment.

We also assume that states are i.i.d. across time. However many applications involve persistent states. The optimal taxation models often assume that an agent's ability evolves according to a Markov process (e.g. Golosov et al. (2003); Farhi and Werning (2013); Kapička (2013)). We could still write down a Bellman equation, but now the value of the Bellman equation depends not only on the promised utilities, but also on the previous state. It is not immediately clear whether our analysis for the i.i.d. case extends to Markovian states. On the other hand if the Markov chain is ergodic, then the state today is essentially uncorrelated with the state a thousand periods later. One open question for future research is whether our results extend to ergodic Markov chains.

Another direction for future research is the reputation problem studied in Section 7.4. We gave a lower bound for the receiver's payoff (Proposition 7.7), and we found an example for which the lower bound is tight (Proposition 7.9). It would be nice to characterize the receiver's payoff for arbitrary sender preference. Moreover we assumed that there are only two types of senders. What is the receiver's payoff if the sender has multiple types?

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A Proofs for Section 3

Theorem 3.1 follows from Theorem 3.6. Theorem 3.6 is a special case of Theorem 6.1, which we prove in Section D. In this section we prove Proposition 3.4. We divide the proposition into three lemmas. For the ease of notation we take x as given and write a_θ instead of $a_{\theta,x}$.

Lemma A.1. *Suppose* $\theta \in \operatorname{argmax}_{\theta'}(1-\delta)\cdot u_S(a_{\theta'},\theta)+\delta\cdot u_{\theta'}$ *for all* θ *and* θ'. *Then* a_θ *is non-decreasing in* θ.

Proof. The single crossing condition implies that u_S has strictly increasing differences. Indeed for all $a > a'$ and $\theta > \theta'$ we have $u_S(a,\theta)-u_S(a,\theta') = \int_{\theta'}^{\theta} \frac{\partial u_S(a,s)}{\partial s} ds > \int_{\theta'}^{\theta} \frac{\partial u_S(a',s)}{\partial s} ds = u_S(a',\theta)-u_S(a',\theta')$.

On the other hand the incentive constraints imply that $(1-\delta)u_S(a_{\theta'},\theta)+\delta u_{\theta'} \leq (1-\delta)u_S(a_\theta,\theta)+\delta u_\theta$ and $(1-\delta)u_S(a_\theta,\theta')+\delta u_\theta \leq (1-\delta)u_S(a_{\theta'},\theta')+\delta u_{\theta'}$. Summing up these two inequalities gives us $u_S(a_\theta,\theta)-u_S(a_\theta,\theta') \geq u_S(a_{\theta'},\theta)-u_S(a_{\theta'},\theta')$. Therefore if $\theta > \theta'$, we must have $a_\theta \geq a_{\theta'}$ (otherwise the preceding inequality contradicts the strictly increasing differences property). □

Lemma A.2. *Suppose* $\theta \in \text{argmax}_{\theta'}(1-\delta)\cdot u_S(a_{\theta'},\theta)+\delta\cdot u_{\theta'}$ *for all* θ *and* θ'. *Let* $U(\theta)=(1-\delta)\cdot u_S(a_\theta,\theta)+\delta\cdot u_\theta$. *We have*

$$U(\theta)=U(\underline{\theta})+\int_{\underline{\theta}}^{\theta}(1-\delta)\frac{\partial u_S(a_s,s)}{\partial s}\cdot ds.$$

Proof. We have $U(\theta)\geq(1-\delta)u_S(a_{\theta'},\theta)+\delta u_{\theta'}$ and $U(\theta')\geq(1-\delta)u_S(a_\theta,\theta')+\delta u_\theta$. We obtain that

$$(1-\delta)[u_S(a_{\theta'},\theta)-u_S(a_{\theta'},\theta')]\leq U(\theta)-U(\theta')\leq(1-\delta)[u_S(a_\theta,\theta)-u_S(a_\theta,\theta')].$$

If $\theta>\theta'$, we have

$$(1-\delta)\frac{u_S(a_{\theta'},\theta)-u_S(a_{\theta'},\theta')}{\theta-\theta'}\leq\frac{U(\theta)-U(\theta')}{\theta-\theta'}\leq(1-\delta)\frac{u_S(a_\theta,\theta)-u_S(a_\theta,\theta')}{\theta-\theta'}.$$

If $\theta<\theta'$, we have

$$(1-\delta)\frac{u_S(a_\theta,\theta)-u_S(a_\theta,\theta')}{\theta-\theta'}\leq\frac{U(\theta)-U(\theta')}{\theta-\theta'}\leq(1-\delta)\frac{u_S(a_{\theta'},\theta)-u_S(a_{\theta'},\theta')}{\theta-\theta'}.$$

Take θ' close to θ. We observe that the left derivative of $U(\theta)$ (if it exists) is at most $(1-\delta)\frac{\partial U_S(a_\theta,\theta)}{\partial\theta}$, and the right derivative of $U(\theta)$ (if it exists) is at least $(1-\delta)\frac{\partial U_S(a_\theta,\theta)}{\partial\theta}$. Hence we deduce that $U'(\theta)=(1-\delta)\frac{\partial U_S(a_\theta,\theta)}{\partial\theta}$ whenever the derivative exists.

Since $\frac{\partial u_S(a,\theta)}{\partial\theta}$ is uniformly bounded, we have $u_S(\cdot,\theta)$ is Lipschitz continuous in θ. Suppose the Lipschitz constant is c. Then the above inequalities imply that $U(\theta)$ is Lipschitz continuous with constant $(1-\delta)c$. Hence $U'(\theta)$ exists almost everywhere, which means the integral formula holds for all θ. □

Lemma A.3. *Fix* a_θ *and* u_θ. *Let* $U(\theta)=(1-\delta)u_S(a_\theta,\theta)+\delta u_\theta$. *Suppose* a_θ *is non-decreasing in* θ. *Suppose* $U(\theta)=U(\underline{\theta})+\int_{\underline{\theta}}^{\theta}(1-\delta)\frac{\partial u_S(a_s,s)}{\partial s}ds$ *for all* θ. *We have* $\theta\in\text{argmax}_{\theta'}(1-\delta)\cdot u_S(a_{\theta'},\theta)+\delta\cdot u_{\theta'}$ *for all* θ *and* θ'.

Proof. It suffices to show that for all θ and θ' we have

$$u_S(a_{\theta'},\theta)-u_S(a_{\theta'},\theta')\leq\frac{U(\theta)-U(\theta')}{1-\delta}\leq u_S(a_\theta,\theta)-u_S(a_\theta,\theta').$$

This inequality is equivalent to

$$\int_{\theta'}^{\theta}\frac{\partial u_S(a_{\theta'},s)}{\partial s}ds\leq\int_{\theta'}^{\theta}\frac{\partial u_S(a_s,s)}{\partial s}ds\leq\int_{\theta'}^{\theta}\frac{\partial u_S(a_\theta,s)}{\partial s}ds.$$

The inequality follows from the fact that a_s is non-decreasing and $\frac{\partial u_S(a,s)}{\partial s}$ is strictly increasing in a. □

B Proof for Section 4

Theorem 4.1 is a special case of Theorem 6.4, which we prove in Section D. In this section we prove Theorem 4.2 and Corollary 4.3.

Lemma B.1. *For all $\widehat{u} \in (\underline{u}, \overline{u})$ there exists a $C > 0$ and $\Delta > 0$ such that if $\int_{\underline{x}}^{\overline{x}} \int_{\underline{\theta}}^{\overline{\theta}} u_R(a_{\theta,x}, \theta) dF(\theta) dx \in [w(\widehat{u}) - \Delta, w(\widehat{u}) + \Delta]$ and $\int_{\underline{x}}^{\overline{x}} \int_{\underline{\theta}}^{\overline{\theta}} u_S(a_{\theta,x}, \theta) dF(\theta) dx \in [\widehat{u} - \Delta, \widehat{u} + \Delta]$, then*

$$\begin{aligned} \min_{t_x : \mathbb{E}_x tx = 0} \mathbb{E}_{x,\theta}[z_{\theta,x} + t_x | z_{\theta,x} + t_x > 0] &> C \\ \min_{t_x : \mathbb{E}_x tx = 0} \mathbb{E}_{x,\theta}[z_{\theta,x} + t_x | z_{\theta,x} + t_x < 0] &< -C \end{aligned}$$

Proof. As $\Delta \to 0$ we have $z_{\theta,x} \to z^*_{\theta,x}$ in distribution. Since $z_{\theta,x}$ is bounded, we have $F(z_{\theta,x}) - F(z^*_{\theta,x})$ uniformly converges to 0. Let $C^* = \min_{t_x : \mathbb{E}_x tx = 0} \mathbb{E}_{x,\theta}[z^*_{\theta,x} + t_x | z^*_{\theta,x} + t_x > 0]$. Since $z^*_{\theta,x}$ is non-degenerate for all x, we have $C^* > 0$. Hence for all $C < C^*$ there is a Δ such that $\min_{t_x : \mathbb{E}_x tx = 0} \mathbb{E}_{x,\theta}[z_{\theta,x} + t_x | z_{\theta,x} + t_x > 0] > C$. Analogous argument holds for the lower bound. □

Lemma B.2. *For all $\widehat{u} \in (\underline{u}, \overline{u})$ for all $C > 0$ there exist $\Delta \in (0, \frac{C}{200})$, $\delta^* < 1$, $u_0 \in (\underline{u}, \widehat{u})$, $u_1 \in (\widehat{u}, \overline{u})$ such that for all $\delta > \delta^*$ for all $u \in [u_0, u_1]$ we have $\int_{\underline{x}}^{\overline{x}} \int_{\underline{\theta}}^{\overline{\theta}} u_R(a_{\theta,x}, \theta) dF(\theta) dx \in [w(\widehat{u}) - \Delta, w(\widehat{u}) + \Delta]$ and $\int_{\underline{x}}^{\overline{x}} \int_{\underline{\theta}}^{\overline{\theta}} u_S(a_{\theta,x}, \theta) dF(\theta) dx \in [\widehat{u} - \Delta, \widehat{u} + \Delta]$, where $a_{\theta,x}$ denote the actions from mechanism $\texttt{OPT}_\delta(u)$.*

Proof. First pick $u_0 \in (\widehat{u} - \frac{\Delta}{2}, \widehat{u})$ and $u_1 \in (\widehat{u}, \widehat{u} + \frac{\Delta}{2})$ such that $|w(u) - w(\widehat{u})| < \Delta$ for all $u \in [u_1 - \Delta, u_0 + \Delta]$. Then pick δ^* such that for all $\delta > \delta^*$ the following conditions hold:

$$\begin{aligned} w^R_\delta(u_1) &> 0.9 \cdot w'(u_1) \\ w^L_\delta(u_0) &< 1.1 \cdot w'(u_0) \\ \frac{w(u_0 + \Delta) - w_\delta(u_1)}{u_0 + \Delta - u_1} &< 0.9 \cdot w'(u_1) \\ \frac{w_\delta(u_0) - w(u_1 - \Delta)}{u_0 - (u_1 - \Delta)} &> 1.1 \cdot w'(u_0) \\ w_\delta(u_0 + \Delta), w_\delta(u_1 - \Delta) &> w(\widehat{u}) - \Delta. \end{aligned}$$

These conditions hold for δ close to 1 because $w_\delta \to w$ and the left and right derivatives of w_δ both go to w'. Note that these conditions imply that $\frac{w(u_0+\Delta)-w_\delta(u_1)}{u_0+\Delta-u_1} < 0.9 \cdot w'(u_1) < w_\delta^R(u_1)$ and $\frac{w_\delta(u_0)-w(u_1-\Delta)}{u_0-(u_1-\Delta)} > 1.1 \cdot w'(u_0) > w_\delta^L(u_0)$. We deduce that

$$\begin{aligned} w(u_0+\Delta)-w_\delta(u_1) &< w_\delta(u_1)-w_\delta(u)+w_\delta^R(u_1)\cdot(u_0+\Delta-u_1) \\ &\leq w_\delta(u_1)-w_\delta(u)+w_\delta^R(u)\cdot(u_0+\Delta-u_1) \\ &\leq w_\delta^R(u)\cdot(u_0+\Delta-u), \end{aligned}$$

and

$$\begin{aligned} w_\delta(u)-w(u_1-\Delta) &> w_\delta(u)-w_\delta(u_0)+w_\delta^L(u_0)\cdot(u_0-(u_1-\Delta)) \\ &\geq w_\delta(u)-w_\delta(u_0)+w_\delta^L(u)\cdot(u_0-(u_1-\Delta)) \\ &\geq w_\delta^L(u)\cdot(u-(u_1-\Delta)). \end{aligned}$$

