Effective Interference and Effective Bandwidth of Linear Multiuser Receivers in Asynchronous CDMA Systems *

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Abstract

The performance of linear multiuser receivers in terms of the signal-to-interference ratio (SIR) achieved by the users has been analyzed in a synchronous CDMA system under random spreading sequences. In this paper, we extend these results to a symbol-asynchronous but chip-synchronous system and characterize the SIR for linear receivers — the matched filter receiver, the MMSE receiver and the decorrelator. For each of the receivers, we characterize the limiting SIR achieved when the processing gain is large and also derive lower bounds on the SIR using the notion of effective interference. Applying the results to a power controlled system, we derive effective bandwidths of the users for these linear receivers and characterize the user capacity region: a set of users is supportable by a system if the sum of the effective bandwidths is less than the processing gain of the system. We show that while the effective bandwidth of the decorrelator and the MMSE receiver is higher in an asynchronous system than that in a synchronous system, it progressively decreases with the increase in the length of the observation window and is asymptotic to that of the synchronous system, when the observation window extends infinitely on both sides of the symbol of interest. Moreover, the performance gap between the MMSE receiver and the decorrelator is significantly wider in the asynchronous setting as compared to the synchronous case.

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1 Introduction

Multiuser receivers which utilize the structure of multi-access interference in order to improve performance among users have gained importance as they perform better than the conventional matched filter receiver in a spread-spectrum system [3, 4, 5, 6, 10, 11, 15, 18]. Much work has already been undertaken on characterizing the performance of multiuser receivers, using measures such as asymptotic efficiency and near-far resistance [16]. One drawback is that these measures are often complicated functions of the signature sequences and received powers of the users, and it is difficult to draw simple analytical insights about the performance of the receivers. Moreover, near-far resistance measures the worst-case performance given arbitrary received powers from interferers, and is not suitable for power-controlled systems.

Recently, [14] characterized the signal-to-interference ratio (SIR) performance of linear multiuser receivers in symbol-synchronous power-controlled systems, under spreading sequences that are random but known perfectly to the receiver. Particular attention was paid to the linear minimum mean-square error (MMSE) receiver which maximizes the output SIR, although parallel results were derived for the conventional matched filter and the decorrelator. The main result showed that in a system with large processing gain and many users, the performance of all these receivers converges to deterministic constants independent of the particular realization of the random signature sequences. Moreover, in this limit, the interference across users at the output of each of these linear receivers can be decoupled by ascribing an effective interference term to each interferer. Based on the notion of effective interference, the paper characterized the user capacity in a power controlled system by the concept of effective bandwidths of the users which is the fraction of resources consumed by a user for attaining the target SIR. Simple expressions for the effective interference and effective bandwidths are obtained for various linear receivers. Related results were independently derived in [17] for a system with equal received powers.

In this paper, we extend the results of [14] to linear multiuser receivers in a symbol-asynchronous but chip-synchronous CDMA system, where we assume that the receiver not only has acquired perfect knowledge of the signature sequences of the users but also their relative delays. The output SIR is random, being a function of the random spreading sequences and the random relative delays between the asynchronous users. For each of the three receivers, we show that the random SIR converges to a deterministic limit and we characterize the limit by the solution of a fixed point equation that depends only on the received power and relative delay distributions of the users and not on the particular realization of signature sequences. Focusing on the MMSE receiver, we obtain a lower bound on the limiting SIR using the notion of effective interference. As in the synchronous case, these results are applied to derive notions of effective bandwidths in an asynchronous power-controlled system.
In the synchronous case, the output SIR $\beta_{1,\text{sync}}$ of user 1 under the MMSE receiver has been shown to satisfy the fixed-point equation, asymptotically in a large system:

$$\beta_{1,\text{sync}} \approx \frac{P_1}{\sigma^2 + \frac{1}{N} \sum_{k=2}^{K} I(P_k, P_1; \beta_{1,\text{sync}})}$$

where $N, K$ are the processing gain and number of users respectively, $\sigma^2$ is the background noise power per degree of freedom, $P_k$ is the received power of user $k$, and

$$I(P_k, P_1; \beta) := \frac{P_k P_1}{P_1 + P_k \beta}$$

This quantity can be interpreted as the effective interference of interferer $k$ on user 1, and it depends only on the received power $P_1$ of the user to be demodulated, the received power $P_k$ of the interferer, and the attained SIR $\beta$. The above fixed-point equation says that in a large system with random spreading sequences, the effect of an interferer at the output of the MMSE receiver depends on the received powers of the other interferers only through the attained SIR. This is surprising given the fact that the MMSE receiver structure depends on the powers and signature sequences of all the users in the system. This result is justified by a limit theorem in the regime of large $K, N$ but fixing their ratio (number of users per degree of freedom).

In the asynchronous case, we will show that an analogous concept arises: the effective interference of an interferer, with received power $P_k$ and whose symbols are delayed by a fraction of $\tau_k$ with respect to the user to be demodulated, is given by:

$$I(\tau_k P_k, P_1; \beta) + I((1 - \tau_k) P_k, P_1; \beta),$$

assuming that the observation window of the MMSE receiver is limited to one symbol duration. The above expression shows that the effect of the asynchrony is approximately equivalent to splitting each interferer into two virtual users, one for each partial symbol interfering with the symbol to be demodulated and with a power proportional to the amount of overlap. Unlike the synchronous case, however, this only yields a lower bound to the limiting SIR achieved, although numerical results will show that this bound is very tight.

We also extend the above result to linear receivers that estimate the transmitted symbol by observing the received signal over a window that spans more than one symbol interval and is symmetrical around the symbol to be demodulated. In such a system, we show that the effective interference of interferer $k$ under the MMSE receiver is given by

$$\frac{1}{T} \left[ I(\tau_k P_k, P_1; \beta) + (T-1)I(P_k, P_1; \beta) + I((1 - \tau_k) P_k, P_1; \beta) \right]$$
where $T$ is the length of the observation window. The first and last term can be attributed to the effect of the two symbols partially overlapping with the observation window at the two ends, while the middle term corresponds to the effect of the $T - 1$ completely overlapping symbols.

In order to characterize the user capacity of the system we extend the concept of effective bandwidth of [14] to the asynchronous system. In an asynchronous system where the relative delays $\tau_k$’s are uniformly distributed, the effective bandwidth of a user is given by:

$$e_{m.f}(\beta; T) = \beta; \quad e_{mmse}(\beta; T) = \frac{1}{T} \left[ (T - 1) \left( \frac{\beta}{1 + \beta} \right) + 2 \left( 1 - \frac{\ln(1 + \beta)}{\beta} \right) \right]; \quad e_{dec}(\beta; T) = \frac{T + 1}{T}$$

where $\beta$ is the SIR requirement of the user and $T$ is the length of the observation window. The effective bandwidths can be interpreted as the fraction of the available degrees of freedom consumed by a user: A set of users can be admitted into the system if the sum of their effective bandwidths is less than the total number of degrees of freedom in the system. The effective bandwidths for the matched filter gives an asymptotically exact characterization of the user capacity region, while the effective bandwidth characterization for the MMSE receiver and the decorrelator yields only an inner bound. As the observation window extends infinitely on both sides, that is, as $T \rightarrow \infty$, we see that the effective bandwidth is asymptotic to that in the synchronous system:

$$e_{m.f}^{\text{sync}}(\beta) = \beta; \quad e_{mmse}^{\text{sync}}(\beta) = \frac{\beta}{1 + \beta}; \quad e_{dec}^{\text{sync}}(\beta) = 1.$$ 

The results in [14] were derived using the random matrix results in [7, 12]. But in the asynchronous system, due to the relative delays between the users, these results are no longer applicable. We use some stronger results on random matrices from [1] in order to derive the SIR achieved by users in an asynchronous system.

The outline of this paper is as follows. In Section 2, we discuss the model of the asynchronous multi-access spread-spectrum system and the structure of linear multiuser receivers. In Section 3, the concept of random spreading sequences is explained, followed by our main result for the MMSE receiver, extending the notion of effective interference to asynchronous systems. Corresponding results for the decorrelator and the matched filter are also presented. In Section 4, we apply the results derived to study the performance under power control and define a notion of effective bandwidths for asynchronous systems. In Section 5, we generalize the above results to a system where the observation window extends over multiple symbols. In Section 6, we present a heuristic extension of our results to the symbol and chip asynchronous system. We conclude the paper with some discussions in Section 7.
Figure 1: User 1 and a typical interferer within the observation window.

2 The Direct-Sequence CDMA Model

In a DS-CDMA system, each user’s information symbols are spread over a larger bandwidth by modulation onto its signature (or spreading) sequence. We consider a system that has a processing gain of $N$. In a symbol synchronous situation, a chip-sampled discrete-time model of the received signal $r \in \mathbb{R}^N$ in a multi-access spread spectrum system with $K$ users is given by,

$$r = \sum_{k=1}^{K} x_k s_k + n$$  

(1)

where $x_k \in \mathbb{R}$ is the transmitted information symbol and $s_k \in \mathbb{R}^N$ is the signature sequence of the $k^{th}$ user and $n$ is the background Gaussian noise $\mathcal{N}(0, \sigma^2 I_N)$. The transmitted symbols, $x_k$ are assumed to be independent with $\mathbb{E}[x_k] = 0$ and $\mathbb{E}[x_k^2] = P_k$, where $P_k$ is the received power of user $k$.