We claim that for these u_0, u_1, δ^* we have $\int_{\underline{x}}^{\overline{x}} \int_{\underline{\theta}}^{\overline{\theta}} u_R(a_{\theta,x},\theta)dF(\theta)dx \in [w(\widehat{u})-\Delta, w(\widehat{u})+\Delta]$ and $\int_{\underline{x}}^{\overline{x}} \int_{\underline{\theta}}^{\overline{\theta}} u_S(a_{\theta,x},\theta)dF(\theta)dx \in [\widehat{u}-\Delta, \widehat{u}+\Delta]$. We first show that $\int_{\underline{x}}^{\overline{x}} \int_{\underline{\theta}}^{\overline{\theta}} u_S(a_{\theta,x},\theta)dF(\theta)dx \in [\widehat{u}-\Delta, \widehat{u}+\Delta]$. Let $u+\epsilon$ denote $\int_{\underline{x}}^{\overline{x}} \int_{\underline{\theta}}^{\overline{\theta}} u_S(a_{\theta,x},\theta)dF(\theta)dx$. We have $\int_{\underline{x}}^{\overline{x}} \int_{\underline{\theta}}^{\overline{\theta}} u_R(a_{\theta,x},\theta)dF(\theta)dx \leq w(u+\epsilon)$ because w is the first-best. Furthermore the expected future promised utility is equal to $u-\frac{1-\delta}{\delta}\epsilon$, and the concavity of w_δ implies that the receiver's continuation payoff is at most $w_\delta(u-\frac{1-\delta}{\delta}\epsilon)$. We get

$$w_\delta(u) \leq (1-\delta)\cdot w(u+\epsilon)+\delta\cdot w_\delta(u-\frac{1-\delta}{\delta}\epsilon).$$

Rearranging the terms yield

$$w(u+\epsilon)-w(u) \geq \epsilon \cdot \frac{w_\delta(u)-w_\delta(u-\frac{1-\delta}{\delta}\epsilon)}{u-(u-\frac{1-\delta}{\delta}\epsilon)}.$$

We deduce that if $\epsilon > 0$, then $\frac{w(u+\epsilon)-w_\delta(u)}{\epsilon} \geq w_\delta^R(u)$, and if $\epsilon < 0$, then $\frac{w(u+\epsilon)-w_\delta(u)}{\epsilon} \leq w_\delta^L(u)$. We deduce that $\epsilon < u_0+\Delta-u$ and $-\epsilon < u-(u_1-\Delta)$, which means $u_1-\Delta < u+\epsilon < u_0+\Delta$, which is in the range of $[\widehat{u}-\Delta, \widehat{u}+\Delta]$. We thus obtain that $\int_{\underline{x}}^{\overline{x}} \int_{\underline{\theta}}^{\overline{\theta}} u_S(a_{\theta,x},\theta)dF(\theta)dx \in [\widehat{u}-\Delta, \widehat{u}+\Delta]$.

We now show that $\int_{\underline{x}}^{\overline{x}} \int_{\underline{\theta}}^{\overline{\theta}} u_R(a_{\theta,x},\theta)dF(\theta)dx \in [w(\widehat{u})-\Delta, w(\widehat{u})+\Delta]$. The concavity of w_δ implies that $w_\delta(u) \geq (1-\delta)\cdot w_\delta(u+\epsilon)+\delta\cdot w_\delta(u-\frac{1-\delta}{\delta}\epsilon)$. Since the continuation payoffs are at most $w_\delta(u-\frac{1-\delta}{\delta}\epsilon)$, we have $\int_{\underline{x}}^{\overline{x}} \int_{\underline{\theta}}^{\overline{\theta}} u_R(a_{\theta,x},\theta)dF(\theta)dx$ is at least $w_\delta(u+\epsilon)$. We know that $u_1-\Delta < u+\epsilon < u_0+\Delta$, and we picked δ^* such that $w_\delta(u_1-\Delta)$ and $w_\delta(u_0+\Delta)$ are both greater than $w(\widehat{u})-\Delta$. Consequently we have $\int_{\underline{x}}^{\overline{x}}\int_{\underline{\theta}}^{\overline{\theta}} u_R(a_{\theta,x},\theta)dF(\theta)dx > w(\widehat{u})-\Delta$. We also have $\int_{\underline{x}}^{\overline{x}}\int_{\underline{\theta}}^{\overline{\theta}} u_R(a_{\theta,x},\theta)dF(\theta)dx < w(u+\epsilon) < w(\widehat{u})+\Delta$. □

Lemma B.3. *For all $\widehat{u} \in (\underline{u}, \overline{u})$ there exist $C > 0$, $\delta^* < 1$, $u_0 \in (\underline{u}, \widehat{u})$, $u_1 \in (\widehat{u}, \overline{u})$ such that for all $\delta > \delta^*$ the following two statements hold:*

1. $\min_{u\in[u_0,u_1]} |w''(u)| > 100 \cdot \max_{u,v\in[u_0-\frac{1-\delta}{\delta}(\frac{C}{100}+C),u_1+\frac{1-\delta}{\delta}(\frac{C}{100}+C)]} |w''(u)-w''(v)|$;
2. *for all $u \in [u_0, u_1]$ there exists an $\epsilon \in (-\frac{C}{100}, \frac{C}{100})$ such that*

$$w_\delta(u) \le (1-\delta)\cdot w(u+\epsilon) + \frac{\delta}{2}\cdot w_\delta(w - \frac{1-\delta}{\delta}\epsilon - \frac{1-\delta}{\delta}C) + \frac{\delta}{2}\cdot w_\delta(u - \frac{1-\delta}{\delta}\epsilon + \frac{1-\delta}{\delta}C).$$

Proof. Part 1: since $w'' < 0$ and w'' is continuous, there is a neighborhood of $\widehat{u}$ such that the inequality holds. Suppose a set of C, δ^*, u_0, u_1 satisfies Part 2. We could shrink $[u_0, u_1]$ and increase δ^* until $[u_0 - \frac{1-\delta}{\delta}(\frac{C}{100}+C), u_1 + \frac{1-\delta}{\delta}(\frac{C}{100}+C)]$ lie inside the appropriate neighborhood of $\widehat{u}$. Notice that shrinking $[u_0, u_1]$ and increasing δ^* keeps Part 2 intact. Hence it remains to prove that there exists a set of C, δ^*, u_0, u_1 that satisfies Part 2.

Part 2: Take the C from Lemma B.1. Take the $\Delta, \delta*, u_0, u_1$ from Lemma B.2. Choose ϵ such that $u+\epsilon = \int_{\underline{x}}^{\overline{x}}\int_{\underline{\theta}}^{\overline{\theta}} u_S(a_{\theta,x},\theta)dF(\theta)dx$. Lemma B.2 implies that $|\epsilon| < 2\Delta < \frac{C}{100}$. To prove the upper bound of w_δ recall that the $a_{\theta,x}$ satisfies

$$(1-\delta)\cdot u_S(a_{\theta,x},\theta) + \delta\cdot u_{\theta,x} = \underline{U}_x + \int_{\underline{\theta}}^{\theta}(1-\delta)\frac{\partial u_S(a_{s,x},s)}{\partial s}$$
$$\int_{\underline{x}}^{\overline{x}}[\underline{U}_x + \int_{\underline{\theta}}^{\overline{\theta}}(1-\delta)\frac{u_S(a_{\theta,x},\theta)}{\partial\theta}\frac{1-F(\theta)}{f(\theta)}dF(\theta)]\cdot dx = u.$$

We rewrite these two constraints as

$$u_{\theta,x} = u - \frac{1-\delta}{\delta}\epsilon - \frac{\underline{U}_x - \int_{\underline{x}}^{\overline{x}}\underline{U}_k dk}{\delta} + \frac{1-\delta}{\delta}z_{\theta,x}.$$

Note that $\frac{\underline{U}_x - \int_{\underline{x}}^{\overline{x}}\underline{U}_k dk}{\delta}$ has mean 0. Lemma B.1 implies that

$$\mathbb{E}[u_{\theta,x}|u_{\theta,x} < u - \frac{1-\delta}{\delta}\epsilon] \quad < \quad u - \frac{1-\delta}{\epsilon} - \frac{1-\delta}{\delta}C$$
$$\mathbb{E}[u_{\theta,x}|u_{\theta,x} > u - \frac{1-\delta}{\delta}\epsilon] \quad > \quad u - \frac{1-\delta}{\epsilon} + \frac{1-\delta}{\delta}C.$$

The concavity of w_δ implies that

$$\int_{\underline{x}}^{\overline{x}}\int_{\underline{\theta}}^{\overline{\theta}} w_\delta(u_{\theta,x})dF(\theta)dx \leq \frac{1}{2}\cdot w_\delta(u-\frac{1-\delta}{\epsilon}-\frac{1-\delta}{\delta}C)+\frac{1}{2}\cdot w_\delta(u-\frac{1-\delta}{\epsilon}+\frac{1-\delta}{\delta}C).$$

Together with the fact that $\int_{\underline{x}}^{\overline{x}}\int_{\underline{\theta}}^{\overline{\theta}} u_R(a_{\theta,x},\theta)dF(\theta)dx < w(u+\epsilon)$ we conclude that

$$w_\delta(u) \leq (1-\delta)\cdot w(u+\epsilon)+\frac{\delta}{2}\cdot w_\delta(w-\frac{1-\delta}{\delta}\epsilon-\frac{1-\delta}{\delta}C)+\frac{\delta}{2}\cdot w_\delta(u-\frac{1-\delta}{\delta}\epsilon+\frac{1-\delta}{\delta}C).$$

This inequality holds for all $u \in [u_0, u_1]$. □

Lemma B.4. *Take the C, δ^*, u_0, u_1 from Lemma B.3. There exists a $K > 0$ such that for all $\delta > \delta^*$ for all $u \in [u_0, u_1]$ we have*

$$\begin{aligned}\frac{w(u)-w_\delta(u)}{1-\delta} \geq K(1-\delta)+\frac{\delta}{2}\cdot\frac{w(u-\frac{1-\delta}{\delta}\epsilon-\frac{1-\delta}{\delta}C)-w_\delta(u-\frac{1-\delta}{\delta}\epsilon-\frac{1-\delta}{\delta}C)}{1-\delta}\\ +\frac{\delta}{2}\cdot\frac{w(u-\frac{1-\delta}{\delta}\epsilon+\frac{1-\delta}{\delta}C)-w_\delta(u-\frac{1-\delta}{\delta}\epsilon+\frac{1-\delta}{\delta}C)}{1-\delta}.\end{aligned}$$

Proof. Lemma B.3 says for all $\delta > \delta^*$ for all $u \in [u_0, u_1]$ we have $w_\delta(u) \leq (1-\delta)\cdot w(u+\epsilon)+\frac{\delta}{2}\cdot w_\delta(w-\frac{1-\delta}{\delta}\epsilon-\frac{1-\delta}{\delta}C)+\frac{\delta}{2}\cdot w_\delta(u-\frac{1-\delta}{\delta}\epsilon+\frac{1-\delta}{\delta}C)$. We transform this inequality as follows:

$$\begin{aligned}\frac{w(u)-w_\delta(u)}{1-\delta} \geq \frac{w(u)-(1-\delta)w(u+\epsilon)-\frac{\delta}{2}w(u-\frac{1-\delta}{\delta}\epsilon-\frac{1-\delta}{\delta}C)-\frac{\delta}{2}w(u-\frac{1-\delta}{\delta}\epsilon+\frac{1-\delta}{\delta}C)}{1-\delta}\\ +\frac{\delta}{2}\cdot\frac{w(u-\frac{1-\delta}{\delta}\epsilon-\frac{1-\delta}{\delta}C)-w_\delta(u-\frac{1-\delta}{\delta}\epsilon-\frac{1-\delta}{\delta}C)}{1-\delta}\\ +\frac{\delta}{2}\cdot\frac{w(u-\frac{1-\delta}{\delta}\epsilon+\frac{1-\delta}{\delta}C)-w_\delta(u-\frac{1-\delta}{\delta}\epsilon+\frac{1-\delta}{\delta}C)}{1-\delta}.\end{aligned}$$

We have

$$\begin{aligned}w(u+\epsilon) &= w(u)+w'(u)\cdot\epsilon+\frac{w''(u)}{2}\epsilon^2+\int_u^\epsilon\int_u^a[w''(s)-w''(u)]dsda\\ w(u)-w(u+\epsilon) &= -w'(u)\cdot\epsilon+\frac{|w''(u)|}{2}\epsilon^2-\int_u^\epsilon\int_u^a[w''(s)-w''(u)]dsda\\ &\geq -w'(u)\cdot\epsilon+\frac{|w''(u)|}{2}\epsilon^2-\frac{\epsilon^2}{2}\cdot\frac{|w''(u)|}{100}.\end{aligned}$$

The last step follows from the condition that $|w''(s)-w''(u)| < \frac{|w''(u)|}{100}$ (from Lemma B.3).