In this paper, we focus on a symbol asynchronous multi-access spread spectrum system. Even though we allow the system to be symbol asynchronous, we will assume the system to be chip-synchronous to make the analysis tractable. In the first part of the paper, we also restrict ourselves to receivers that have an observation window of one symbol, $N$ chips in length. Without loss of generality, we focus on a symbol of user 1 and notice that a typical interferer will have two different symbols interfering with the symbol of user 1 within the observation window, as shown in Figure 1. The observation window is marked in thick lines and the two interfering symbols of a typical interferer are shown in solid and dotted line below the reference symbol of user 1. Since the system is corrupted by white noise and is assumed to be chip synchronous, the projections onto $N$ waveforms which are matched to the $N$ chip pulses (an orthogonal basis set) form a sufficient statistic for the received signal [16]. Thus, the sampled discrete-time model for the received signal $r \in \mathbb{R}^N$ is given by,

$$r = x_1 s_1 + \sum_{k=2}^{K} x_k u_k + \sum_{k=2}^{K} y_k v_k + n$$  

(2)

where $x_k, y_k \in \mathbb{R}$ are the two consecutive symbols of the $k^{th}$ user which overlap with user 1 in the observation window, as shown in Figure 1. These have effective signature sequences $u_k \in \mathbb{R}^N$ and
\( \mathbf{v}_k \in \mathbb{R}^N \) respectively. The effective signature sequences are completely determined by the original signature sequences \( \mathbf{s}_k \) and the delays relative to user 1. If \( d_k \in \mathbb{Z}^+ \) denotes the relative delay in terms of number of chips of the \( k^{th} \) user with respect to user 1, then \( \mathbf{u}_k \) has its first \( d_k \) elements to be the last \( d_k \) elements of \( \mathbf{s}_k \) and the rest zeros. Similarly, \( \mathbf{v}_k \) has the first \( d_k \) elements zero and the last \( N - d_k \) elements to be the first \( N - d_k \) elements of \( \mathbf{s}_k \). That is,

\[
\begin{align*}
(\mathbf{u}_k)_i &= \begin{cases} 
(\mathbf{s}_k)_{(N - d_k + i)} & i \leq d_k \\
0 & d_k < i \leq N
\end{cases} \\
(\mathbf{v}_k)_i &= \begin{cases} 
0 & i \leq d_k \\
(\mathbf{s}_k)_{(i - d_k)} & d_k < i \leq N
\end{cases}
\end{align*}
\]

The background noise \( \mathbf{n} \) is white Gaussian, \( \mathcal{N}(0, \sigma^2 \mathbf{I}_N) \). The transmitted symbols \( x_k \) and \( y_k \) are assumed to be independent of each other and of the transmitted symbols of other users with \( \mathbb{E}[x_k] = \mathbb{E}[y_k] = 0 \) and \( \mathbb{E}[x_k^2] = \mathbb{E}[y_k^2] = P_k \), where \( P_k \) is the received power of user \( k \). Notice that the model for the synchronous system in eqn (1) can be derived by setting the relative delays \( d_k = 0 \) in eqn (2).

In this paper, we restrict ourselves to the study of linear demodulators (receivers). If \( \hat{x}_1 \) is the estimate of \( x_1 \), the symbol transmitted by user 1, then, a linear demodulator is captured by,

\[ \hat{x}_1(\mathbf{r}) = \mathbf{c}_1^t \mathbf{r} \]

The information symbols transmitted by the user may be coded, in which case the linear receiver returns soft decisions to the channel decoder. From this point of view, the signal-to-interference ratio (SIR) of the estimates is a relevant performance measure [14] and the SIR achieved by the linear receiver in the symbol-asynchronous case is given by,

\[
\text{SIR}_1 = \frac{(\mathbf{c}_1^t \mathbf{s}_1)^2 P_1}{(\mathbf{c}_1^t \mathbf{c}_1) \sigma^2 + \sum_{k=2}^K \left( (\mathbf{c}_1^t \mathbf{u}_k)^2 + (\mathbf{c}_1^t \mathbf{v}_k)^2 \right) P_k}
\]

The conventional matched filter receiver simply projects the received vector onto the user’s signature signature sequence, \( \mathbf{s}_1 \). This matched filter demodulator is optimal only if the total interference is white, which may not necessarily be the case in a multi-access system. In general, the MMSE receiver is the optimal linear receiver in the sense that it maximizes the SIR of user 1 by exploiting the structure of the interference [5, 10, 11]. The estimate \( \hat{x}_{\text{MMSE}} \) of the MMSE demodulator [5] is given by,

\[
\hat{x}_{\text{MMSE}}(\mathbf{r}) = \frac{P_1 \mathbf{s}_1^t \left( \mathbf{S}_1 \mathbf{D} \mathbf{S}_1^t + \sigma^2 \mathbf{I} \right)^{-1} \mathbf{r}}{1 + P_1 \mathbf{s}_1^t \left( \mathbf{S}_1 \mathbf{D} \mathbf{S}_1^t + \sigma^2 \mathbf{I} \right)^{-1} \mathbf{s}_1}
\]
and the SIR achieved by user 1 is given by,

$$SIR_1 = P_1 s_1^T (S_1D_1 s_1^T + \sigma^2 I)_1^{-1} s_1$$

(5)

where $S_1$ is a $N \times 2(K - 1)$ matrix that has the effective signature sequences of the interferers, $u_2, \ldots, u_K, v_2, \ldots, v_K$ for its columns and $D_1 = \text{diag}(P_2, \ldots, P_K, P_2, \ldots, P_K)$ which is the covariance matrix of $(x_2, \ldots, x_K, y_2, \ldots, y_K)^t$.

3 Performance under Random Spreading Sequences

The SIR described in Section 2 can be used to calculate the performance achieved with specific set of signature sequences assigned to the users. But it does not give much insight as to how the users interfere with each other to affect the performance because the output of the MMSE receiver has a complicated dependence on the powers and signature sequences of all the interferers, despite the fact that the interferers are additive at the input of the receiver. In practice, it is often more reasonable to assume that the spreading sequences are randomly and independently chosen [6, 14, 17]. These random sequences can be from a long pseudo-random code or sequences picked at random from a large look-up table. Since the signature sequences are random, the SIR achieved $\beta$, being a function of the signature sequences is a random variable. In this paper, the signature sequences, though chosen randomly, are assumed to be known to the receiver.

The random signature sequence of the $k^{th}$ user can be modeled as $s_k = \frac{1}{\sqrt{N}} (\nu_{k1}, \nu_{k2}, \ldots, \nu_{kN})^t$, where $\nu_{kd}$ are i.i.d, zero-mean unit variance random variables. The normalization of $\frac{1}{\sqrt{N}}$ ensures that $E[||s_k||^2] = 1$. For technical reasons, we also require that the random variables have a finite fourth moment, $E[\nu_{kd}^4] < \infty$. Practical situations like choice of $\pm 1$ signature sequences can be obtained as particular cases of the final result. The results in this paper show that in the asymptotic regime, the SIR achieved is independent of the distribution of the random variables that constitute the signature sequences.

In an asynchronous system, the various relative delays $d_k$ are also assumed to be random. Hence, the performance measure as a random variable is a function of the random spreading sequences and the random delays. Again, though the delays are random, it is assumed that the receiver has information of the delays of all the users. Practically, this means that the receiver has attained timing information of different users and this varies at a rate considerably lower than the symbol rate.

All the results described in this paper are asymptotic in nature: we consider the limiting regime where the number of users is large, $K \rightarrow \infty$. This means that the processing gain also needs to be scaled up, else we will have the SIR to be zero with probability 1. Therefore, we have a system where $\alpha, K \rightarrow \infty$, but with a fixed number of users per degree of freedom, $\alpha$, that is,
$K = \lfloor \alpha N \rfloor$. As we scale up the system, the empirical distribution of the powers of the users is assumed to converge to a fixed distribution (Cumulative Distribution Function), $F(P)$. The empirical distribution of the delays of the different users, relative to the observation window is also assumed to converge to a fixed distribution (CDF), $G(\tau)$, where the delay $d$ relative to the reference user is given by $d = \lfloor \tau N \rfloor$. In a typical asynchronous system, we can assume that the arrivals are equally probable to be anywhere in $[0, N)$ and hence uniform delay distribution serves as a good model for the delays relative to a particular user. This paper analyzes the performance for a general delay distribution (which is useful in certain cases such as quasi-asynchronous systems [16]), and later specializes to the uniform delay distribution.

In the synchronous system, the matrix whose columns are the interferers’ signature sequences have i.i.d. entries and the random matrix results of [7] are applied to prove main theorem of [14]. But in the asynchronous setting, some entries of $S_1$ are zero (depending on the delays $d_k$) and we therefore need more powerful results than [7]. The following theorem gives the asymptotic SIR achieved by user 1. Its proof requires some random matrix results for matrices with independent but not necessarily identically distributed entries [1], and appears in Appendix A.

**Theorem 3.1** Let $\beta_1^{(N)}$ is the SIR attained by user 1 for the MMSE receiver in an asynchronous system with a processing gain of $N$ and an observation window of one symbol ($N$ chips). As $N \to \infty$ with $\frac{K}{N} \to \alpha$, $\beta_1^{(N)}$ converges in probability to a deterministic constant $\beta_1^*$, where

$$
\beta_1^* = \int_0^1 w(x)dx
$$

and

$$
w(x) = \frac{P_1}{\sigma^2 + \alpha \text{E}_P \text{E}_\tau \left\{ I\left(P, P_1, \int_0^\tau w(z)dz\right) 1_{\{\tau \geq x\}} + I\left(P, P_1, \int_\tau^1 w(z)dz\right) 1_{\{\tau \leq x\}} \right\}}
$$

where $\text{E}_P$ and $\text{E}_\tau$ denote the expectation with respect to the power distribution $F(P)$ and the delay distribution $G(\tau)$ respectively. $I(P, P_1, \beta) = \frac{PP_1}{P_1 + P\beta}$ is the effective interference function introduced in the synchronous case and $1_{\{\tau \geq x\}}$ is an indicator function that is 1 if $\tau \geq x$ and 0 otherwise. The solution to $w(x)$ exists and is unique in a class of function $w(x) \geq 0$.

Given distributions of delays and powers of the users, the function $w(\cdot)$ can be solved numerically by iterating the fixed point equation (7). Convergence is guaranteed from any initial positive function. It is important to note that, as in the synchronous system, the limiting SIR is independent of the specific realizations of the signature sequences. However, the complexity of the
expression makes it difficult to draw interesting analytical insights. The following theorem (derived in Appendix B) gives a lower bound on the SIR achieved and also provides simpler insights on how the interference from other users affects the performance by reducing the dependence to the overall SIR achieved. To derive the bound, we need to make a weak symmetry assumption on the delay distribution, satisfied in most cases of interest.

**Theorem 3.2** In an asynchronous system, if the relative delay distribution \( G(\tau) = 1 - G(1 - \tau) \) satisfies the condition \( G(\tau) \leq 1 - G(1 - \tau) \), then the asymptotic SIR \( \beta_1^* \) attained is lower bounded by \( \gamma_1^* \), which is the unique solution of the fixed point equation,

\[
\gamma_1^* = \frac{P_1}{\sigma^2 + a \mathbb{E}[\mathbb{E}\tau \{I(\tau P_1, P_1, \gamma_1^*) + I((1 - \tau)P_1, P_1, \gamma_1^*)\}^n]}
\]

(8)

where

\[
I(P, P_1, \gamma_1^*) = \frac{PP_1}{P_1 + P_1 \gamma_1^*}
\]

is the effective interference of an user of power \( P \) at SIR \( \gamma_1^* \), as developed in the synchronous case.

Heuristically, in a large system,

\[
\gamma_1^* \approx \frac{P_1}{\sigma^2 + \frac{1}{N} \sum_{k=2}^{K} I(\tau_k P_k, P_1, \gamma_1^*) + I((1 - \tau_k)P_k, P_1, \gamma_1^*)}
\]

In the synchronous case, an interferer with received power \( P_k \) contributes an effective interference of \( I(P_k, P_1, \gamma_1^*) \). In the asynchronous case, we can then interpret \( I((1 - \tau_k)P_k, P_1, \gamma_1^*) \) and \( I(\tau_k P_k, P_1, \gamma_1^*) \) as the effective interference contributed by the two partial symbols of interferer \( k \) within the observation window. Therefore, the effective interference of the \( k^{th} \) interferer on user 1 at SIR requirement \( \beta_1 \) is given by,

\[
I(\tau_k P_k, P_1, \beta_1) + I((1 - \tau_k)P_k, P_1, \beta_1),
\]

(9)

We observe that the first symbol of the \( k^{th} \) interferer has \( \tau_k N \) part of its signature sequence overlapping within the observation window. This means that it has an effective power of \( \tau_k P_k \) interfering in the observation window. Similarly, the second part of the \( k^{th} \) interferer has an effective power of \((1 - \tau_k)P_k\). But since the two parts do not spread over the entire range of \( N \), Theorem 3.1 of [14] for the synchronous case cannot be applied. But the above theorem shows that the effective interference, assuming that the two parts spread over the entire range, but with a proportionally reduced power serves as a lower bound to the SIR achieved.
It is somewhat surprising that $\gamma^*$ is a lower bound and not the exact limiting SIR, as one might have first guessed. The proof of Theorem 3.2 reveals that the part of a virtual interferer's received signal near the center of the observation window has a stronger interfering effect than the part near the edge of the window. Hence, the overall effect of an interferer's signal cannot be just proportional to the amount of overlap with the observation window. We will provide a more detailed intuitive discussion of this phenomenon at the end of Appendix B.

It is straightforward to show that the matched filter receiver in the asynchronous case has the same asymptotic performance as in the synchronous situation [14]: the $k$th interferer contributes an effective interference of $P_k$. We observe that $I(\tau_k P_k, P_1, \gamma^*_1) + I((1 - \tau_k) P_k, P_1, \gamma^*_1) \leq P_k$ and hence the MMSE receiver performs better than the matched filter receiver (which is consistent with the fact that the MMSE receiver maximizes SIR among all the linear receivers). As in the synchronous case, note that the effective interference under the MMSE receiver remains bounded as the interferer's power grows, in contrast to that under the matched filter receiver.

For the special case of uniform relative delay distribution and equal received powers, the fixed point equation for the lower bound $\gamma^*_1$ becomes:

$$\gamma^*_1 = \frac{P}{\sigma^2 + \frac{2\alpha P}{\gamma^*_1} \left(1 - \ln(1 + \frac{\gamma^*_1}{\gamma^*_1})\right)}$$ (10)

The first plot in the top left corner of Figure 2 compares this lower bound $\gamma^*$ to the actual limiting SIR $\beta^*$ obtained by numerically solving $w(x)$ given by eqn (7) by method of iterations. The signal-to-noise ratio $\frac{P}{\sigma^2}$ is 20dB, which will be the same for all the numerical results in this paper. The maximum difference between the proposed bound and the limiting SIR is less than half a dB over a wide range of $\alpha$. The limiting SIR for the synchronous case is also plotted for comparison. In the remaining three plots, we would like to give a sense of the rate of convergence to the asymptotic limit $\beta^*$. The plots compare the actual SIRs attained for several realizations with the asymptotic SIR achieved. The simulations assume that the elements of the signature sequences are equiprobable $\pm \frac{1}{\sqrt{N}}$ random variables and the relative delays are uniformly distributed in $[0, N)$. For different spreading lengths and for each value of $\alpha$, 1000 samples of the SIR attained are computed using eqn (5). We plot the average of the sample points of the simulated SIRs and also the 1 standard deviation spread around the mean (simulated $\beta \pm \sigma_\beta$). From the plots, we notice that the average of the simulated SIRs is very close to the asymptotic SIR given by eqn (7). Moreover, by observing the variance spread in $N = 32, 64, 128$, we notice that the variance progressively reduces as $N$ increases (roughly halved on doubling $N$). For $N = 128$, the spread is less than 1 dB around the mean.

Another linear receiver to consider is the decorrelator [3, 4]. This receiver was shown to be optimal in the worst case scenario, when the powers of the interferers tend to infinity, in the sense
Limiting SIR ($\beta^*$) attained in the Asynchronous MMSE Demodulator.
Average of the simulated SIRs in the Asynchronous MMSE Demodulator.
Bound ($\gamma^*$) on the limiting SIR achieved in the MMSE receiver attained using the notion of Effective Interference.
Limiting SIR achieved in the Synchronous MMSE demodulator.
Mean + one Standard Deviation of the simulated SIRs ($\beta + \sigma_\beta$, simulated)
Mean - one Standard Deviation of the simulated SIRs ($\beta - \sigma_\beta$, simulated)

Figure 2: Comparison of the randomly generated SIRs (for processing gains $N = 32, 64, 128$) with limiting SIR achieved and the bound proposed.
that it attains the optimal near-far resistance, both in the synchronous and asynchronous systems [3, 4]. In the asynchronous system, the decorrelating filter [3, 4] can be described by an overall filter \((S^tS)^{-1}S^t\) (if the inverse does not exist, then the pseudo-inverse is used in its place). Here, \(S\) is the \(N \times (2K - 1)\) matrix whose columns are the signature sequences \(s_1, u_2, \ldots, u_K, v_2, v_3, \ldots, v_K\). In the absence of the external noise \(n\), this would give perfect estimates of the information symbols and hence is a zero-forcing linear filter [3, 16]. If \(w = (S^tS)^{-1}S^tn\) denotes the colored noise at the output of the filter, then it has a covariance matrix \(\Sigma = (S^tS)^{-1}\sigma^2\). Therefore, we can describe the effect of the overall system as corrupting the information symbol by noise \(w_k\), a zero mean Gaussian random variable of variance \(\Sigma_{kk}\). The SIR attained by user 1 is therefore given by,

\[
SIR_1 = \frac{P_1}{\Sigma_{11}}
\]  

(11)

**Theorem 3.3** In an asynchronous system, if \(\beta_{1,\text{dec}}^{(N)}\) is the SIR of a decorrelator for user 1, then, as the processing gain \(N \to \infty\) with \(K/N \to \alpha\), \(\beta_{1,\text{dec}}^{(N)}\) converges in probability to \(\beta_{1,\text{dec}}^*\) given by,

\[
\beta_{1,\text{dec}}^* = \begin{cases} 
\frac{P_1(1-2\alpha)}{\sigma^2} & \alpha < \frac{1}{2} \\
0 & \alpha \geq \frac{1}{2}
\end{cases}
\]  

(12)

**Proof:** See Appendix C

The interpretation is that each interferer occupies precisely two degrees of freedom asymptotically. Contrast this with the synchronous case, where the corresponding limit is \(P_1(1 - \alpha)/\sigma^2\) for \(\alpha < 1\) and 0 otherwise [14, 17]. Each interferer occupies only one degree of freedom in the synchronous system.

In order to compare the performance of the MMSE receiver and the decorrelator, we plot the limiting SIR achieved in the two receivers for the special case of equal received powers and uniform delay distribution in Figure 3. The limiting SIR achieved by the MMSE receiver is plotted by numerically solving for \(w(x)\) in eqn (7) in Theorem 3.1 and the limiting SIR for the decorrelator is given by Theorem 3.3. The bound as given by eqn (8) is also plotted in the figure. We notice that the performance of the MMSE and the decorrelator are very close when the number of users per degree of freedom is small. But as the number of users per degree of freedom increases, the decorrelator degrades very rapidly and \(\beta_{\text{dec}} \to 0\) as \(\alpha \to \frac{1}{2}\), whereas, the MMSE receiver performs significantly better than the decorrelator in that region and beyond \(\alpha = \frac{1}{2}\). The MMSE receiver has a more graceful degradation than the decorrelator and is therefore preferred in a system that needs to operate when the number of users form a significant fraction (roughly around 0.4 or higher) of the total processing gain of the system.
Asymptotic SIR achieved by the MMSE receiver in the Asynchronous system.
Asymptotic SIR achieved by the decorrelator in the Asynchronous system.
Lower bound on the SIR achieved by the MMSE receiver in the
Asynchronous system, derived from the notion of effective interference.