We deduce that

$$w(u) - w(u+\epsilon) \geq -w'(u)\cdot\epsilon + \frac{\epsilon^2}{2}\cdot\frac{99}{100}|w''(u)|.$$

The same argument implies that

$$\begin{aligned} w(u) - w(u - \frac{1-\delta}{\delta}\epsilon - \frac{1-\delta}{\delta}C) \geq \\ - w'(u)\cdot(-\frac{1-\delta}{\delta}\epsilon - \frac{1-\delta}{\delta}C) + \frac{(-\frac{1-\delta}{\delta}\epsilon - \frac{1-\delta}{\delta}C)^2}{2}\cdot\frac{99}{100}|w''(u)| \end{aligned}$$

and

$$\begin{aligned} w(u) - w(u - \frac{1-\delta}{\delta}\epsilon + \frac{1-\delta}{\delta}C) \geq \\ - w'(u)\cdot(-\frac{1-\delta}{\delta}\epsilon + \frac{1-\delta}{\delta}C) + \frac{(-\frac{1-\delta}{\delta}\epsilon + \frac{1-\delta}{\delta}C)^2}{2}\cdot\frac{99}{100}|w''(u)|. \end{aligned}$$

We obtain that

$$\begin{aligned} & w(u) - (1-\delta)w(u+\epsilon) - \frac{\delta}{2}w(u - \frac{1-\delta}{\delta}\epsilon - \frac{1-\delta}{\delta}C) - \frac{\delta}{2}w(u - \frac{1-\delta}{\delta}\epsilon + \frac{1-\delta}{\delta}C) \\ = \; & (1-\delta)(w(u) - w(u+\epsilon)) + \frac{\delta}{2}(w(u) - w(u - \frac{1-\delta}{\delta}\epsilon - \frac{1-\delta}{\delta}C)) \\ & + \frac{\delta}{2}(w(u) - w(u - \frac{1-\delta}{\delta}\epsilon + \frac{1-\delta}{\delta}C)) \\ \geq \; & \left[(1-\delta)\frac{\epsilon^2}{2} + \frac{\delta}{2}\frac{(-\frac{1-\delta}{\delta}\epsilon - \frac{1-\delta}{\delta}C)^2}{2} + \frac{\delta}{2}\frac{(-\frac{1-\delta}{\delta}\epsilon + \frac{1-\delta}{\delta}C)^2}{2}\right]\cdot\frac{99}{100}|w''(u)| \\ \geq \; & K(1-\delta)^2, \end{aligned}$$

where

$$K = \left[\frac{(-\frac{C}{100} - C)^2}{4\delta^*} + \frac{(-\frac{C}{100} + C)^2}{4\delta^*}\right]\cdot\frac{99}{100}\cdot\min_{u\in[u_0,u_1]}|w''(u)|.$$

(The last step of the inequality uses the fact that $\epsilon \in (-\frac{C}{100}, \frac{C}{100})$ and $\frac{\epsilon^2}{2} \geq 0$.) We are left to show that $K > 0$. By construction we have $C > 0$ and $\min_{u\in[u_0,u_1]}|w''(u)| > 0$, so $K > 0$. The inequality holds for all $\delta > \delta^*$ and for all $u \in [u_0, u_1]$. □

Lemma B.5. *Take C, δ^*, u_0, u_1 from Lemma B.3. Take the K from Lemma B.4. Let $\widehat{u} = \frac{u_0+u_1}{2}$. For all $\delta > \delta^*$ we have* $\frac{w(\widehat{u}) - w_\delta(\widehat{u})}{1-\delta} \geq \min\{K/2, (\frac{u_1-u_0}{2} - \frac{1-\delta}{\delta}(C + \frac{C}{100}))\frac{K/2}{C+\frac{C}{100}}\}$.

Proof. Suppose on the contrary that $\frac{w(\widehat{u}) - w_\delta(\widehat{u})}{1-\delta} < \min\{K/2, (\frac{u_1-u_0}{2} - \frac{1-\delta}{\delta}(C + \frac{C}{100}))\frac{K/2}{C+\frac{C}{100}}\}$. For the ease of notation let $r(u) = \frac{w(u) - w_\delta(u)}{1-\delta}$. We know from Lemma B.4 that for all

$u \in [u_0, u_1]$ we have

$$r(u) \geq K(1-\delta) + \frac{\delta}{2} \cdot r(u - \frac{1-\delta}{\delta}\epsilon - \frac{1-\delta}{\delta}C) + \frac{\delta}{2} \cdot r(u - \frac{1-\delta}{\delta}\epsilon + \frac{1-\delta}{\delta}C).$$

If $r(u) < K/2$, then either $r(u - \frac{1-\delta}{\delta}\epsilon - \frac{1-\delta}{\delta}C) < r(u) - \frac{1-\delta}{\delta}K/2$ or $r(u - \frac{1-\delta}{\delta}\epsilon + \frac{1-\delta}{\delta}C) < r(u) - \frac{1-\delta}{\delta}K/2$. In particular if $r(u) < K/2$, there exists a v such that $|u-v| < \frac{1-\delta}{\delta}(C + \frac{C}{100})$ and $r(v) < r(u) - \frac{1-\delta}{\delta}K/2$. Start from $u^0 = \widehat{u}$. Define u^{t+1} as the value such that $|u^{t+1} - u^t| < \frac{1-\delta}{\delta}(C + \frac{C}{100})$ and $r(u^{t+1}) < r(u^t) - \frac{1-\delta}{\delta}K/2$. Since in each iteration the value of $r(u^t)$ drops by more than $\frac{1-\delta}{\delta}K/2$, we have $r(u^t) < 0$ for all $t > \frac{r(u)}{\frac{1-\delta}{\delta}K/2}$. However in each iteration we have $|u^{t+1} - u^t| < \frac{1-\delta}{\delta}(C + \frac{C}{100})$, so for $t \in (\frac{r(u)}{\frac{1-\delta}{\delta}K/2}, \frac{r(u)}{\frac{1-\delta}{\delta}K/2} + 1)$ we have $|u^t - \widehat{u}| < (\frac{r(u)}{\frac{1-\delta}{\delta}K/2} + 1) \cdot \frac{1-\delta}{\delta}(C + \frac{C}{100})$, which is less than $\frac{u_1 - u_0}{2}$ because $r(u) < (\frac{u_1-u_0}{2} - \frac{1-\delta}{\delta}(C + \frac{C}{100}))\frac{K/2}{C + \frac{C}{100}}$. Hence there exists a $u^t \in [u_0, u_1]$ such that $r(u^t) < 0$. Contradiction (since $r(u) \geq 0$ for all u). □

Proof of Theorem 4.2. It follows from Lemma B.5 that as $\delta \to 1$ we have $\frac{w(u) - w_\delta(u)}{1-\delta}$ is bounded below, so $w(u) - w_\delta(u) = \Omega(1-\delta)$. □

Proof of Corollary 4.3. In Lemma B.5 consider any $u \in (\frac{u_0 + \widehat{u}}{2}, \frac{\widehat{u} + u_1}{2})$ and consider the interval $(u - \frac{u_1 - u_0}{4}, u + \frac{u_1 - u_0}{4})$. We deduce that there is a δ^* such that for all $\delta > \delta^*$ and for all $u \in (\frac{u_0 + \widehat{u}}{2}, \frac{\widehat{u} + u_1}{2})$ we have $\frac{w(u) - w_\delta(u)}{1-\delta} \geq \min\{K/2, (\frac{u_1 - u_0}{4} - \frac{1-\delta}{\delta}(C + \frac{C}{100}))\frac{K/2}{C + \frac{C}{100}}\}$.

Now take $\widehat{u} = u^* = \text{argmax}_u w(u)$. Since $w''(u) > 0$, the maximizer is unique. As $\delta \to 1$ we have $\text{argmax}_u w_\delta(u)$ converges to u^*; otherwise $w_\delta(u) < w(u) < w(u^*)$. Let δ be close to 1 such that $\text{argmax}_u w_\delta(u)$ is in the interior of $(\frac{u_0 + u^*}{2}, \frac{u^* + u_1}{2})$. Take δ close to 1 such that $\delta > \delta^*$. We obtain that $\frac{w(u) - w_\delta(u)}{1-\delta}$ is bounded below. Since $w(u^*) > w(u)$, we have $\frac{w(u^*) - \max_u w_\delta(u)}{1-\delta}$ is bounded below as $\delta \to 1$. □

C Proofs for Section 5

C.1 The Bellman equation

We prove Proposition 5.1. Proposition 5.2 is a special case of Theorem 6.1.

Lemma C.1. *For all u_1 and u_2 we have $\frac{w_\delta(u_1) - w_\delta(u_2)}{u_1 - u_2} \in [-1, 1]$.*

Proof. We know that $w_\delta(u) \leq w(u) = \min\{u + \frac{1}{2}, -u - \frac{1}{2}\}$, and equality holds for $u = -1$ and $u = 0$. We have $w'(u) = 1$ for $u < -\frac{1}{2}$ and $w'(u) = -1$ for $u > -\frac{1}{2}$. The lemma follows from the concavity of w_δ. □

Let $p_l = \int_{\underline{x}}^{\overline{x}} (1-a_{l,x})dx$ and $p_h = \int_{\underline{x}}^{\overline{x}} a_{h,x}dx$ denote the probability that the receiver follows the sender's recommendation in each state. Let $u_l = \int_{\underline{x}}^{\overline{x}} u_{l,x}dx$ and $u_h = \int_{\underline{x}}^{\overline{x}} u_{h,x}dx$. The incentive constraints and the promise-keeping constraint imply that

$$\begin{gathered}
-(1-\delta)p_h + \delta u_h \geq -(1-\delta)(1-p_l) + \delta u_l \\
-(1-\delta)(1-p_l) + \delta u_l \geq -(1-\delta)p_h + \delta u_h \\
\tfrac{-(1-\delta)(1-p_l)+\delta u_l}{2} + \tfrac{-(1-\delta)p_h+\delta u_h}{2} = u
\end{gathered}$$

Since the first and second inequalities are the reverse of each other, we have $-(1-\delta)(1-p_l) + \delta u_l \geq -(1-\delta)p_h + \delta u_h$, and the promise-keeping constraint implies that $u_l = \frac{u+(1-\delta)(1-p_l)}{\delta}$ and $u_h = \frac{u+(1-\delta)p_h}{\delta}$. The receiver's payoff in the current period is equal to $-\frac{1-p_l+1-p_h}{2}$. Moreover since w_δ is concave, we have $\int_{\underline{x}}^{\overline{x}} w_\delta(u_{\theta,x})dx \leq w_\delta(\int_{\underline{x}}^{\overline{x}} u_{\theta,x}dx)$. We obtain that

$$\begin{aligned}
w_\delta(u) &\leq \max_{p_l,p_h} w_\delta(u,p_l,p_h) \\
&= \max_{p_l,p_h} -(1-\delta)\cdot\frac{1-p_l+1-p_h}{2} + \frac{\delta}{2}\cdot w_\delta(\frac{u+(1-\delta)(1-p_l)}{\delta}) + \frac{\delta}{2}\cdot w_\delta(\frac{u+(1-\delta)p_h}{\delta}).
\end{aligned}$$

Lemma C.2. *We have $w_\delta(u,p_l,p_h)$ is non-decreasing in p_l and p_h.*

Proof. We have

$$\begin{aligned}
& w_\delta(u,p_l,p_h+\epsilon) - w_\delta(u,p_l,p_h) \\
= \;& (1-\delta)\frac{\epsilon}{2} + \frac{\delta}{2}\cdot w_\delta\left(\frac{u+(1-\delta)p_h}{\delta} + \frac{(1-\delta)\epsilon}{\delta}\right) - \frac{\delta}{2}\cdot w_\delta\left(\frac{u+(1-\delta)p_h}{\delta}\right) \\
= \;& (1-\delta)\frac{\epsilon}{2}\cdot\left(1 + \frac{w_\delta\left(\frac{u+(1-\delta)p_h}{\delta}+\frac{(1-\delta)\epsilon}{\delta}\right) - w_\delta\left(\frac{u+(1-\delta)p_h}{\delta}\right)}{\left(\frac{u+(1-\delta)p_h}{\delta}+\frac{(1-\delta)\epsilon}{\delta}\right) - \left(\frac{u+(1-\delta)p_h}{\delta}\right)}\right),
\end{aligned}$$

which is non-negative by Lemma C.1.

We similarly have

$$\begin{aligned}
& w_\delta(u,p_l+\epsilon,p_h) - w_\delta(u,p_l,p_h) \\
= \;& (1-\delta)\frac{\epsilon}{2} + \frac{\delta}{2}\cdot w_\delta\left(\frac{u+(1-\delta)(1-p_l)}{\delta} - \frac{(1-\delta)\epsilon}{\delta}\right) - \frac{\delta}{2}\cdot w_\delta\left(\frac{u+(1-\delta)(1-p_l)}{\delta}\right) \\
= \;& (1-\delta)\frac{\epsilon}{2}\cdot\left(1 - \frac{w_\delta\left(\frac{u+(1-\delta)(1-p_l)}{\delta}-\frac{(1-\delta)\epsilon}{\delta}\right) - w_\delta\left(\frac{u+(1-\delta)(1-p_l)}{\delta}\right)}{\left(\frac{u+(1-\delta)(1-p_l)}{\delta}-\frac{(1-\delta)\epsilon}{\delta}\right) - \left(\frac{u+(1-\delta)(1-p_l)}{\delta}\right)}\right),
\end{aligned}$$

which is non-negative by Lemma C.1. □

Proof of Proposition 5.1. Suppose $u \in [-\delta, \delta - 1]$. Lemma C.2 implies that $w_\delta(u, p_l, p_h)$ is maximized when $p_l = p_h = 1$, and for $u \in [-\delta, \delta - 1]$ we have $\frac{u+(1-\delta)(1-p_l)}{\delta}$ and $\frac{u+(1-\delta)p_h}{\delta}$ are both within $[-1, 0]$. Therefore

$$w_\delta(u) \leq w_\delta(u, 1, 1) = \frac{\delta}{2} \cdot w_\delta(\frac{u}{\delta}) + \frac{\delta}{2} \cdot w_\delta(\frac{u + 1 - \delta}{\delta}).$$

Furthermore equality holds if we set $a_{\theta,x} = \theta$ and $u_{\theta,x} = u_\theta$ for all θ and x. □

C.2 Rate of convergence

Theorem 5.3 is a special case of Theorem 6.4. In this section we prove Theorem 5.4 in two parts: the $O(\sqrt{1-\delta})$ bound (Theorem C.8) and the $\Omega(\sqrt{1-\delta})$ bound (Theorem C.15). Both parts rely on the following lemma.

Lemma C.3. *Part 1: let v_δ be a function such that $v_\delta(u) = u + \frac{1}{2}$ for all $u \in [-1, -\delta]$ and $v_\delta(u) = -u - \frac{1}{2}$ for all $u \in [\delta - 1, 0]$. If for all $u \in [-\delta, \delta - 1]$ we have*

$$v_\delta(u) \geq \frac{\delta}{2} \cdot v_\delta(\frac{u}{\delta}) + \frac{\delta}{2} \cdot v_\delta(\frac{u + 1 - \delta}{\delta}),$$

then $v_\delta(u) \geq w_\delta(u)$ for all u.

Part 2: let v_δ be a function such that $v_\delta(u) \leq 0$ for all $u \in [-1, -\delta] \cup [\delta - 1, 0]$. If for all $u \in [-\delta, \delta - 1]$ we have

$$v_\delta(u) \leq \frac{\delta}{2} \cdot v_\delta(\frac{u}{\delta}) + \frac{\delta}{2} \cdot v_\delta(\frac{u + 1 - \delta}{\delta}),$$

then $v_\delta(u) \leq w_\delta(u)$ for all u.

Proof. For the first part let $K = \min_u v_\delta(u) - w_\delta(u)$. Suppose $K < 0$. Then the minimizer u must lie in $[-\delta, \delta - 1]$. However in this interval we have $w_\delta(u) = \frac{\delta}{2} \cdot w_\delta(\frac{u}{\delta}) + \frac{\delta}{2} \cdot w_\delta(\frac{u+1-\delta}{\delta})$, which means

$$K = v_\delta(u) - w_\delta(u) \geq \frac{\delta}{2} \cdot \left(v_\delta(\frac{u}{\delta}) - w_\delta(\frac{u}{\delta})\right) + \frac{\delta}{2} \cdot \left(v_\delta(\frac{u + 1 - \delta}{\delta}) - w_\delta(\frac{u + 1 - \delta}{\delta})\right).$$

By the definition K both $v_\delta(\frac{u}{\delta}) - w_\delta(\frac{u}{\delta})$ and $v_\delta(\frac{u+1-\delta}{\delta}) - w_\delta(\frac{u+1-\delta}{\delta})$ must be greater than or equal to K. Then the right hand side is at least δK, which is impossible because if

$K < 0$, then $K < \delta K$. Hence $K \geq 0$. For the second part take $K = \sup_u v_\delta(u) - w_\delta(u)$ and analogous argument shows $K \leq 0$. $\square$

C.2.1 The $O(1-\delta)^{1/2}$ bound

We construct a function v_δ such that $v_\delta(u) \leq w_\delta(u)$ for all u and $v_\delta(-\frac{1}{2}) = -c(1-\delta)^{1/2}$ for some c.

Pick $c > \sqrt{2} - 1$. Again let $w(u) = \min\{u + \frac{1}{2}, -u - \frac{1}{2}\}$ denote the first-best. Define v_δ as follows:

$$v_\delta(u) = \begin{cases} \sqrt{1 - \delta - (u + \frac{1}{2})^{1/2}} - (c+1)(1-\delta)^{1/2} & \text{if } |u + \frac{1}{2}| \leq \frac{(1-\delta)^{1/2}}{\sqrt{2}} \\ w(u) - (c + 1 - \sqrt{2})(1-\delta)^{1/2} & \text{otherwise} \end{cases}$$

As shown in Figure 3 we draw a circle with center $(-\frac{1}{2}, -(c+1)(1-\delta)^{1/2})$ and radius $(1-\delta)^{1/2}$. For $c > \sqrt{2} - 1$ this circle is below $w(u)$. Take the upper semi-circle. Draw tangents to this semi-circle with slope 1 and -1. The function v_δ consists of a line with slope 1 (for $u < -\frac{1}{2} - \frac{(1-\delta)^{1/2}}{\sqrt{2}}$), an arc of the circle (for $|u + \frac{1}{2}| \leq \frac{(1-\delta)^{1/2}}{\sqrt{2}}$), and a line with slope -1 (for $u > -\frac{1}{2} + \frac{(1-\delta)^{1/2}}{\sqrt{2}}$).

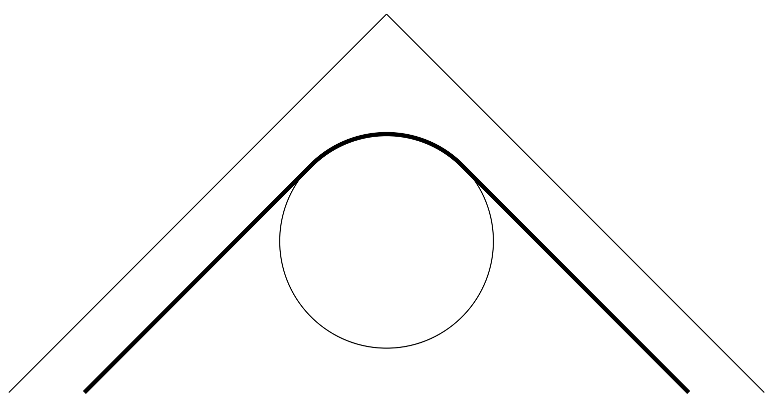

Figure 3: $v_\delta(u)$ is in bold

Assumptions:

- $c > \sqrt{2} - 1$
- $c > \sqrt{2} - 1 + \frac{\sqrt{2}}{\delta}(\frac{1}{2} - \frac{(1-\delta)^{1/2}}{\sqrt{2}})^2$
- $2\sqrt{2}\delta - \sqrt{2} > (1-\delta)^{1/2}$

- $c > \frac{\sqrt{2}}{4\delta} + \frac{1}{\sqrt{2}}$.

We claim that under these assumptions we have

$$v_\delta(u) \leq \frac{\delta}{2} \cdot v_\delta(\frac{u}{\delta}) + \frac{\delta}{2} \cdot v_\delta(\frac{u+1-\delta}{\delta}). \tag{C.1}$$

Then it follows from Lemma C.3 that $v_\delta(u) \leq w_\delta(u)$ for all u.

Lemma C.4. *Assume that* $c > \sqrt{2} - 1 + \frac{\sqrt{2}}{\delta}(\frac{1}{2} - \frac{(1-\delta)^{1/2}}{\sqrt{2}})^2$. *If* $|u + \frac{1}{2}| \geq \frac{(1-\delta)^{1/2}}{\sqrt{2}}$, *then* $v_\delta(u) \leq \frac{\delta}{2} \cdot v_\delta(\frac{u}{\delta}) + \frac{\delta}{2} \cdot v_\delta(\frac{u+1-\delta}{\delta})$.

Proof. Assume $u < -\frac{1}{2}$. The other case follows analogously. If $\frac{u+1-\delta}{\delta} \leq -\frac{1}{2} - \frac{(1-\delta)^{1/2}}{\sqrt{2}}$, then the inequality is trivial:

$$\begin{aligned} v_\delta(u) &= u + \frac{1}{2} - (c+1-\sqrt{2})(1-\delta)^{1/2} \\ &\leq \frac{\delta}{2}(\frac{u}{\delta} + \frac{1}{2} - (c+1-\sqrt{2})(1-\delta)^{1/2}) + \frac{\delta}{2}(\frac{u+1-\delta}{\delta} + \frac{1}{2} - (c+1-\sqrt{2})(1-\delta)^{1/2}) \\ &= \frac{\delta}{2} \cdot v_\delta(\frac{u}{\delta}) + \frac{\delta}{2} \cdot v_\delta(\frac{u+1-\delta}{\delta}). \end{aligned}$$

Now suppose $u \leq -\frac{1}{2} - \frac{(1-\delta)^{1/2}}{\sqrt{2}} < \frac{u+1-\delta}{\delta}$. Let

$$\epsilon(u) = \frac{u+1-\delta}{2} + \frac{1}{2} - (c+1-\delta)(1-\delta)^{1/2} - (\sqrt{1-\delta-(\frac{u+1-\delta}{\delta}+\frac{1}{2})^2} - (c+1)(1-\delta)^{1/2}).$$

The inequality is equivalent to proving

$$\frac{\epsilon(u)}{2} \leq \frac{1-\delta}{\delta}(c+1-\sqrt{2})(1-\delta)^{1/2}. \tag{C.2}$$

The RHS is independent of u, and the LHS is maximized at $u = -\frac{1}{2} - \frac{(1-\delta)^{1/2}}{\sqrt{2}}$. We prove this inequality for $u = -\frac{1}{2} - \frac{(1-\delta)^{1/2}}{\sqrt{2}}$ using power of a point.

Let X denote the point $(\frac{u+1-\delta}{\delta}, v_\delta(\frac{u+1-\delta}{\delta}) + \epsilon)$. Let A denote the point $(u, v_\delta(u))$. Let B and c denote the intersections of the circle with the vertical line passing through X. We have XA is tangent to the circle, and $XA = \sqrt{2}(\frac{u+1-\delta}{\delta} - u)$. We have $BC > \sqrt{2}(1-\delta)^{1/2}$. We deduce that

$$\epsilon = \frac{XA^2}{\epsilon + BC} < \frac{(\sqrt{2}(\frac{u+1-\delta}{\delta} - u))^2}{\sqrt{2}(1-\delta)^{1/2}} = \frac{\sqrt{2}}{\delta}(\frac{1}{2} - \frac{(1-\delta)^{1/2}}{\sqrt{2}})^2(1-\delta)^{1/2}\frac{1-\delta}{\delta}.$$

Inequality (C.2) follows from the assumption that $c > \sqrt{2} - 1 + \frac{\sqrt{2}}{\delta}(\frac{1}{2} - \frac{(1-\delta)^{1/2}}{\sqrt{2}})^2$. □

We next focus on the case when $|u+\frac{1}{2}| < \frac{(1-\delta)^{1/2}}{\sqrt{2}}$. We rewrite (C.1) as follows:

$$v_\delta(u) - \frac{v_\delta(\frac{u}{\delta}) + v_\delta(\frac{u+1-\delta}{\delta})}{2} \le \frac{1-\delta}{\delta}(-v_\delta(u)).$$

The RHS can be easily bounded by $\frac{1-\delta}{\delta}(-v_\delta(u)) > \frac{1-\delta}{\delta}c(1-\delta)^{1/2}$. We next bound the LHS for those u for which $|u+\frac{1}{2}| < \frac{(1-\delta)^{1/2}}{\sqrt{2}}$.

Lemma C.5. *We have*

$$v_\delta(u) - \frac{v_\delta(\frac{u}{\delta}) + v_\delta(\frac{u+1-\delta}{\delta})}{2} \le v_\delta(u) - \frac{\frac{u+1-\delta}{\delta} - u}{\frac{u+1-\delta}{\delta} - \frac{u}{\delta}} v_\delta(\frac{u}{\delta}) - \frac{u - \frac{u}{\delta}}{\frac{u+1-\delta}{\delta} - \frac{u}{\delta}} v_\delta(\frac{u+1-\delta}{\delta}) + |-u-\frac{1}{2}|\frac{1-\delta}{\delta}.$$

Proof. Assume $u \le -\frac{1}{2}$. The other case follows analogously. The RHS minus LHS is equal to $(-u-\frac{1}{2})\frac{1-\delta}{\delta}(1 - \frac{v_\delta(\frac{u+1-\delta}{\delta}) - v_\delta(\frac{u}{\delta})}{\frac{u+1-\delta}{\delta} - \frac{u}{\delta}})$. The inequality is equivalent to showing $\frac{v_\delta(\frac{u+1-\delta}{\delta}) - v_\delta(\frac{u}{\delta})}{\frac{u+1-\delta}{\delta} - \frac{u}{\delta}} \le 1$, which follows from the fact that v_δ is concave and $v'_\delta(x) = 1$ for small x. □

Lemma C.6. *Suppose $2\sqrt{2}\delta - \sqrt{2} > (1-\delta)^{1/2}$. For all u such that $|u+\frac{1}{2}| < \frac{(1-\delta)^{1/2}}{\sqrt{2}}$ we have*

$$v_\delta(u) - \frac{v_\delta(\frac{u}{\delta}) + v_\delta(\frac{u+1-\delta}{\delta})}{2} \le \frac{\sqrt{2}}{4\delta^2}(1-\delta)^{3/2} + \frac{(1-\delta)^{3/2}}{\delta\sqrt{2}}.$$

Proof. We first apply the bound from Lemma C.5. We have $(-u-\frac{1}{2})\frac{1-\delta}{\delta} < \frac{(1-\delta)^{3/2}}{\delta\sqrt{2}}$. We are left to show that $v_\delta(u) - \frac{\frac{u+1-\delta}{\delta} - u}{\frac{u+1-\delta}{\delta} - \frac{u}{\delta}} v_\delta(\frac{u}{\delta}) - \frac{u - \frac{u}{\delta}}{\frac{u+1-\delta}{\delta} - \frac{u}{\delta}} v_\delta(\frac{u+1-\delta}{\delta}) < \frac{\sqrt{2}}{4\delta^2}(1-\delta)^{3/2}$.

Let A denote point $(u, v_\delta(u))$. Let B denote the point $(u, -\sqrt{1-\delta-(u+\frac{1}{2})^2} - (c+1)(1-\delta)^{1/2})$. Let C denote point $(\frac{u}{\delta}, v_\delta(\frac{u}{\delta}))$. Let D denote point $(\frac{u+1-\delta}{\delta}, v_\delta(\frac{u+1-\delta}{\delta}))$. Let X denote the intersection of AB and CD. We use the power of a point from X.