Figure 3: Comparison of the limiting SIR achieved in the MMSE receiver and the decorrelator in
an asynchronous system with equal received powers and $\frac{P}{\sigma^2} = 20$ dB.

4 User Capacity and Performance under Power Control

In the previous section, we analyzed the SIR achieved when the interferers had arbitrary received
powers. We now apply the results to analyze the performance in a power-controlled system. For
given SIR requirements of the users and given received power constraints, we characterize the
number of users that can be admitted into the system under appropriate power control. We define
the user capacity of the system as the number of users that can be admitted into the system
without any power constraints; this can be thought of as the interference-limited capacity of the
system. The user capacity regions of the various demodulators will be analyzed in the asymptotic
regime considered so far. We first focus on the case when all users have a common SIR requirement
$\beta$, and then extend the results to the case when users can have different requirements.

The matched filter receiver has exactly the same SIR in the synchronous and asynchronous
cases and hence the derivations for the user capacity are identical to those found in [14]. It is
shown there that the asymptotic minimal power solution is to assign equal received powers to
all users. For a given power constraint $\bar{P}$, the maximum number of users with requirement $\beta$
supportable is given by,

$$ a_{\text{max}, MF}(\bar{P}, \beta) = \frac{1}{\beta} - \frac{\sigma^2}{\bar{P}} \text{ users per degree of freedom}. $$
Therefore, the user capacity of the matched filter receiver as $\bar{P} \to \infty$ is,

$$C_{MF}(\beta) = \frac{1}{\beta} \text{ users per degree of freedom.} \quad (13)$$

For the MMSE receiver, we find that the actual limiting SIR achieved, given by Theorem 3.1, does not allow easy calculations of the user capacity. Therefore, we use the lower bound on the SIR proposed in Theorem 3.2 to compute a lower bound on the user capacity. In typical situations, the difference between the bound and the actual SIR attained is shown by numerical calculations (Figure 2) to be small and hence we can expect the user capacity to be only slightly greater than the bound derived below.

To ensure that any pair of users has the same relative delay distribution, we will only consider the uniform delay distribution in the analysis here. Moreover, in a large system, the removal of any one user does not affect the empirical power distribution, and hence we have similar expressions for the SIRs of each user, with the ratio of the SIRs being equal to the ratio of the received powers. Thus, for the scenario when the users have common SIR requirement, it suffices to assign equal received powers to all users. Since the SIR achieved in the MMSE receiver is a decreasing function of $\alpha$, we can expect it to saturate just like the matched filter receiver, but at a possibly higher value of $\alpha$. Indeed, for a given power constraint $\bar{P}$, a lower bound on the number of users with requirement $\beta$ supportable can be obtained by setting $P = \bar{P}$ and solving eqn (10) for $\alpha$:

$$\alpha_{\text{max}}(\bar{P}, \beta) = \frac{1}{2 \left(1 - \frac{\ln(1+\beta)}{\beta}\right)} \left(1 - \frac{\sigma^2 \beta}{\bar{P}}\right) \text{ users per degree of freedom.}$$

Hence, the user capacity of the MMSE receiver without power constraints is lower bounded by

$$C_{MMSE}(\beta) = \frac{1}{2 \left(1 - \frac{\ln(1+\beta)}{\beta}\right)} \text{ users per degree of freedom.} \quad (14)$$

Contrasting eqns (13) and (14), we note that if $\alpha$ is feasible for both the matched filter and the MMSE receivers, the MMSE receiver achieves the target SIR with a lower power consumption than the matched filter receiver and also yields a higher user capacity. Also, if $\alpha < 1/2$, then arbitrarily high SIRs $\beta$ can be achieved without saturating the MMSE receiver, whereas the matched filter receiver saturates as the required SIR $\beta \to \frac{1}{\alpha}$.

In case of the decorrelator, if all the users require an SIR of $\beta$ and power control is employed, then we can deduce from Theorem 3.3 that the maximum number of users supportable at requirement $\beta$ and power constraint $\bar{P}$ is

$$\alpha_{\text{max, dec}}(\bar{P}, \beta) = \frac{1}{2} - \frac{\beta \sigma^2}{2 \bar{P}} \text{ users per degree of freedom.}$$
Thus, the user capacity of the decorrelator is,

\[ C_{\text{dec}} = \frac{1}{2} \text{ users per degree of freedom.} \]  

(15)

It is important to note that the above expressions for the decorrelator are exact unlike those for the MMSE receiver. Contrasting with the user capacity of the decorrelator in the synchronous system [14], we note that the effect of asynchrony in the decorrelator is to reduce the user capacity of the system by a factor of 2.

The above results for the single class of users can be generalized to a system where there are \( J \) classes of users, with the \( j^\text{th} \) class requiring an SIR of \( \beta_j \). Let the number of users in the \( j^\text{th} \) class be denoted by \( [\alpha_j N] \) and we consider the regime where \( N \rightarrow \infty \). The synchronous situation was analyzed and the effective bandwidths of the matched filter, the MMSE receiver and the decorrelator were derived in [14].

Using the effective interference lower bound on the SIR under the MMSE receiver, we can obtain an inner bound on the user capacity region. Assuming a uniform delay distribution, it can be deduced from Theorem 3.2 and the monotonicity property of the fixed-point equation that an SIR of at least \( \beta_j \) can be achieved if we assign powers \( P_j \) to users in class \( j \) such that:

\[ \frac{P_k}{\sigma^2 + \sum_{j=1}^{J} \frac{2\alpha_j P_k}{\beta_j} \left[ 1 - \frac{P_k}{\beta_k P_j} \ln \left( 1 + \frac{\beta_k P_j}{P_k} \right) \right]} = \beta_k, \quad \text{for } k = 1, \ldots, J \]  

(16)

The above equations implies \( \frac{\beta_j}{P_j} \) is a constant across the classes and further simplification yields,

\[ P_k = \frac{\beta_k \sigma^2}{1 - \sum_{j=1}^{J} \frac{2\alpha_j}{1 - \frac{\ln(1 + \beta_j)}{\beta_j}}} \quad \text{for } k = 1, \ldots, J \]  

(17)

Hence, if the linear constraint:

\[ \sum_{j=1}^{J} 2 \left( 1 - \frac{\ln(1 + \beta_j)}{\beta_j} \right) \alpha_j < 1 \]  

(18)

is satisfied, then the SIR requirements can be attained with some received powers. Thus, this constraint specifies an inner bound on the interference-limited user capacity region, i.e., when there are no power limitations. If class \( j \) user has a power constraint of \( P_j \) such that \( P_j \leq \bar{P}_j \) for all users in class \( j \), then the sufficient condition for the SIR requirements to be satisfied becomes

\[ \sum_{j=1}^{J} 2 \left( 1 - \frac{\ln(1 + \beta_j)}{\beta_j} \right) \alpha_j \leq \min_{1 \leq r \leq J} \left[ 1 - \frac{\beta_j \sigma^2}{\bar{P}_j} \right] \]  

(19)
Figure 4: Comparison of the effective bandwidths of the linear multiuser receivers in synchronous and asynchronous systems.

Note that the constraints are linear in \((\alpha_1, \ldots, \alpha_J)\) and therefore, the MMSE receiver (under uniform delay distribution) has an effective bandwidth,

\[
\epsilon_{\text{mmse}}(\beta) = 2 \left( 1 - \frac{\ln(1 + \beta)}{\beta} \right) \text{ degrees of freedom per user.} \tag{20}
\]

The actual user capacity region for a requirement of \(\beta_k\) will contain the region given above, and the effective bandwidth is an upper bound to the amount of degrees of freedom consumed by the user (hence the upperbar in the notation).

The user capacity regions under the matched filter receiver and the decorrelator have identical forms but with different expressions for the effective bandwidths:

\[
\epsilon_{\text{mf}}(\beta) = \beta; \quad \epsilon_{\text{dec}}(\beta) = 2 \tag{21}
\]

However, in contrast to the MMSE receiver, these effective bandwidths yield (asymptotically) exact capacity regions rather than inner bounds. We see that the effect of asynchrony on the decorrelator is precisely to double the effective bandwidth.

Figure 4 displays the effective bandwidths of the three linear receivers analyzed in the synchronous and asynchronous systems. From the plot, we notice that when the SIR requirement is
small, the matched filter receiver has a lower effective bandwidth than the decorrelator, but when the SIR requirement is large, the decorrelator outperforms the matched filter. This cross-over occurs at a higher value of required SIR in the asynchronous system than in the synchronous system. The MMSE receiver follows the matched filter receiver for small values of SIR and is asymptotic to that of the decorrelator when the SIR requirement is high. Observe also that the improvement gain of the MMSE receiver over the decorrelator is more significant in the asynchronous case than in the synchronous case. This is because the decorrelator loses an extra degree of freedom in the asynchronous system due to an additional interfering symbol per user, while the MMSE receiver fares better as it takes advantage of the fact that the overlapping symbols are only partial and hence their energy is only a fraction of that of the interfering symbol in the synchronous case. This can also be seen directly from the expression for the effective interference eqn (9).

5 Multiple Symbol Observation Window

In the previous sections, we considered the observation window limited to the symbol to be demodulated and analyzed the performance of different receivers. In this section, we extend the results to a situation where the symbol of interest is estimated by observing over $T$ symbol intervals.

As shown in Figure 5, the observation window is assumed to be symmetric about the symbol to be demodulated. The symbols transmitted by the users are assumed to be independent, with the power of the $k^{th}$ user being constant over the observation interval. As explained in Section 3, the spreading sequences are assumed to be randomly chosen. In what follows, we assume that the spreading sequence of any user is independent from symbol to symbol, in addition to being independent of the signature sequences of the other users. This is a valid assumption in the case of long spreading codes which extend over many symbols. But in some situations, it is more reasonable to choose a signature sequence and repeat it from symbol to symbol. We will analyze the system where the signature sequences are independent from symbol to symbol and then compare the results by means of simulation to that of a system with repeated signature sequences. As usual, the receiver is assumed to have knowledge of the signature sequences and relative delays.
Theorem 5.1 Let $\beta_1^{(N)}$ denote the SIR attained by user 1 for the MMSE receiver in the asynchronous system of processing gain $N$ and an observation window of $T \geq 1$ symbols ($T$ being an odd integer), symmetric about the symbol to be demodulated. As $N, K \rightarrow \infty$ with $\frac{K}{N} \rightarrow \alpha$, $\beta_1^{(N)}$ converges in probability to $\beta_1^*$, where $\beta_1^*$ is given by,

$$\beta_1^* = \int_{-\frac{T+1}{2}}^{\frac{T+1}{2}} w(x)dx$$

(22)

where $w(x) \geq 0$ in $[0, T]$ is given by

$$w(x) = \frac{P_1}{\sigma^2 + \alpha \mathbb{E}_P \mathbb{E}_\tau I(P, P_1, \int_{C(x, \tau)} w(z)dz)}$$

(23)

The region of integration $C(x, \tau)$ is given by,

$$C(x, \tau) = \begin{cases} [0, \tau] & x \in [0, \tau] \\ [\tau + i - 1, \tau + i] & x \in [\tau + i - 1, \tau + i] \text{ for } i = 1, \ldots, (T - 1) \\ [\tau + T - 1, T] & x \in [\tau + T - 1, T] \end{cases}$$

and $\mathbb{E}_P, \mathbb{E}_\tau$ denote the expectations with respect to the power distribution $F(P)$ and the delay distribution $G(\tau)$ respectively. $I(P, P_1, \beta) = \frac{PP_1}{P_1 + P_1\beta}$ is the effective interference introduced in the synchronous case. The solution to $w(x)$ exists and is unique in a class of functions $w(x) \geq 0$.

The above theorem is an extension of the results in Section 3. The proof of Theorem 5.1 is analogous to the proof of Theorem 3.1 and follows the same lines of Lemma A.1 and Theorem A.2, with the regions of integration $C(x, \tau)$ depending on the regions where the signature sequence of interferer’s symbols have non-zero interfering power, which in turn depends on the relative delays. The derivation in the last part of Appendix B yields a lower bound for the SIR achieved, which is given by the following theorem.

Theorem 5.2 If the delay distribution satisfies $G(\tau) = G(1 - \tau)$ and the observation window is an odd integer $T \geq 1$, symmetric about the symbol to be demodulated, then the asymptotic SIR $\beta_1^*$ achieved by the MMSE receiver can be lower bounded by $\gamma_1^*$, which is the unique solution of the fixed point equation,

$$\gamma_1^* = \frac{P_1}{\sigma^2 + \alpha \mathbb{E}_P \mathbb{E}_\tau [I(\tau P, P_1, \gamma_1^*) + (T - 1)I(P, P_1, \gamma_1^*) + (1 - \tau)P, P_1, \gamma_1^*)]}$$

(24)

where $I(P, P_1, \gamma_1^*) = \frac{P_1P_1}{P_1 + P_1\gamma_1^*}$ is the effective interference of an user of power $P$, at SIR $\gamma_1^*$.
Heuristically, in a large system,
\[
\gamma_{1}^{*} \approx \frac{P_{1}}{\sigma^{2} + \frac{1}{NT} \sum_{k=2}^{K} \left[ I(\tau_{k}P_{k}, P_{1}, \gamma_{1}^{*}) + (T-1)I(P_{k}, P_{1}, \gamma_{1}^{*}) + I((1-\tau_{k})P_{k}, P_{1}, \gamma_{1}^{*}) \right]}
\]
and we have the interpretation that
\[
\frac{1}{T} \left[ I(\tau_{k}P_{k}, P_{1}, \beta) + (T-1)I(P_{k}, P_{1}, \beta) + I((1-\tau_{k})P_{k}, P_{1}, \beta) \right]
\]
is the effective interference of the \( k \)th user at SIR requirement \( \beta \). As observed in the other cases, the effective interference depends only on the received power of the interferer, the received power of the user and the SIR achieved. The three terms can be explained by referring back to the Figure 5: the first term is due to \( d_{k} = \lfloor \tau_{k}N \rfloor \) part of the symbol interfering within the observation window, the second term is due to the \((T-1)\) complete symbols interfering and the last term is due to the \( N - d_{k} \) chips of the interfering symbol in the observation window. Therefore, just as in the single symbol observation case, the bound can be interpreted in terms of the effects of virtual interferers which have power proportional to the fraction of the interfering symbol within the observation window. Since
\[
I(P_{k}, P_{1}, \gamma_{1}^{*}) \leq I(\tau_{k}P_{k}, P_{1}, \gamma_{1}^{*}) + I((1-\tau_{k})P_{k}, P_{1}, \gamma_{1}^{*})
\]
a better limiting SIR is achieved by a longer observation window.

Since the decorrelator can be obtained from the MMSE receiver when the SNR \( \frac{P_{1}}{\sigma^{2}} \to \infty \) for each user, we can use Theorem 5.1 to obtain a lower bound for the performance achieved by the decorrelator when the observation window spans over \( T \) symbols.

**Proposition 5.3** In an asynchronous system of processing gain \( N \), if the decorrelator estimating the symbol of user 1 by observing over \( T \) symbols attains an SIR \( \beta_{dec}^{(N)} \), then, as the processing gain \( N \to \infty \), \( \beta_{dec}^{(N)} \) converges in probability to \( \beta_{dec}^{*} \) which is lower bounded by \( \gamma_{dec}^{*} \) given by,
\[
\gamma_{dec}^{*} = \left\{ \begin{array}{ll}
\frac{P_{1}}{\sigma^{2}} \left[ 1 - \frac{(T+1)}{T} \right] & \alpha < \frac{T}{T+1} \\
0 & \alpha \geq \frac{T}{T+1}
\end{array} \right.
\]

From eqns (24) and (27), we notice that as the observation window extends infinitely on both sides of the symbol of interest \( (T \to \infty) \), the SIR achieved is asymptotic to the corresponding SIR achieved in the synchronous system. It is also not difficult to show that the limiting SIR performance of the matched filter performance does not depend on the window size \( T \).
We now specialize the above results to a typical system where the empirical relative delay distribution is uniform. In addition, if we consider all users to have the same power $P$, the MMSE receiver attains an SIR that is lower bounded by $\gamma^*$, which satisfies the fixed point equation,

$$\gamma^* = \frac{P}{\sigma^2 + \alpha \left( \frac{T-1}{T} \right) (\frac{P}{1+\gamma^*}) + \frac{2\alpha P}{T\gamma^*} \left( 1 - \frac{\ln(1+\gamma^*)}{\gamma^*} \right)}$$  \hspace{1cm} (28)

In order to get a feel for the advantage of increasing the observation window to more than one symbol, we plot some numerical results in Figure 6. In these, we consider a system that has $\alpha$ users per degree of freedom, all with equal received powers and the relative delay distribution is uniform. The limiting SIR achieved for a particular $T$ is computed by numerically solving for $w(x)$ in eqn (23) for a specific case of uniform delay distribution. The bound was evaluated by solving for $\gamma^*$ in eqn (28). In each of the four plots, apart from the actual SIR and the bound on the SIR for a particular $T$, we also plot the SIR achieved in the synchronous system, in order to help in comparison. From the plots, we note that when the observation window is increased from $T = 1$ to $T = 3$, there is a very large gain in the performance achieved by the user and the incremental gain achieved reduces as $T$ is increased further. Therefore, we note that a delay of 2-3 symbols in demodulation and increased complexity may be a good compromise as we can achieve an SIR that is very close to the one achieved in a synchronous system.

We also note that the effective interference bound in $T = 3, 5$ is not as close as in the single symbol observation window. This is because the effective interference contribution of an interfering symbol depends not only on how much it overlaps with the window but also on how far away it is from the symbol to be demodulated. Thus, the resulting bound does not take into account the fact that the interfering symbols at the edge of the window have a smaller effect than the interfering symbols overlapping with the demodulated symbol. But as $T$ increases, the relative contribution from these edge symbols goes down and the bound gets tighter, and finally converges to the SIR achieved in the synchronous system. Similar reasoning explains why the bound for the decorrelator is not tight for $T > 1$.

In order to compare the various kinds of receivers analyzed so far, we derive the effective bandwidths for the three linear receivers for an observation window that extends over $T$ symbols. The results here follow from similar arguments as in Section 4. We consider a system consisting of $J$ classes of users, with the $j^{th}$ class requiring an SIR of $\beta_j$ and the number of users in the $j^{th}$ class equal to $[\alpha_j N]$. The effective bandwidth characterization of the interference-limited user capacity region is of the form:

$$\sum_{j=1}^{J} e(\beta_j; T)\alpha_j \leq 1$$
SIR achieved by the MMSE receiver over $T$-symbol observation window in an Asynchronous system.