We have $|\frac{u}{\delta} + \frac{1}{2}| < \sqrt{2}(1-\delta)^{1/2}$ and $|\frac{u+1-\delta}{\delta} + \frac{1}{2}| < \sqrt{2}(1-\delta)^{1/2}$ due to the assumption that $2\sqrt{2}\delta - \sqrt{2} > (1-\delta)^{1/2}$ and the fact that $|u+\frac{1}{2}| < \frac{(1-\delta)^{1/2}}{\sqrt{2}}$. Hence points C and D are both above the horizontal line $v_\delta(-\frac{1}{2} - (1-\delta)^{1/2})$. Consequently $XB > \frac{(1-\delta)^{1/2}}{\sqrt{2}}$.

Note that points C and D either lie on the circle or lie outside the circle, so the power of a point implies that $AX \cdot XB \le CX \cdot XD$. Since the slope of CD is at most 1, we deduce that

$$AX \le \frac{CX \cdot XD}{XB} \le \frac{(u - \frac{u}{\delta})(\frac{u+1-\delta}{\delta})}{\frac{(1-\delta)^{1/2}}{\sqrt{2}}} \le \frac{\sqrt{2}}{4\delta^2}(1-\delta)^{3/2}.$$

We finish with the observation that $AX = v_\delta(u) - \frac{\frac{u+1-\delta}{\delta} - u}{\frac{u+1-\delta}{\delta} - \frac{u}{\delta}} v_\delta(\frac{u}{\delta}) - \frac{u - \frac{u}{\delta}}{\frac{u+1-\delta}{\delta} - \frac{u}{\delta}} v_\delta(\frac{u+1-\delta}{\delta})$. □

Corollary C.7. *Suppose* $c > \frac{\sqrt{2}}{4\delta} + \frac{1}{\sqrt{2}}$. *Suppose* $2\sqrt{2}\delta - \sqrt{2} > (1-\delta)^{1/2}$. *Then for all* u *such that* $|u + \frac{1}{2}| < \frac{(1-\delta)^{1/2}}{\sqrt{2}}$ *we have*

$$v_\delta(u) - \frac{v_\delta(\frac{u}{\delta}) + v_\delta(\frac{u+1-\delta}{\delta})}{2} \leq \frac{1-\delta}{\delta}(-v_\delta(u)).$$

Proof. We have $\frac{1-\delta}{\delta}(-v_\delta(u)) \geq \frac{1-\delta}{\delta}c(1-\delta)^{1/2}$. When $c > \frac{\sqrt{2}}{4\delta} + \frac{1}{\sqrt{2}}$, we have $\frac{1-\delta}{\delta}c(1-\delta)^{1/2} > \frac{\sqrt{2}}{4\delta^2}(1-\delta)^{3/2} + \frac{(1-\delta)^{3/2}}{\delta\sqrt{2}}$. Result follows from Lemma C.6. □

Theorem C.8. *If* $\delta > 0.695$ *and* $c > 1.21582$, *then we have* $v_\delta(u) \leq w_\delta(u)$ *for all* u. *In particular we have* $w_\delta(-\frac{1}{2}) \geq -c(1-\delta)^{1/2}$.

Proof. These values of δ and c satisfy the conditions for Lemma C.3, Lemma C.4, and Corollary C.7. Thus for all u we have $v_\delta(u) - \frac{v_\delta(\frac{u}{\delta})+v_\delta(\frac{u+1-\delta}{\delta})}{2} \leq \frac{1-\delta}{\delta}(-v_\delta(u))$, which rearranges into

$$v_\delta(u) \leq \frac{\delta}{2} \cdot v_\delta(\frac{u}{\delta}) + \frac{\delta}{2} \cdot v_\delta(\frac{u+1-\delta}{\delta}).$$

Lemma C.3 implies that $v_\delta(u) \leq w_\delta(u)$ for all u. □

C.2.2 The $\Omega(1-\delta)^{1/2}$ bound

We construct a function v_δ such that $v_\delta(u) \geq w_\delta(u)$ for all u and $v_\delta(-\frac{1}{2}) = -c(1-\delta)^{1/2}$ for some c.

Pick a $c < \sqrt{2} - 1$. Let $w(u) = \min\{u + \frac{1}{2}, -u - \frac{1}{2}\}$ denote the first-best. Define

$$v_\delta(u) = \begin{cases} \min\{w(u), \sqrt{1-\delta-(u+\frac{1}{2})^2} - (c+1)(1-\delta)^{1/2}\} & \text{if } |u+\frac{1}{2}| < \frac{(1-\delta)^{1/2}}{\sqrt{2}} \\ w(u) & \text{otherwise} \end{cases}$$

Intuitively we draw a circle centered at $(-\frac{1}{2}, -(c+1)(1-\delta)^{1/2})$ with radius $(1-\delta)^{1/2}$. Consider only the upper semi-circle. If $c < \sqrt{2} - 1$, this semi-circle intercepts $w(u)$, and we take the arc that's below $w(u)$. Figure 4 illustrates $v_\delta(u)$.

Assumptions:

- $c < \sqrt{2} - 1$
- $c < \frac{\delta}{2\sqrt{2}} - \frac{(1-\delta)^{1/2}}{4}$
- $4c^2 - \frac{4\delta}{1-\delta}c < \frac{1}{4}$.

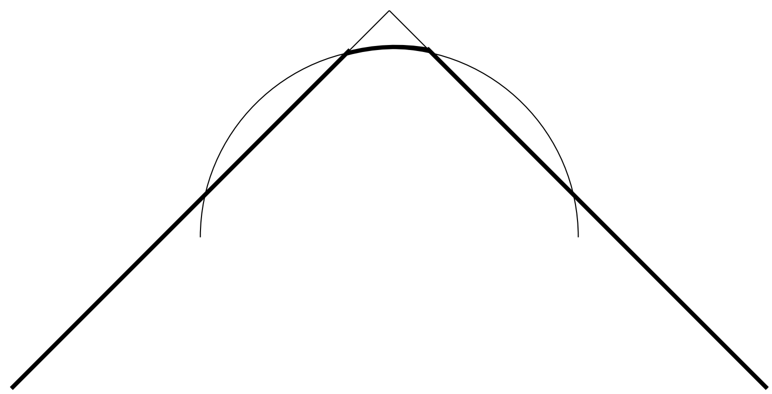

Figure 4: $v_\delta(u)$ is in bold

We claim that under these assumptions we have

$$v_\delta(u) \geq \frac{\delta}{2} \cdot v_\delta(\frac{u}{\delta}) + \frac{\delta}{2} \cdot v_\delta(\frac{u+1-\delta}{\delta}).$$

Then it follows from Lemma C.3 that $v_\delta(u) \geq w_\delta(u)$.

Lemma C.9. *If* $v_\delta(u) = w(u)$, *then* $v_\delta(u) \geq \frac{\delta}{2} \cdot v_\delta(\frac{u}{\delta}) + \frac{\delta}{2} \cdot v_\delta(\frac{u+1-\delta}{\delta})$.

Proof. Assume $u < -\frac{1}{2}$. The other case follows analogously. We have $v_\delta(u) = u + \frac{1}{2} = \frac{\delta}{2}(\frac{u}{\delta} + \frac{1}{2}) + \frac{\delta}{2}(\frac{u+1-\delta}{\delta} + \frac{1}{2}) \geq \frac{\delta}{2} \cdot v_\delta(\frac{u}{\delta}) + \frac{\delta}{2} \cdot v_\delta(\frac{u+1-\delta}{\delta})$. □

Next we prove the inequality for the u for which v_δ is on the arc. We rewrite the inequality $v_\delta(u) \geq \frac{\delta}{2} v_\delta(\frac{u}{\delta}) + \frac{\delta}{2} v_\delta(\frac{u+1-\delta}{\delta})$ as follows:

$$v_\delta(u) - \frac{v_\delta(\frac{u}{\delta}) + v_\delta(\frac{u+1-\delta}{\delta})}{2} \geq \frac{1-\delta}{\delta}(-v_\delta(u)).$$

We claim that the LHS is at least $\frac{\frac{1}{4} - 4c^2(1-\delta)}{2\delta^2}(1-\delta)^{3/2}$, and the RHS is at most $\frac{2c}{\delta}(1-\delta)^{3/2}$. The inequality then follows from the assumption that $4c^2 - \frac{4\delta}{1-\delta}c < \frac{1}{4}$.

We first prove the upper bound for $\frac{1-\delta}{\delta}(-v_\delta(u))$. The next lemma provides a bound for all u for which $v_\delta(u)$ is on the arc.

Lemma C.10. *If* $v_\delta(u) < w(u)$, *then* $|u + \frac{1}{2}| < 2c(1-\delta)^{1/2}$.

Proof. Suppose $w(u)$ intersects the semi-circle at $u = -\frac{1}{2} - x(1-\delta)^{1/2}$. Then $\frac{x}{c}$ is maximized when $c = \sqrt{2} - 1$, which is when the circle is tangent to $w(u)$. At this point we have $x = \frac{1}{2-\sqrt{2}} < 2$. If $v_\delta(u) < w(u)$, then $|u + \frac{1}{2}| < x(1-\delta)^{1/2} < 2c(1-\delta)^{1/2}$. □

Corollary C.11. *If* $v_\delta(u) < w(u)$*, then* $\frac{1-\delta}{\delta}(-v_\delta(u)) < \frac{1-\delta}{\delta} \cdot 2c(1-\delta)^{1/2}$.

Proof. We only need to prove the bound for the two intersection points of $w(u)$ and the semi-circle. Let u be an intersection. Then $-v_\delta(u) = |u+\frac{1}{2}|$, which is less than $2c(1-\delta)^{1/2}$. Hence $\frac{1-\delta}{\delta}(-v_\delta(u)) < \frac{1-\delta}{\delta} \cdot 2c(1-\delta)^{1/2}$. □

We next prove the lower bound for $v_\delta(u) - \frac{v_\delta(\frac{u}{\delta})+v_\delta(\frac{u+1-\delta}{\delta})}{2}$, where $|u+\frac{1}{2}| < 2c(1-\delta)^{1/2}$.

Lemma C.12. *For all* u *we have*

$$v_\delta(u) - \frac{v_\delta(\frac{u}{\delta}) + v_\delta(\frac{u+1-\delta}{\delta})}{2} \geq v_\delta(u) - \frac{\frac{u+1-\delta}{\delta} - u}{\frac{u+1-\delta}{\delta} - \frac{u}{\delta}} v_\delta(\frac{u}{\delta}) - \frac{u - \frac{u}{\delta}}{\frac{u+1-\delta}{\delta} - \frac{u}{\delta}} v_\delta(\frac{u+1-\delta}{\delta}).$$

Proof. Assume $u < -\frac{1}{2}$. The other case follows analogously. We have $\frac{u+1-\delta}{\delta} - u < u - \frac{u}{\delta}$. Since v_δ is concave and symmetric about $-\frac{1}{2}$, we have $v_\delta(\frac{u}{\delta}) \leq v_\delta(\frac{u+1-\delta}{\delta})$. Moreover since $\frac{u-\frac{u}{\delta}}{\frac{u+1-\delta}{\delta} - \frac{u}{\delta}} > \frac{1}{2}$, we have

$$\frac{1}{2} v_\delta(\frac{u}{\delta}) + \frac{1}{2} v_\delta(\frac{u+1-\delta}{\delta}) \leq \frac{\frac{u+1-\delta}{\delta} - u}{\frac{u+1-\delta}{\delta} - \frac{u}{\delta}} v_\delta(\frac{u}{\delta}) + \frac{u - \frac{u}{\delta}}{\frac{u+1-\delta}{\delta} - \frac{u}{\delta}} v_\delta(\frac{u+1-\delta}{\delta}),$$

which establishes the inequality in the lemma. □

Lemma C.13. *Suppose* $c < \frac{\delta}{2} - \frac{(1-\delta)^{1/2}}{4}$*. If* $v_\delta(u) < w(u)$*, then*

$$v_\delta(u) - \frac{\frac{u+1-\delta}{\delta} - u}{\frac{u+1-\delta}{\delta} - \frac{u}{\delta}} v_\delta(\frac{u}{\delta}) - \frac{u - \frac{u}{\delta}}{\frac{u+1-\delta}{\delta} - \frac{u}{\delta}} v_\delta(\frac{u+1-\delta}{\delta}) \geq \frac{\frac{1}{4} - 4c^2(1-\delta)}{2\delta^2}(1-\delta)^{3/2}.$$

Proof. Let A denote point $(u, v_\delta(u))$. Let $B = (u, -\sqrt{1-\delta-(u+\frac{1}{2})^{1/2}} - (c+1)(1-\delta)^{1/2})$. Let $C = (\frac{u}{\delta}, v_\delta(\frac{u}{\delta}))$ and $D = (\frac{u+1-\delta}{\delta}, v_\delta(\frac{u+1-\delta}{\delta}))$. Let X denote the intersection of AB and CD.

Points A and B lie on the circle with radius $(1-\delta)^{1/2}$ and center $(-\frac{1}{2}, -(c+1)(1-\delta)^{1/2})$. Moreover points C and D either lies on the circle or inside the circle. Indeed we know from Lemma C.10 that $|u+\frac{1}{2}| < 2c(1-\delta)^{1/2}$. Together with $c < \frac{\delta}{2} - \frac{(1-\delta)^{1/2}}{4}$ we obtain that $|\frac{u}{\delta} + \frac{1}{2}| < \frac{(1-\delta)^{1/2}}{\sqrt{2}}$ and $|\frac{u+1-\delta}{\delta} + \frac{1}{2}| < \frac{(1-\delta)^{1/2}}{\sqrt{2}}$.