The bound on the SIR attained using Effective Interference for the MMSE receiver over $T$-symbol observation window in an Asynchronous system.

SIR achieved by the MMSE receiver in the Synchronous system.

Figure 6: Comparison of the SIR achieved by users in a system of equal received powers and $P_{tx} = 20$dB for observation windows spanning over 1, 3, 5 and 7 symbols.
and with received power constraints \( P_j \) is given by

\[
\sum_{j=1}^{J} e(\beta_j; T)\alpha_j \leq \min_{1 \leq j \leq J} \left[ 1 - \frac{\beta_j^2}{P_j} \right]
\]

where the effective bandwidth function \( e(\beta, T) \) depends on the receiver:

\[
e_{mf}(\beta, T) = \beta \text{ degrees of freedom per user.} \tag{29}
\]

\[
e_{mmse}(\beta, T) = \frac{1}{T} \left[ (T - 1) \left( \frac{\beta}{1 + \beta} \right) + 2 \left( 1 - \frac{\ln(1 + \beta)}{\beta} \right) \right] \text{ degrees of freedom per user.} \tag{30}
\]

\[
e_{dec}(\beta, T) = \frac{T + 1}{T} \text{ degrees of freedom per user.} \tag{31}
\]

It is worth emphasizing again that while the characterization for the matched filter receiver is asymptotically exact, those for the MMSE receiver and the decorrelator are inner bounds on the user capacity region (except for the decorrelator when \( T = 1 \)).

If we look at the expressions for the effective bandwidths of the MMSE receiver and the decorrelator given by eqns (30), (31), we notice that the effective bandwidths decrease with the increase in the length of the observation window \( T \) and as \( T \to \infty \), the MMSE and the decorrelator have the effective bandwidth asymptotically approaching the respective effective bandwidths in a synchronous system:

\[
e_{mf}^{\text{sync}}(\beta) = \beta; \quad e_{mmse}^{\text{sync}}(\beta) = \frac{\beta}{1 + \beta}; \quad e_{dec}^{\text{sync}}(\beta) = 1.
\]

All the analysis in this section was made with the assumption that the signature sequences are independent across users and across symbols of any particular user. Here, we compare the theoretical results obtained above with simulation results for the sequences that are random and independent across users, but repeated for each user. In Figure 7, we consider an asynchronous system in which the users have equal received powers, with an observation window of \( T = 3 \). The asymptotic SIR, on the assumption that the signature sequences are independent across symbols is calculated by numerically solving for \( u(x) \) in eqn (23). In the plot, we compare this with the average of the simulated SIRs (500 sample points) for the case when the signature sequences were randomly chosen once and then repeated for the other symbols transmitted by the user within the observation window. The figure shows that the results derived above are almost identical to the performance in the case of repeated signature sequences. The variance of the simulated SIRs for the case of repeated codes is found to reduce with increasing \( N \), similar to those shown in Figure 2.
Average of the simulated SIR achieved by the MMSE receiver over a 3-symbol observation window in an Asynchronous system, when the signature sequences are repeated.

Theoretically predicted SIR achieved by the MMSE receiver over a 3-symbol observation window in an asynchronous system, assuming independent signature sequences.

Figure 7: A comparison of the SIR achieved for an observation window of $T = 3$ with the system when the signature sequences are repeated.

6 Chip-Synchronous versus Completely Asynchronous

In all the sections so far, we analyzed the asynchronous CDMA system with the assumption that the users were chip-synchronous. In this section, we compare the results derived for the chip-synchronous system with some simulations of a chip-asynchronous system with rectangular chip waveform. In addition to this, based on some insights gained in the previous sections, we also propose a heuristic lower bound for the SIR achieved in the chip-asynchronous system.

In Figure 8, we compare the SIR achieved by the MMSE receiver under different assumptions, when the observation window is one symbol duration and the received powers of all the users are equal. The curve plotted in the chip-synchronous system is the result derived in this paper for a single symbol observation window, when the relative delay between users in terms of number of chips is an integer which is uniformly picked in $\{0, 1, \ldots, N - 1\}$. The SIR achieved in the synchronous system as derived in [14] is also plotted for comparison. The solid curve for the completely asynchronous system is the average of 500 sample values of the SIR achieved by user 1 in a simulation when the processing gain is $N = 64$ and the relative delay is a real number uniform in $[0, N)$. From the figure, we notice that the results in the chip-synchronous system, is tight for small $\alpha$ and in general form a conservative estimate for the SIR achieved in the completely asynchronous system.
SIR achieved by the MMSE receiver in an Asynchronous system, when the users are chip-asynchronous.

SIR achieved by the matched filter receiver in an Asynchronous system, when the users are chip-asynchronous.

SIR achieved by the MMSE receiver in the Synchronous system.

Figure 8: Comparison of the SIR achieved by users in a system of equal received powers and $\frac{P}{\sigma^2} = 20$dB.

Heuristic bound on the SIR achieved in the completely asynchronous system, based on the notion of effective interference.

SIR achieved by the matched filter receiver in a Synchronous system.

Figure 9: Comparison of the SIR achieved by users in a system of equal received powers and $\frac{P}{\sigma^2} = 20$dB.
In order to show the dependence of the SIR achieved by the receivers for large values of $\alpha$, in Figure 9, we plot the performance achieved by the matched filter and the MMSE receivers in the chip-synchronous and the chip-asynchronous systems when all the users have equal received powers. The average of the simulated (500 sample points each) SIRs for the chip-synchronous and the completely asynchronous systems with $N = 32$ under the assumptions stated in the previous paragraph is also plotted. From the figure, we notice that for small values of $\alpha$, the SIR achieved by the MMSE receiver in the completely asynchronous system is close to the SIR achieved in the chip-synchronous system and for large values of $\alpha$, the performance is similar to that of the matched filter receiver.

In Figure 9, we notice that at large $\alpha$, the completely asynchronous system has an SIR about 1.76dB greater than that in the chip-synchronous system. This can be explained as follows: when $\alpha$ is large and the attained SIR is small, the MMSE receiver has a performance close to that of the matched filter receiver.

In [13, 9], it was shown that the matched filter receiver in a completely asynchronous system with rectangular chip waveforms has interferers with an effective interference of $\frac{2P}{3}$.

Therefore, for large values of $\alpha$, the matched filter (and hence the MMSE receiver) in the completely asynchronous system has an SIR which is about 1.76dB (10log$_{10}$($\frac{3}{2}$)) greater than that in the chip synchronous system. Based on this and the notion of effective interference, we propose a heuristic lower bound for the SIR achieved by the MMSE receiver in the completely asynchronous case. Since a user of power $P$ interferers with an effective power of $\frac{2P}{3}$ in a chip asynchronous system for the matched filter, we can heuristically predict the effective interference, at the output of the MMSE receiver for user 1 due to an interferer of power $P$ and relative delay $\tau$, at a target SIR of $\beta_1$ to be,

$$I \left( \frac{2P}{3}, P_1; \beta_1 \right) + I \left( 1 - \frac{2P}{3}, P_1; \beta_1 \right)$$

This is plotted as the dotted line in Figure 8. As seen in the figure, it is a decent lower bound to the simulation results, with the maximum difference being less than 1 dB. The heuristic bound as expected approaches that of the matched filter receiver as $\alpha \rightarrow \infty$.

In the case of a decorrelator, as shown in Figure 10, we plot the SIR achieved when the system is chip-synchronous and chip-asynchronous. The plots show that the results derived in the previous sections under the assumption that the users are chip-synchronous is coincident with the mean SIR (of 500 sample points simulated at $N = 64$) achieved when the users are completely asynchronous. Therefore, with the decorrelator as the receiver, in an asynchronous system of processing gain $N$, we can admit approximately $\frac{N}{2}$ users.
SIR achieved by the decorrelator when the users are allowed to be chip asynchronous.

SIR achieved by the decorrelator when the users are chip synchronous.

Figure 10: Comparison of the SIR achieved by users in a decorrelator, when the users have equal received powers and $\frac{E}{N_0} = 20$dB.

7 Summary of Results and Conclusion

In an asynchronous CDMA system, we have characterized the Signal-to-Interference ratio of the users by the notion of effective interference and the user capacity of the system by the notion of effective bandwidths. The effective interference and effective bandwidths of the three receivers are given by eqns (26), (29) and (30), and the results are summarized in Figures 11 and 12. We emphasize that while the effective interference and effective bandwidth characterizations are exact for the matched filter for any $T$, they are exact for the decorrelator only when $T = 1$. For the decorrelator when $T > 1$ and for the MMSE for any $T$, the characterizations yield only lower bounds to performance. However, as $T \to \infty$, these bounds approach the synchronous case and hence they are tight for large observation windows. Another observation is that as the SIR requirement $\beta$ increases, the effective bandwidth under the MMSE receiver approaches that under the decorrelator. However, the performance gap between these two receivers is wider when $T$ is small.

This paper analyzed the asynchronous CDMA system with the assumption that the users were chip-synchronous and characterized the limiting SIR for linear multiuser receivers. We provided a heuristic lower bound for the completely asynchronous system and compared the results derived by means of simulation (Figure 8). The bounds derived in the multisymbol observation window ($T \geq 3$, Figure 6) are not as close as in the single symbol observation window. The notion of
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<td><strong>$T = 5$</strong></td>
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Figure 11: Effective Interference for the MMSE receiver in Synchronous and Asynchronous systems.

Figure 12: Effective Bandwidths for the MMSE receiver in Synchronous and Asynchronous systems.
effective interference might prove of great utility in deriving tighter bounds on the asymptotic SIR, which will give better characterization of user capacity regions.

**Acknowledgments**

We would like to thank Professor Dilip Sarwate for pointing out reference [9] regarding the performance difference of the matched filter receiver in the chip-asynchronous and completely asynchronous cases.

**References**


A  Proof of the Theorem 3.1

The SIR achieved by user 1 in the MMSE receiver for an asynchronous system is given by eqn (5). If \( \mathbf{K}_z = \mathbf{S}_1 \mathbf{D}_1 \mathbf{S}_1^H + \sigma^2 \mathbf{I} \) denotes the covariance matrix of the interference, then the SIR of user 1 is given by,

\[
\beta_1 = P_1 s_1^H \mathbf{K}_z^{-1} s_1
\]  

(32)

The following lemma (proved in [7]) indicates that when the signature sequences are randomly chosen, the above performance measure asymptotically depends in a simple way only on the eigenvalues of the covariance matrix.

**Lemma A.1** If \( s_1 = \frac{1}{\sqrt{N}} (\nu_{11}, \ldots, \nu_{iN})^T \), where \( \nu_{ij} \) are i.i.d zero mean, unit variance random variables with finite fourth moment, and independent of \( \mathbf{A} \), a \( N \times N \) symmetric random matrix, then,

\[
\mathbb{E} \left[ \left( s_1^H A s_1 - \frac{1}{N} \operatorname{tr} A \right)^2 \right] \leq \frac{C_1}{N} \mathbb{E} \left[ \lambda_{\text{max}}^2 (A) \right]
\]

for some constant \( C_1 \) which depends only on the fourth moment of \( \nu_{11} \).

Since \( s_1 \) is independent of \( \mathbf{K}_z \), applying the above lemma to the SIR of user 1 in eqn (32) and noting that \( \lambda_{\text{max}} (\mathbf{K}_z^{-1}) \leq \frac{1}{\sigma^2} \), we have

\[
s_1^H \mathbf{K}_z^{-1} s_1 - \frac{1}{N} \operatorname{tr} \mathbf{K}_z^{-1} \xrightarrow{P} 0
\]  

(33)

Therefore, the problem reduces to showing that \( \frac{1}{N} \operatorname{tr} \mathbf{K}_z^{-1} \) converges and computing the limiting value. Observe that this quantity is actually the Stieltjes transform of the empirical eigenvalue distribution \( \mathcal{H}_N \) of \( \mathbf{S}_1 \mathbf{D}_1 \mathbf{S}_1^H \), evaluated at \( -\sigma^2 \):

\[
\frac{1}{N} \operatorname{tr} \mathbf{K}_z^{-1} = \int_0^\infty \frac{1}{\lambda + \sigma^2} d\mathcal{H}_N (\lambda),
\]  

(34)

(See [14] for more discussions on this). Fortunately for us, many random matrices results characterize the limiting empirical eigenvalue distribution in terms of the limit of the corresponding Stieltjes’ transform. This was how we proved Theorem 3.1 in [14], by appealing to a result in [7, 12].

Since the elements of \( \mathbf{S}_1 \) are neither independent nor identically distributed, the results of [7, 12] cannot be applied for the asynchronous problem. The spirit of the proof for the present result is as follows. Condition on the relative delays and the received powers of the users, the matrix \( \mathbf{S}_1 \) has independent though not identically distributed elements (since some entries are zeros). We
then apply Theorem A.2, a result from [1] on the limiting eigenvalue distribution of such type of random matrices. From this, we infer that the limiting empirical eigenvalue distribution of $S_1S_1^t$, condition on the delays and powers, exists if the empirical distribution of the conditioning delays and powers converges. Moreover, the limit depends only on the limiting empirical distribution of the delays and powers. Hence, the unconditional limit is the same as the conditional limit, almost surely.

The following theorem captures the essence of Corollary 10.1.2 of [1], which is the key random matrix result we need.

**Theorem A.2** Let $A$ be a $n \times [cn]$ random matrix with independent entries which are zero-mean and satisfy the condition

$$n \text{Var}(A_{ij}) < B$$

for some uniform bound $B < \infty$. Moreover, suppose we define for each $n$ a function $v_n : [0, 1] \times [0, c] \rightarrow \mathbb{R}$ by:

$$v_n(x, y) = n \text{Var}(A_{ij}), \quad i, j \text{ satisfying } \frac{i}{n} \leq x \leq \frac{i+1}{n}, \frac{j}{n} \leq y \leq \frac{j+1}{n},$$

and that $v_n$ converges uniformly to a limiting bounded function $v$. Then the limiting eigenvalue distribution $H^*$ of $A_n A_n^t$ exists and for every $t \geq 0$ satisfies:

$$\int_0^\infty (1 + t\lambda)^{-1} d H^*(\lambda) = \frac{1}{\int_0^1 u(x, t) dx}$$

(35)

and $u(x, t)$ satisfies the equation,

$$u(x, t) = \frac{1}{1 + t \int_0^c \frac{v(x, y) dy}{\int_0^1 u(z, t) v(z, y) dz}}$$

(36)

The solution of eqn (36) exists and is unique in the class of functions $u(x, t) \geq 0$, analytical on $t$ and continuous on $x \in [0, 1]$.

In an asynchronous system, if the received powers of the users belong to a set with finite number of discrete power levels, then we can use the above theorem to find the limiting SIR achieved by user 1. Assume that as $N \rightarrow \infty$, the limiting power distribution of the users has $M$ discrete power levels, $P_1, \ldots, P_M$ occurring with probability masses $q_1, \ldots, q_M$ respectively. We also have that the empirical relative delay distribution converges to $G(\tau)$. Now, if we rearrange the $S_1S_1^\frac{1}{2}$ matrix
by grouping users of different power levels into different blocks and within each block, arranging the users in the increasing order of their delays, then from the Strong Law of Large Numbers,

\[
v(x,y) = \begin{cases} 
P_m & 0 \leq x \leq G \left( \frac{y-2\alpha \sum_{i=1}^{m} q_i + 2\alpha q_m}{\alpha q_m} \right) \\
0 & \text{Otherwise}
\end{cases}
\]

\[
v(x,y) = \begin{cases} 
P_m & G \left( \frac{y-2\alpha \sum_{i=1}^{m} q_i + \alpha q_m}{\alpha q_m} \right) < x \leq 1 \\
0 & \text{Otherwise}
\end{cases}
\]

for \( m = 1, \ldots, M \). Therefore, for a system that has \( M \) discrete power levels, from Theorem A.2,

\[
u_M(x,t) = \frac{1}{1 + t \int_0^c v(x,y) \, dy} \int_0^c \frac{1}{1 + t \int_0^t u_M(z,t)v(z,y)dz} dz
\]

\[
= \frac{1}{1 + \alpha t \sum_{m=1}^M \mathbb{E}_p \left[ \frac{P_m}{1 + tP_m \int_0^c u_M(z,t)dz} 1_{\tau \geq x} + \frac{P_m}{1 + tP_m \int_0^1 u_M(z,t)dz} 1_{\tau < x} \right]} q_m
\]

\[
= \frac{1}{1 + \alpha t \mathbb{E}_p \mathbb{E}_\tau \left[ \frac{P}{1 + tP \int_0^c u_M(z,t)dz} 1_{\tau \geq x} + \frac{P}{1 + tP \int_0^1 u_M(z,t)dz} 1_{\tau < x} \right]}
\]

where \( \mathbb{E}_p \) is the expectation with respect to the probability masses \( q_i \). Therefore, from Theorem A.2, the limiting eigenvalue distribution of \( S_1D_1S_1^t \) converges to a limit \( H_M^* \) for the system which has finite number of power levels. Combining this with (33) and (34), it follows that:

\[
\beta_1^{(N)} \xrightarrow{P} \int_0^\infty \frac{P_1}{\lambda + \sigma^2} dH_M^*(\lambda) := \beta_M^*
\]

Using eqn (35), this limit can be calculated:

\[
\beta_M^* = \frac{P_1}{\sigma^2} \int_0^1 u_M(x, \frac{1}{\sigma^2}) dx
\]
Now, if we define \( w_M(x) = \frac{P_1}{\sigma^2} u(x, 1/\sigma^2) \), then,

\[
\begin{align*}
    w_M(x) &= \frac{P_1}{\sigma^2 + \alpha E_P E_T} \left\{ I \left( P, P_1, \int_0^\tau w_M(z) dz \right) 1_{\{\tau \geq x\}} + I \left( P, P_1, \int_\tau^1 w_M(z) dz \right) 1_{\{\tau \leq x\}} \right\}
\end{align*}
\]  

(37)

and the SIR of user 1 converges in probability to

\[
\beta^*_M = \int_0^1 w_M(x) dx.
\]

(38)

This is precisely Theorem 3.1 for the case when the number of power levels are finite.