Power of a point implies that $AX \cdot XB \geq CX \cdot XD$. We have $AX = v_\delta(u) - \frac{\frac{u+1-\delta}{\delta} - u}{\frac{u+1-\delta}{\delta} - \frac{u}{\delta}} v_\delta(\frac{u}{\delta}) - \frac{u-\frac{u}{\delta}}{\frac{u+1-\delta}{\delta} - \frac{u}{\delta}} v_\delta(\frac{u+1-\delta}{\delta})$. We have $BX < 2(1-\delta)^{1/2}$. We have $CX > u - \frac{u}{\delta}$ and $DX > \frac{u+1-\delta}{\delta}$. Hence

$$AX \geq \frac{CX \cdot DX}{BX} > \frac{(u - \frac{u}{\delta})(\frac{u+1-\delta}{\delta} - u)}{2(1-\delta)^{1/2}}.$$

The RHS simplifies to $\frac{(-u)(u+1)}{2\delta^2}(1-\delta)^{3/2}$. We know from Lemma C.10 that $|u+\frac{1}{2}| < 2c(1-\delta)^{1/2}$, which means $(-u)(u+1) > (-\frac{1}{2} - 2c(1-\delta)^{1/2})(-\frac{1}{2} + 2c(1-\delta)^{1/2})$. We deduce that $AX > \frac{\frac{1}{4}-4c^2(1-\delta)}{2\delta^2}(1-\delta)^{3/2}$. □

Corollary C.14. *Suppose* $c < \frac{\delta}{2\sqrt{2}} - \frac{(1-\delta)^{1/2}}{4}$. *In addition suppose* $4c^2 - \frac{4\delta}{1-\delta}c < \frac{1}{4}$. *If* $v_\delta(u) < w(u)$, *then*

$$v_\delta(u) - \frac{v_\delta(\frac{u}{\delta}) + v_\delta(\frac{u+1-\delta}{\delta})}{2} \geq \frac{1-\delta}{\delta}(-v_\delta(u)).$$

Proof. Lemma C.12 and Lemma C.13 imply that the LHS is at least $\frac{\frac{1}{4}-4c^2(1-\delta)}{2\delta^2}(1-\delta)^{3/2}$. Corollary C.11 says the RHS is at most $\frac{1-\delta}{\delta}2c(1-\delta)^{1/2}$. Inequality follows from the assumption that $4c^2 - \frac{4\delta}{1-\delta}c < \frac{1}{4}$. □

Theorem C.15. *If* $\delta > 0.695$ *and* $c < 0.107653$, *we have* $v_\delta(u) \geq w_\delta(u)$ *for all* u. *In particular we have* $w_\delta(-\frac{1}{2}) \leq -c(1-\delta)^{1/2}$.

Proof. These values of δ and c satisfy the conditions for Lemma C.3, Lemma C.9, and Corollary C.14 imply that for all u we have $v_\delta(u) - \frac{v_\delta(\frac{u}{\delta})+v_\delta(\frac{u+1-\delta}{\delta})}{2} \geq \frac{1-\delta}{\delta}(-v_\delta(u))$, which is equivalent to

$$v_\delta(u) \geq \frac{\delta}{2} \cdot v_\delta(\frac{u}{\delta}) + \frac{\delta}{2} \cdot v_\delta(\frac{u+1-\delta}{\delta}).$$

It follows from Lemma C.3 that $v_\delta(u) \geq w_\delta(u)$. □

D Proofs of Section 6

D.1 Proof of Theorem 6.1

We first prove that $\lim_{\delta\to 1} w_\delta$ exists. We then prove that the limit is equal to w.

Lemma D.1. *If* $\delta' > \delta$, *then* $w_{\delta'}(u) \geq w_\delta(u)$ *for all* $u \in (\underline{u}, \overline{u})$.

Proof. Fix u. Let $a_{\theta,x}$ and $u_{\theta,x}$ denote the mechanism that gives payoff $w_\delta(u)$. Then under δ' the receiver could implement the same actions $a_{\theta,x}$ using $\frac{\delta'-\delta}{1-\delta}u + \frac{1-\delta'}{1-\delta}u_{\theta,x}$. Indeed if $\theta \in \text{argmax}_{\theta'}(1-\delta)u_S(a_{\theta',x},\theta) + \delta u_{\theta',x}$, then

$$\theta \in \text{argmax}_{\theta'}(1-\delta') \cdot u_S(a_{\theta',x},\theta) + \delta' \cdot (\frac{\delta'-\delta}{1-\delta}u + \frac{1-\delta'}{1-\delta}u_{\theta,x}).$$

Furthermore since both u and $u_{\theta,x}$ are in the domain of $[\underline{u},\overline{u}]$, we have $\frac{\delta'-\delta}{1-\delta}u+\frac{1-\delta'}{1-\delta}u_{\theta,x}$ is also in $[\underline{u},\overline{u}]$. Concavity of $w_{\delta'}$ implies that

$$w_{\delta'}(\frac{\delta'-\delta}{1-\delta}u+\frac{1-\delta'}{1-\delta}u_{\theta,x}) \geq \frac{\delta'-\delta}{1-\delta}w_{\delta'}(u)+\frac{1-\delta'}{1-\delta}w_{\delta'}(u_{\theta,x}).$$

We deduce that

$$\begin{aligned} w_{\delta'}(u) &\geq \mathbb{E}_{\theta,x}(1-\delta)\cdot u_R(a_{\theta,x},\theta)+\delta\cdot w_{\delta'}(\frac{\delta'-\delta}{1-\delta}u+\frac{1-\delta'}{1-\delta}u_{\theta,x}) \\ &\geq \mathbb{E}_{\theta,x}(1-\delta)\cdot u_R(a_{\theta,x},\theta)+\delta\cdot\left(\frac{\delta'-\delta}{1-\delta}w_{\delta'}(u)+\frac{1-\delta'}{1-\delta}w_{\delta'}(u_{\theta,x})\right), \end{aligned}$$

which rearranges into

$$w_{\delta'}(u)\geq \mathbb{E}_{\theta,x}[(1-\delta)\cdot u_R(a_{\theta,x},\theta)+\delta\cdot w_{\delta'}(u_{\theta,x})].$$

To finish the proof recall that $w_\delta(u)=\mathbb{E}_{\theta,x}[(1-\delta)u_R(a_{\theta,x},\theta)+\delta w_\delta(u_{\theta,x})]$. We obtain that

$$w_{\delta'}(u)-w_\delta(u)\geq\delta\cdot\mathbb{E}_{\theta,x}[w_{\delta'}(u_{\theta,x})-w_\delta(u_{\theta,x})].$$

Let $K=\min_u w_{\delta'}(u)-w_\delta(u)$. Since the domain of u is bounded, and both $w_{\delta'}$ and w_δ are continuous, there exists a minimizer u. Suppose $K<0$. Then there exists a $u_{\theta,x}$ such that $w_{\delta'}(u_{\theta,x})-w_\delta(u_{\theta,x})\leq\frac{K}{\delta}<K$. Contradiction. Therefore we must have $K\geq 0$, which means $w_{\delta'}(u)\geq w_\delta(u)$ for all u. □

Corollary D.2. *We have* $\lim_{\delta\to1}w_\delta(u)$ *exists.*

Lemma D.3. *Suppose the limit of* w_δ *is differentiable at* u*. Then* $\frac{w_\delta(u^*_{\theta,x})-w_\delta(u)}{u^*_{\theta,x}-u}$ *converges to the derivative* $\frac{d}{du}\lim_{\delta\to1}w_\delta(u)$*.*

Proof. We claim that for any $\Delta>0$ if δ is close enough to 1, then

$$\frac{d}{du}\lim_{\delta\to1}w_\delta(u)-\Delta\leq\frac{w_\delta(u^*_{\theta,x})-w_\delta(u)}{u^*_{\theta,x}-u}\leq\frac{d}{du}\lim_{\delta\to1}w_\delta(u)+\Delta.$$

Since $\lim_{\delta\to1}w_\delta(u)$ is concave and differentiable at u, there exists an $\epsilon>0$ such that

$$\frac{d}{du}\lim_{\delta\to1}w_\delta(u)-\Delta<\lim_{\delta\to1}\frac{w_\delta(u+\epsilon)-w_\delta(u)}{\epsilon}\leq\lim_{\delta\to1}\frac{w_\delta(u)-w_\delta(u-\epsilon)}{\epsilon}<\frac{d}{du}\lim_{\delta\to1}w_\delta(u)+\Delta.$$

Consider δ close to 1 such that $u-\epsilon<u^*_{\theta,x}<u+\epsilon$ (this inequality holds for δ sufficiently

close to 1 because $(u^*_{\theta,x} - u)\frac{\delta}{1-\delta}$ is bounded by assumption). Since w_δ is concave, we have

$$\frac{w_\delta(u+\epsilon) - w_\delta(u)}{\epsilon} \leq \frac{w_\delta(u^*_{\theta,x}) - w_\delta(u)}{u^*_{\theta,x} - u} \leq \frac{w_\delta(u) - w_\delta(u-\epsilon)}{\epsilon}.$$

Taking $\Delta \to 0$ and $\delta \to 1$ finishes the proof. □

Proof of Theorem 6.1. Since w_δ is concave, the limit of w_δ is also concave and is therefore differentiable almost everywhere. Take a u such that $\frac{d}{du}\lim_{\delta\to 1} w_\delta(u)$ exists. We have $w_\delta(u)$ is at least the payoff from mechanism $\texttt{GREEDY}_\delta(u)$:

$$w_\delta(u) \geq \mathbb{E}_{\theta,x}[(1-\delta)\cdot u_R(a^*_{\theta,x}, \theta) + \delta \cdot w_\delta(u^*_{\theta,x})].$$

Divide by $(1-\delta)$ and rearrange the terms:

$$\begin{aligned} w_\delta(u) \quad &\geq \quad \mathbb{E}_{\theta,x}\left[u_R(a^*_{\theta,x}, \theta) + \frac{\delta}{1-\delta}\cdot [w_\delta(u^*_{\theta,x}) - w_\delta(u)]\right] \\ &= \quad w(u) + \mathbb{E}_{\theta,x}\frac{\delta}{1-\delta}(u^*_{\theta,x} - u)\cdot \frac{w_\delta(u^*_{\theta,x}) - w_\delta(u)}{u^*_{\theta,x} - u}. \end{aligned}$$

We are left to show that the second term goes to 0. We know from (6.1) that $\frac{\delta}{1-\delta}(u^*_{\theta,x} - u)$ is independent of δ and is bounded. The term $\frac{w_\delta(u^*_{\theta,x}) - w_\delta(u)}{u^*_{\theta,x} - u}$ converges to $\frac{d}{du}\lim_{\delta\to 1} w_\delta(u)$ by Lemma D.3. Hence the second term on the second line converges to

$$\lim_{\delta\to 1}\mathbb{E}_{\theta,x}\left[\frac{\delta}{1-\delta}(u^*_{\theta,x} - u)\cdot \frac{w_\delta(u^*_{\theta,x}) - w_\delta(u)}{u^*_{\theta,x} - u}\right] = \left(\frac{d}{du}\lim_{\delta\to 1} w_\delta(u)\right)\cdot \mathbb{E}_{\theta,x}\frac{\delta}{1-\delta}(u^*_{\theta,x} - u).$$

By construction we have $\mathbb{E}_{\theta,x} u^*_{\theta,x} = u$. Indeed the mechanism $\texttt{GREEDY}_\delta(u)$ gives the sender utility u in the first period, so the continuation payoffs in expectation must equal to u as well. We deduce that $\mathbb{E}_{\theta,x}\frac{\delta}{1-\delta}(u^*_{\theta,x} - u) = 0$. As a result we have $\lim_{\delta\to 1} w_\delta(u) \geq w(u)$, but we know that $w(u)$ is the upper bound, so we must have $\lim_{\delta\to 1} w_\delta(u) = w(u)$. □

D.2 Proof of Theorem 6.4

D.2.1 The $o(1-\delta)$ and $O(\sqrt{1-\delta})$ bounds

Let $D = \max_{a,\theta} u_S(a, \theta)$. For all $u \in (\underline{u}, \overline{u})$ we have

$$w_\delta(u) \geq (1-\delta)\cdot w(u) + \frac{\delta}{2}\cdot w(u - \frac{1-\delta}{\delta}D) + \frac{\delta}{2}\cdot w(u + \frac{1-\delta}{\delta}D) \qquad \text{(D.1)}$$

whenever δ is close to 1 such that both $u - \frac{1-\delta}{\delta}D$ and $u + \frac{1-\delta}{\delta}D$ are in the domain $[\underline{u}, \overline{u}]$. Indeed the mechanism $\texttt{GREEDY}_\delta(u)$ gives a payoff $(1-\delta)\cdot w(u) + \delta \cdot \mathbb{E}_{\theta,x} w_\delta(u^*_{\theta,x})$. Since $\mathbb{E}_{\theta,x} u^*_{\theta,x} = u$ and $|u - u^*_{\theta,x}| \leq D$, the concavity of w_δ implies that $\mathbb{E}_{\theta,x} w_\delta(u^*_{\theta,x}) \geq \frac{1}{2}\cdot w(u - \frac{1-\delta}{\delta}D) + \frac{1}{2}\cdot w(u + \frac{1-\delta}{\delta}D)$.

Lemma D.4. *If f is a continuous and non-decreasing function, then*

$$\int_a^b \frac{f(x+\epsilon)-f(x)}{\epsilon}dx \leq f(b+\epsilon) - f(a).$$

Proof. Since f is non-decreasing, we have

$$\int_a^b (f(x+\epsilon)-f(x))dx = \epsilon(f(b+\epsilon)-f(a)) - \int_a^{a+\epsilon}(f(x)-f(a))dx - \int_b^{b+\epsilon}(f(b+\epsilon)-f(x))dx.$$

The last two terms on the right are non-negative. □

Lemma D.5. *Suppose $\underline{u} < u_0 < u_1 < \overline{u}$, and $w'(u_0)$ and $w'(u_1)$ both exist. We have*

$$\limsup_{\delta\to 1} \int_{u_0}^{u_1} \frac{w(u)-w_\delta(u)}{1-\delta}du \leq D^2(w'(u_0) - w'(u_1)).$$

Proof. Equation (D.1) implies that

$$w_\delta(u) \geq w(u) + \frac{\delta}{1-\delta}\cdot\left(\frac{w_\delta(u-\frac{1-\delta}{\delta}D) - w_\delta(u)}{2} + \frac{w_\delta(u+\frac{1-\delta}{\delta}D) - w_\delta(u)}{2}\right).$$

We deduce that

$$\begin{aligned} w(u) - w_\delta(u) &\leq \frac{\delta}{1-\delta}\cdot\left(\frac{w_\delta(u) - w_\delta(u-\frac{1-\delta}{\delta}D)}{2} - \frac{w_\delta(u+\frac{1-\delta}{\delta}D) - w_\delta(u)}{2}\right) \\ &= \frac{D}{2}\cdot\left(\frac{w_\delta(u) - w_\delta(u-\frac{1-\delta}{\delta}D)}{u-(u-\frac{1-\delta}{\delta}D)} - \frac{w_\delta(u+\frac{1-\delta}{\delta}D) - w_\delta(u)}{(u+\frac{1-\delta}{\delta}D)-D}\right) \\ &\leq \frac{D}{2}\cdot\left(w'_\delta(u-\frac{1-\delta}{\delta}D) - w'_\delta(u+\frac{1-\delta}{\delta}D)\right). \end{aligned}$$

(We use w'_δ as the left derivative of w_δ, which exists because w is concave.)

We obtain that

$$\int_{u_0}^{u_1}(w(u)-w_\delta(u))du \leq \int_{u_0}^{u_1}\frac{D}{2}\cdot\left(w'_\delta(u-\frac{1-\delta}{\delta}D) - w'_\delta(u+\frac{1-\delta}{\delta}D)\right)du.$$

Since w'_δ is monotonic, Lemma D.4 implies that

$$\int_{u_0}^{u_1}(w(u)-w_\delta(u))du \le \frac{D}{2}\cdot 2\frac{1-\delta}{\delta}D\cdot\left(w'_\delta(u_0-\frac{1-\delta}{\delta}D)-w'_\delta(u_1+\frac{1-\delta}{\delta}D)\right).$$

Lemma D.3 implies that $w'_\delta(u_0-\frac{1-\delta}{\delta}D)-w'_\delta(u_1+\frac{1-\delta}{\delta}D)$ converges to $w'(u_0)-w'(u_1)$. □

Proof of Theorem 6.4: the $o(1-\delta)$ and $O(\sqrt{1-\delta})$ bounds. If $w''(u)=0$, then there exists a neighborhood (u_0,u_1) such that $w'(u_0)=w'(u_1)$. The right hand side of Lemma D.5 becomes 0. Hence if $w(u)$ is locally linear, then the rate of convergence is $o(1-\delta)$.

Now let's prove the $O(\sqrt{1-\delta})$ bound. First consider the case when $u\in(\underline{u},\overline{u})$. Take any neighborhood (u_0,u_1) of u such that $w'(u_0)$ and $w'(u_1)$ both exist. Let $\alpha=\frac{w(u)-w(u_0)}{u-u_0}$. Let c be a constant such that $\frac{c^2}{2\alpha}>D^2(w'(u_0)-w'(u_1))$. For δ close to 1 there exists a $\widehat{u}\in[u-\frac{c}{\alpha}\sqrt{1-\delta},u]$ such that $w_\delta(\widehat{u})>w(u)-c\sqrt{1-\delta}$; otherwise

$$\int_{u-\frac{c}{\alpha}\sqrt{1-\delta}}^{u}[w(\widehat{u})-w_\delta(\widehat{u})]d\widehat{u}\ge\int_{u-\frac{c}{\alpha}\sqrt{1-\delta}}^{u}([w(u)-\alpha(u-\widehat{u})]-[w(u)-c\sqrt{1-\delta}]d\widehat{u}=\frac{c^2}{2\alpha}(1-\delta),$$

which is more than $D^2(w'(u_0)-w'(u_1)(1-\delta)$, contradicting Lemma D.5. Next we take a $\widehat{u}\in[u-\frac{c}{\alpha}\sqrt{1-\delta},u]$ such that $w_\delta(\widehat{u})>w(u)-c\sqrt{1-\delta}$, and the concavity of w_δ means $w_\delta(u)\ge\frac{u_1-u}{u_1-\widehat{u}}w_\delta(\widehat{u})+\frac{u-\widehat{u}}{u_1-\widehat{u}}w_\delta(u_1)$. Consequently we have

$$w(u)-w_\delta(u)\le\frac{u_1-u}{u_1-\widehat{u}}c\sqrt{1-\delta}+\frac{\frac{c}{\alpha}\sqrt{1-\delta}}{u_1-\widehat{u}}(w(u)-w_\delta(u_1)),$$

and the right hand side is on the order of $\sqrt{1-\delta}$. □

D.2.2 The $O(1-\delta)$ bound

Suppose there exists a $\widehat{u}$ such that $\limsup_{\delta\to1}\frac{w(\widehat{u})-w_\delta(\widehat{u})}{1-\delta}=+\infty$. We show that there is a neighborhood $[u_0,u_1]$ of $\widehat{u}$ such that $\limsup_{\delta\to1}\int_{u_0}^{u_1}\frac{w(u)-w_\delta(u)}{1-\delta}du=+\infty$, which contradicts Lemma D.5.

Take a neighborhood $[u_0,u_1]$ of $\widehat{u}$ for which $w(u)$ is locally Lipschitz continuous. Let K denote the Lipschitz constant on $[u_0,u_1]$. Let δ be close to 1 such that both $u_0-\frac{1-\delta}{\delta}2D$ and $u_1+\frac{1-\delta}{\delta}2D$ are in the domain $[\underline{u},\overline{u}]$. Moreover choose u_0 and u_1 such that $\widehat{u}=\frac{u_0+u_1}{2}$.

Lemma D.6. *For all $C\in[D,2D]$ and for all $u\in[u_0+\frac{1-\delta}{\delta}C,u_1-\frac{1-\delta}{\delta}C]$ we have*

$$\frac{w(u)-w_\delta(u)}{1-\delta}\le\frac{1-\delta}{\delta}C^2K+\frac{\delta}{2}\frac{w(u-\frac{1-\delta}{\delta}C)-w_\delta(u-\frac{1-\delta}{\delta}C)}{1-\delta}+\frac{\delta}{2}\frac{w(u+\frac{1-\delta}{\delta}C)-w_\delta(u+\frac{1-\delta}{\delta}C)}{1-\delta}.$$

Proof. Since $C \geq D$, we deduce from Equation (D.1) that

$$
\begin{aligned}
w_\delta(u) &\geq (1-\delta)\cdot w(u) + \delta\cdot\left(\frac{w_\delta(u-\frac{1-\delta}{\delta}D)}{2} + \frac{w_\delta(u+\frac{1-\delta}{\delta}D)}{2}\right) \\
&\geq (1-\delta)\cdot w(u) + \delta\cdot\left(\frac{w_\delta(u-\frac{1-\delta}{\delta}C)}{2} + \frac{w_\delta(u+\frac{1-\delta}{\delta}C)}{2}\right).
\end{aligned}
$$

Rearrange the terms:

$$
\begin{aligned}
w(u) - w_\delta(u) &\leq \delta\cdot w(u) - \delta\cdot\left(\frac{w_\delta(u-\frac{1-\delta}{\delta}C)}{2} + \frac{w_\delta(u+\frac{1-\delta}{\delta}C)}{2}\right) \\
&= \delta\cdot\left(w(u) - \frac{w(u-\frac{1-\delta}{\delta}C)}{2} - \frac{w(u+\frac{1-\delta}{\delta}C)}{2}\right) + \\
&\quad \delta\cdot\left(\frac{w(u-\frac{1-\delta}{\delta}C) - w_\delta(u-\frac{1-\delta}{\delta}C)}{2} + \frac{w(u+\frac{1-\delta}{\delta}C) - w_\delta(u+\frac{1-\delta}{\delta}C)}{2}\right).
\end{aligned}
$$

We obtain that

$$
\begin{aligned}
\frac{w(u) - w_\delta(u)}{1-\delta} &\leq \frac{\delta}{2}\cdot\left(\frac{w(u) - w(u-\frac{1-\delta}{\delta}C)}{1-\delta} + \frac{w(u) - w(u+\frac{1-\delta}{\delta}C)}{1-\delta}\right) + \\
&\quad \frac{\delta}{2}\cdot\frac{w(u-\frac{1-\delta}{\delta}C) - w_\delta(u-\frac{1-\delta}{\delta}C)}{1-\delta} + \frac{\delta}{2}\cdot\frac{w(u+\frac{1-\delta}{\delta}C) - w_\delta(u+\frac{1-\delta}{\delta}C)}{1-\delta}
\end{aligned}
$$

We are left to show that $\frac{\delta}{2}\cdot\left(\frac{w(u)-w(u-\frac{1-\delta}{\delta}C)}{1-\delta} + \frac{w(u)-w(u+\frac{1-\delta}{\delta}C)}{1-\delta}\right) \leq \frac{1-\delta}{\delta}C^2K$. We have

$$
\begin{aligned}
&\frac{\delta}{2}\cdot\left(\frac{w(u) - w(u-\frac{1-\delta}{\delta}C)}{1-\delta} + \frac{w(u) - w(u+\frac{1-\delta}{\delta}C)}{1-\delta}\right) \\
&= \frac{C}{2}\cdot\left(\frac{w(u) - w(u-\frac{1-\delta}{\delta}C)}{u-(u-\frac{1-\delta}{\delta}C)} - \frac{w(u+\frac{1-\delta}{\delta}C) - w(u)}{(u+\frac{1-\delta}{\delta}C)-u}\right) \leq \frac{C}{2}\left(w'(u-\frac{1-\delta}{\delta}C) - w'(u+\frac{1-\delta}{\delta}C)\right)
\end{aligned}
$$

Since w' is locally Lipschitz continuous on $[u_0, u_1]$ with constant K, we have

$$
\begin{aligned}
&\frac{C}{2}\left(w'(u-\frac{1-\delta}{\delta}C) - w'(u+\frac{1-\delta}{\delta}C)\right) = \\
&\qquad \frac{C}{2}2C\frac{1-\delta}{\delta}\cdot\frac{w'(u-\frac{1-\delta}{\delta}C) - w'(u+\frac{1-\delta}{\delta}C)}{(u+\frac{1-\delta}{\delta}C)-(u-\frac{1-\delta}{\delta}C)} \leq \frac{1-\delta}{\delta}C^2K.
\end{aligned}
$$

This inequality finishes the proof. ☐

Lemma D.7. *For all $C \in [D, 2D]$ and for all $u \in [u_0 + \frac{1-\delta}{\delta}C, u_1 - \frac{1-\delta}{\delta}C]$ if $\frac{w(u)-w_\delta(u)}{1-\delta} > \frac{C^2K}{\delta}$, then one of the following statements is true:*

- $\frac{w(u+\frac{1-\delta}{\delta}C\cdot n)-w_\delta(u+\frac{1-\delta}{\delta}C\cdot n)}{1-\delta} > \frac{w(u)-w_\delta(u)}{1-\delta}$ *for all positive integer n such that $u+\frac{1-\delta}{\delta}C\cdot n \leq u_1$;*
- $\frac{w(u-\frac{1-\delta}{\delta}C\cdot n)-w_\delta(u-\frac{1-\delta}{\delta}C\cdot n)}{1-\delta} > \frac{w(u)-w_\delta(u)}{1-\delta}$ *for all positive integer n such that $u-\frac{1-\delta}{\delta}C\cdot n \geq u_0$.*

Proof. We know from Lemma D.6 that

$$\frac{w(u)-w_\delta(u)}{1-\delta} \leq \frac{1-\delta}{\delta}C^2K+\frac{\delta}{2}\frac{w(u-\frac{1-\delta}{\delta}C)-w_\delta(u-\frac{1-\delta}{\delta}C)}{1-\delta}+\frac{\delta}{2}\frac{w(u+\frac{1-\delta}{\delta}C)-w_\delta(u+\frac{1-\delta}{\delta}C)}{1-\delta}.$$

If $\frac{w(u)-w_\delta(u)}{1-\delta} > \frac{C^2K}{\delta}$, then either $\frac{w(u-\frac{1-\delta}{\delta}C)-w_\delta(u-\frac{1-\delta}{\delta}C)}{1-\delta} > \frac{w(u)-w_\delta(u)}{1-\delta}$ (strict inequality) or $\frac{w(u+\frac{1-\delta}{\delta}C)-w_\delta(u+\frac{1-\delta}{\delta}C)}{1-\delta} > \frac{w(u)-w_\delta(u)}{1-\delta}$ (strict inequality).

Suppose $\frac{w(u-\frac{1-\delta}{\delta}C)-w_\delta(u-\frac{1-\delta}{\delta}C)}{1-\delta} > \frac{w(u)-w_\delta(u)}{1-\delta}$ (strict inequality); the other case follows analogously. We have $\frac{w(u-\frac{1-\delta}{\delta}C)-w_\delta(u-\frac{1-\delta}{\delta}C)}{1-\delta} > \frac{C^2K}{\delta}$, so by the same argument we have either $\frac{w(u-\frac{1-\delta}{\delta}2C)-w_\delta(u-\frac{1-\delta}{\delta}2C)}{1-\delta} > \frac{w(u-\frac{1-\delta}{\delta}C)-w_\delta(u-\frac{1-\delta}{\delta}C)}{1-\delta}$ (strict inequality), or we have $\frac{w(u)-w_\delta(u)}{1-\delta} > \frac{w(u-\frac{1-\delta}{\delta}C)-w_\delta(u-\frac{1-\delta}{\delta}C)}{1-\delta}$ (strict inequality). However we know that the second inequality is false, so we must have $\frac{w(u-\frac{1-\delta}{\delta}2C)-w_\delta(u-\frac{1-\delta}{\delta}2C)}{1-\delta} > \frac{w(u-\frac{1-\delta}{\delta}C)-w_\delta(u-\frac{1-\delta}{\delta}C)}{1-\delta}$. Apply the same argument for $u-\frac{1-\delta}{\delta}2C$ we get $\frac{w(u-\frac{1-\delta}{\delta}3C)-w_\delta(u-\frac{1-\delta}{\delta}3C)}{1-\delta} > \frac{w(u-\frac{1-\delta}{\delta}2C)-w_\delta(u-\frac{1-\delta}{\delta}2C)}{1-\delta}$. We can iteratively apply this argument as long as $u-\frac{1-\delta}{\delta}C\cdot n \geq u_0$. ☐

Lemma D.8. *If $\frac{w(\widehat{u})-w_\delta(\widehat{u})}{1-\delta} = M > \frac{(2D)^2K}{\delta}$, then*

$$\int_{u_0}^{u_1} \frac{w(u)-w_\delta(u)}{1-\delta}du > \frac{M}{2}(u_1-u_0-\frac{1-\delta}{\delta}2D).$$

Proof. For all $u \in [u_0, \widehat{u}-\frac{1-\delta}{\delta}D] \cup [\widehat{u}+\frac{1-\delta}{\delta}D, u_1]$ there exists a $C \in [D, 2D]$ such that $\frac{u-\widehat{u}}{\frac{1-\delta}{\delta}C}$ is an integer. Lemma D.7 implies that either $\frac{w(u)-w_\delta(u)}{1-\delta} > \frac{w(\widehat{u})-w_\delta(\widehat{u})}{1-\delta}$ or $\frac{w(2\widehat{u}-u)-w_\delta(2\widehat{u}-u)}{1-\delta} > \frac{w(\widehat{u})-w_\delta(\widehat{u})}{1-\delta}$. We deduce that

$$\frac{w(u)-w_\delta(u)}{1-\delta}+\frac{w(2\widehat{u}-u)-w_\delta(2\widehat{u}-u)}{1-\delta} > M$$

for all $u \in [u_0, \widehat{u}-\frac{1-\delta}{\delta}D] \cup [\widehat{u}+\frac{1-\delta}{\delta}D, u_1]$. The inequality follows trivially. ☐

Corollary D.9. *If* $\limsup_{\delta\to 1} \frac{w(\widehat{u})-w_\delta(\widehat{u})}{1-\delta} = +\infty$, *then* $\limsup_{\delta\to 1} \int_{u_0}^{u_1} \frac{w(u)-w_\delta(u)}{1-\delta} du = +\infty$.

Theorem 6.4 follows from Corollary D.9 and Lemma D.5.

D.3 Proof of Proposition 6.5

Proof of Proposition 6.5. In the proof of Theorem 4.2 only Lemma B.1 and Lemma B.3 assume that states and actions are one-dimensional, and that the sender's preference satisfies the single crossing condition. All other lemmas apply to the general model.

For Lemma B.1 replace the $z_{\theta,x}$ with $t_{\theta,x}$, and the lemma becomes Condition X. For Lemma B.3 replace the line $u_{\theta,x} = u - \frac{1-\delta}{\delta}\epsilon - \frac{U_x - \int_{\underline{x}}^{\overline{x}} U_k dk}{\delta} + \frac{1-\delta}{\delta} z_{\theta,x}$ with $u_{\theta,x} = u - \frac{1-\delta}{\delta}\epsilon + \frac{1-\delta}{\delta} t_{\theta,x}$. Condition X implies that

$$\begin{aligned}
\mathbb{E}[u_{\theta,x} | u_{\theta,x} < u - \frac{1-\delta}{\delta}\epsilon] &< u - \frac{1-\delta}{\epsilon} - \frac{1-\delta}{\delta} C \\
\mathbb{E}[u_{\theta,x} | u_{\theta,x} > u - \frac{1-\delta}{\delta}\epsilon] &> u - \frac{1-\delta}{\epsilon} + \frac{1-\delta}{\delta} C.
\end{aligned}$$

The rest of the proof exactly follows the proof of Theorem 4.2. □

E Proofs for Section 7

E.1 Proof of Proposition 7.2

Lemma E.1. *Let* $D = \max_{a,\theta} u_S(a,\theta)$. *Suppose* $u_1 \in (\underline{u} + \frac{1-\delta}{\delta} D, \overline{u})$. *We have*

$$\limsup_{\delta\to 1} \int_{\underline{u}}^{u_1} \frac{w(u) - w_\delta(u)}{1-\delta} du \le D w(\underline{u}).$$

Proof. From Lemma D.5 we have $\limsup_{\delta\to 1} \int_{\underline{u}+\frac{1-\delta}{\delta}D}^{u_1} \frac{w(u)-w_\delta(u)}{1-\delta} du \le D^2(w'(\underline{u} + \frac{1-\delta}{\delta} D) - w'(u_1)) = 0$. Since $w(\underline{u})$ is the maximal value of w, we have $\int_{\underline{u}}^{\underline{u}+\frac{1-\delta}{\delta}D} (w(u) - w_\delta(u)) du \le \frac{1-\delta}{\delta} D w(\underline{u})$, which means $\limsup_{\delta\to 1} \int_{\underline{u}}^{\underline{u}+\frac{1-\delta}{\delta}D} \frac{w(u)-w_\delta(u)}{1-\delta} du \le D w(\underline{u})$. □

Proof of Proposition 7.2. Let r_δ denote the distance $w(\underline{u}) - \max_u w_\delta(u)$. We know that $w(\underline{u})$ is the maximal point of w. We have $\int_{\underline{u}}^{\underline{u}+r_\delta} (w(u) - w_\delta(u)) du \ge \int_{\underline{u}}^{\underline{u}+r_\delta} r_\delta du = r_\delta^2$, which implies that

$$\int_{\underline{u}}^{\underline{u}+r_\delta} \frac{w(u) - w_\delta(u)}{1-\delta} du \ge \frac{r_\delta^2}{1-\delta}.$$

We know from Theorem 6.1 that $r_\delta \to 0$ as $\delta \to 1$. Fix $u_1 > \underline{u}$. Take δ close to 1 such that $\underline{u} + \frac{1-\delta}{\delta} D < u_1$ and $\underline{u} + r_\delta < u_1$. We have

$$\int_{\underline{u}}^{u_1} \frac{w(u) - w_\delta(u)}{1-\delta} du \geq \frac{r_\delta^2}{1-\delta}.$$

We deduce from Lemma E.1 that $\limsup_{\delta \to 1} \frac{r_\delta^2}{1-\delta} \leq Dw(\underline{u})$, which implies that r_δ is on the order of $O(\sqrt{1-\delta})$. $\square$

E.2 Proof of Proposition 7.6

Proof of Proposition 7.6. Let $a(h_t)$ denote the action taken in the greedy mechanism at history h_t. Since the greedy mechanism extracts full surplus for type $\overline{b}$, we have

$$\begin{aligned} \mathbb{E}_{h_t} - (a(h_t) - \theta_t - \overline{b})^2 &= -\overline{b}^2 \\ \mathbb{E}_{h_t} - (a(h_t) - \theta_t)^2 &= -\epsilon \end{aligned}$$

where $-\epsilon$ is arbitrarily close to 0 as δ goes to 1. These two equations imply that $2\overline{b} \cdot \mathbb{E}_{h_t}(a(h_t) - \theta_t) = \epsilon$. Consequently when type b participates in this greedy mechanism, if type b reports truthfully, her expected payoff is equal to

$$\mathbb{E}_{h_t} - (a(h_t) - \theta_t - b)^2 = -\epsilon - b^2 + 2b \cdot \mathbb{E}_{h_t}(a(h_t) - \theta_t) = -b^2 - \frac{\overline{b} - b}{\overline{b}} \epsilon.$$

Hence if type b misreports in this greedy mechanism for $\overline{b}$, type b must get an expected payoff at least $-b^2 - \frac{\overline{b}-b}{\overline{b}}\epsilon$.

Lemma E.2. *Suppose type b gets $-b^2 + K$ in some mechanism. Then type $\overline{b}$ could get at least $-\overline{b}^2 + \frac{\overline{b}}{b} K$ by mimicking type b's reporting strategy.*

Proof. Let $\tilde{a}(h_t)$ denote the action at history h_t (note that b may not report truthfully). We have

$$\mathbb{E}_{h_t} - (\tilde{a}(h_t) - \theta_t - b)^2 = -b^2 + K.$$

We deduce that $\mathbb{E}_{h_t} - (\tilde{a}(h_t) - \theta_t)^2 + 2b(\tilde{a}(h_t) - \theta_t) = K$.

Suppose type $\overline{b}$ uses exactly the same reporting strategy as type b. Then type $\overline{b}$ gets an expected payoff of

$$\begin{aligned}
\mathbb{E}_{h_t} - (\tilde{a}(h_t) - \theta_t - \overline{b})^2 &= -\overline{b}^2 - \mathbb{E}_{h_t}(\tilde{a}(h_t) - \theta_t)^2 + 2\overline{b} \cdot \mathbb{E}_{h_t}(\tilde{a}(h_t) - \theta_t) \\
&= -\overline{b}^2 + \frac{\overline{b}}{b}K + \frac{\overline{b} - b}{b}\mathbb{E}_{h_t}(\tilde{a}(h_t) - \theta_t)^2 \\
&\geq -\overline{b}^2 + \frac{\overline{b}}{b}K
\end{aligned}$$

Hence type $\overline{b}$ gets at least $-\overline{b}^2 + \frac{\overline{b}}{b}K$ by mimicking type b's reporting strategy. □

We apply Lemma E.2 to bound the payoffs for types b and $\overline{b}$. Let $-b^2 + K$ denote optimal payoff when type b participates in the greedy mechanism for type $\overline{b}$. Lemma E.2 says type $\overline{b}$ gets at least $-\overline{b}^2 + \frac{\overline{b}}{b}K$ in this mechanism, but the greedy mechanism gives exactly $-\overline{b}^2$ to type $\overline{b}$. Hence $K \leq 0$. We deduce that

$$\mathbb{E}_{h_t} - (\tilde{a}(h_t) - \theta_t - b)^2 \in [-b^2 - \frac{\overline{b} - b}{\overline{b}}\epsilon, -b^2],$$

where $\tilde{a}$ denotes the action after history h_t under type b's reporting. Now apply Lemma E.2 again with $K = -\frac{b}{\overline{b}}\epsilon$. Type $\overline{b}$ gets at least $-\overline{b}^2 - \frac{\overline{b}-b}{b}\epsilon$ by inducing $\tilde{a}$. We obtain that

$$\mathbb{E}_{h_t} - (\tilde{a}(h_t) - \theta_t - \overline{b})^2 \in [-\overline{b}^2 - \frac{\overline{b} - b}{b}\epsilon, -\overline{b}^2].$$

The upper bound is because the greedy mechanism promises type $\overline{b}$ a payoff of $-\overline{b}^2$. From the bounds on the payoffs of type b and type $\overline{b}$ we have

$$\begin{aligned}
\mathbb{E}_{h_t} - (\tilde{a}(h_t) - \theta_t)^2 + 2b(\tilde{a}(h_t) - \theta_t) &\in [-\frac{\overline{b} - b}{\overline{b}}\epsilon, 0] \\
\mathbb{E}_{h_t} - (\tilde{a}(h_t) - \theta_t)^2 + 2\overline{b}(\tilde{a}(h_t) - \theta_t) &\in [-\frac{\overline{b} - b}{b}\epsilon, 0]
\end{aligned}$$

We deduce that

$$(\overline{b} - b) \cdot \mathbb{E}_{h_t} - (\tilde{a}(h_t) - \theta_t)^2 \in [-(\overline{b} - b)\epsilon, (\overline{b} - b)\epsilon],$$

which implies that $\mathbb{E}_{h_t} - (\tilde{a}(h_t) - \theta_t)^2 > -\epsilon$. The receiver gets at least $-\epsilon$ when type b participates in the greedy mechanism for type $\overline{b}$. □