In order to extend the above result to a more general case where the limiting power distribution is \( F(P) \), we start by approximating \( F(P) \) by staircase functions, thus reducing it to the case where there are finitely many power levels. For a given \( M \), we can define upper and lower staircase approximations for \( F(P) \), of step height \( \frac{1}{M} \) by

\[
\begin{align*}
    \bar{P}_m &= \sup \{ P : F(P) \leq \frac{m - 1}{M} \} \text{ occurring with probability } q_m = \frac{1}{M} \text{ for } m = 1, \ldots, M \\
    \underline{P}_m &= \inf \{ P : F(P) \geq \frac{m}{M} \} \text{ occurring with probability } q_m = \frac{1}{M} \text{ for } m = 1, \ldots, M
\end{align*}
\]

From the above construction of the staircase functions, we have that \( \bar{P}_m - \underline{P}_m \leq \frac{1}{M} \). These two approximations will have corresponding \( \bar{\nu}^M(x, y), \bar{\nu}_M^*(x), \bar{w}_M(x) \) and \( \underline{\nu}^M(x, y), \underline{\nu}_M^*(x), \underline{w}_M(x) \) defined and from the derivations above, as \( N \to \infty \),

\[
\beta^{(N)}_M \xrightarrow{P} \beta^*_M = \int_0^1 \bar{w}_M(x) dx
\]

where \( \bar{w}_M(x) \) satisfies the fixed point equation,

\[
\begin{align*}
    \bar{w}_M(x) &= \frac{P_1}{\sigma^2 + \alpha E_P E_T} \left\{ I \left( \bar{P}, P_1, \int_0^\tau \bar{w}_M(z) dz \right) 1_{\{\tau \geq x\}} + I \left( \bar{P}, P_1, \int_\tau^1 \bar{w}_M(z) dz \right) 1_{\{\tau \leq x\}} \right\}
\end{align*}
\]

and

\[
\beta^{(N)}_M \xrightarrow{P} \beta^*_M = \int_0^1 \underline{w}_M(x) dx
\]

with

\[
\begin{align*}
    \underline{w}_M(x) &= \frac{P_1}{\sigma^2 + \alpha E_P E_T} \left\{ I \left( \underline{P}, P_1, \int_0^\tau \underline{w}_M(z) dz \right) 1_{\{\tau \geq x\}} + I \left( \underline{P}, P_1, \int_\tau^1 \underline{w}_M(z) dz \right) 1_{\{\tau \leq x\}} \right\}
\end{align*}
\]
and \( \mathbb{E}_T, \mathbb{E}_P \) are expectations with respect to the upper and lower staircase distributions. From the definition of \( I(P, P_1, \beta) \), and the fact that \( P_m - P_m \leq \frac{1}{M} \),

\[
\left| I \left( P, P_1, \int_0^\tau w_M(z)dz \right) - I \left( P, P_1, \int_0^\tau w_M(z)dz \right) \right| \leq \frac{1}{M}
\]

and

\[
\left| I \left( P, P_1, \int_0^1 w_M(z)dz \right) - I \left( P, P_1, \int_0^1 w_M(z)dz \right) \right| \leq \frac{1}{M}
\]

It follows that

\[
\lim_{M \to \infty} \beta_M = \lim_{M \to \infty} \beta^* = \beta^*
\]

where

\[
\beta^* = \int_0^1 w(x)dx
\]

and

\[
w(x) = \frac{P_1}{\sigma^2 + \alpha \mathbb{E}_P \mathbb{E}_T \left\{ I \left( P, P_1, \int_0^\tau w(z)dz \right) 1_{\{\tau \geq x\}} + I \left( P, P_1, \int_0^1 w(z)dz \right) 1_{\{\tau \leq x\}} \right\}}
\]

and \( \mathbb{E}_F \) is expectation with respect to \( F(P) \) to which the upper and lower staircases converge weakly to as \( M \to \infty \).

The SIR achieved by user 1 at any spreading length \( N \), as given by eqn (5) is monotonically non-increasing with the increasing powers of the interferers. Therefore, if \( \beta^{(N)} \) denotes the SIR achieved by user 1 in the system which has a power distribution \( F(P) \), then it follows that for any \( M \),

\[
\beta^{(N)}_M \leq \beta^{(N)} \leq \beta^{(N)}_M
\]

A simple sandwich argument now shows that

\[
\beta^{(N)} \xrightarrow{P} \beta^*
\]

for general power distribution \( F \). \(\square\)
B Proofs of Theorems 3.2 and 5.2

We prove the bound proposed for the limiting SIR in Theorem 3.2 by first noting some of the properties of \( w(x) \) when \( G(\tau) = 1 - G(1 - \tau) \).

**Lemma B.1** The function \( w(x) \) is symmetric about \( \frac{1}{2} \) for all delay distributions which satisfy \( G(\tau) = 1 - G(1 - \tau) \) (probability density function symmetric about \( \frac{1}{2} \)).

**Proof:**

To prove that \( w(x) \) is symmetric about \( \frac{1}{2} \), we need to prove \( w(x) = w(1 - x) \). From eqn (7),

\[
w(1 - x) = \frac{P_1}{\sigma^2 + \alpha \mathbb{E}_{P_1} \mathbb{E}_{\tau} \left\{ I \left( P, P_1, \int_0^\tau w(z)dz \right) 1_{\{\tau \geq 1 - x\}} + I \left( P, P_1, \int_0^1 w(z)dz \right) 1_{\{\tau \leq 1 - x\}} \right\}}
\]

\[
= \frac{P_1}{\sigma^2 + \alpha \mathbb{E}_{P_1} \mathbb{E}_{\tau} \left\{ I \left( P, P_1, \int_0^{1 - \tau} w(z)dz \right) 1_{\{\tau \leq x\}} + I \left( P, P_1, \int_1^{1 - \tau} w(z)dz \right) 1_{\{\tau \geq x\}} \right\}}
\]

\[
= \frac{P_1}{\sigma^2 + \alpha \mathbb{E}_{P_1} \mathbb{E}_{\tau} \left\{ I \left( P, P_1, \int_{\tau}^1 w(1 - z)dz \right) 1_{\{\tau \leq x\}} + I \left( P, P_1, \int_0^\tau w(1 - z)dz \right) 1_{\{\tau \geq x\}} \right\}}
\]

The second equality follows from the fact that \( G(\tau) = 1 - G(1 - \tau) \). Since \( w(1 - x) \) also satisfies the above fixed point equation which has a unique solution \( w(x) \), we have, \( w(1 - x) = w(x) \). \( \square \)

**Lemma B.2** The function \( w(x) \) is a non-decreasing function in \( x \in [0, \frac{1}{2}] \).

From lemma B.1, this implies that the function \( w(x) \) is non-increasing in \( [\frac{1}{2}, 1] \).

**Proof:**

Notice that the dependence on the argument \( x \) in eqn (7) appears only within the two expectations in the interference terms. Denote \( \mathcal{G}_{x}(P, P_1, \tau) \) to be

\[
\mathcal{G}_{x}(P, P_1, \tau) = \left[ I \left( P, P_1, \int_{\tau}^1 w(z)dz \right) 1_{\{\tau \geq x\}} + I \left( P, P_1, \int_0^\tau w(z)dz \right) 1_{\{\tau \leq x\}} \right].
\]

If \( x_1 \leq x_2 \), then,

\[
\mathcal{G}_{x_1}(P, P_1, \tau) - \mathcal{G}_{x_2}(P, P_1, \tau) = \left[ I \left( P, P_1, \int_{\tau}^1 w(z)dz \right) - I \left( P, P_1, \int_0^\tau w(z)dz \right) \right] 1_{\{x_1 \leq \tau \leq x_2\}}
\]
The function \( w(x) \) is symmetric about \( \frac{1}{2} \) and so \( 2 \int_0^{1/2} w(z)dz = \int_0^1 w(z)dz \). From the fact that \( w(x) \geq 0 \) for \( x \in [0,1] \), we have \( \int_0^1 w(z)dz \geq 2 \int_0^\tau w(z)dz \) if \( \tau \leq \frac{1}{2} \), that is, \( \int_0^\tau w(z)dz \leq \frac{1}{2} \int_0^1 w(z)dz \) if \( \tau \leq \frac{1}{2} \). Now from the decreasing property of \( I(P,P_1,y) \) with respect to \( y \), we have,

\[
I \left( P, P_1, \int_0^\tau w(z)dz \right) - I \left( P, P_1, \int_0^1 w(z)dz \right) \geq 0 \text{ for all } \tau \leq \frac{1}{2}
\]

and thus \( G_{x_1}(P, P_1, \tau) - G_{x_2}(P, P_1, \tau) \geq 0 \) for all \( \tau \leq \frac{1}{2} \) and \( x_1 \leq x_2 \leq \frac{1}{2} \).

Therefore, if \( x_1 \leq x_2 \leq \frac{1}{2} \), then \( w(x_1) \leq w(x_2) \leq w(\frac{1}{2}) \). \( \square \)

The following is a key lemma.

**Lemma B.3** Let \( G(x) \) denote the distribution of the random variable \( X \) and for some odd integer \( T \), let \( g(t) \) denote a non-negative function for \( t \in [0,T] \), non-decreasing in \( [0,\frac{T}{2}] \), symmetric about \( \frac{T}{2} \) and has an area \( \int_0^T g(t)dt = \mu \). Among all such functions, \( \bar{g}(t) = \frac{\mu}{T} \) for all \( t \in [0,T] \) minimizes the function,

\[
\mathcal{F}(g) = \mathbb{E}_X \left\{ \frac{1}{X} \int_0^X g(t)dt + \sum_{i=1}^{T-1} \frac{1}{X+i} \int_{X+i}^{X+i+1} g(t)dt + \frac{1}{X+T} \int_X^{X+T-1} g(t)dt \right\}
\]

**Proof:**

The above lemma is proved using the concept of majorization. \([8] : \) \( u = (u_1, \ldots, u_n) \in \mathbb{R}^n \) is said to majorize \( v = (v_1, \ldots, v_n) \in \mathbb{R}^n \),

\[
u \preceq v \quad \text{if} \quad \begin{cases} \sum_{i=1}^k u_i \leq \sum_{i=1}^k v_i, & k = 1, \ldots, n-1 \\ \sum_{i=1}^n u_i = \sum_{i=1}^n v_i \end{cases}
\]

where \((u_1, \ldots, u_n)\) denotes the elements of \( u \) arranged in decreasing order, \( u_1 \geq \cdots \geq u_n \).

Now, define \( y(g) \) such that,

\[
y_k(g) = \begin{cases} \int_0^x g(t)dt & k = 1 \\ \int_{x+k}^x g(t)dt & k = 2, \ldots, T-1 \\ \int_{x+T-1}^x g(t)dt & k = T \end{cases}
\]

and \( \mathcal{F}(g) = \sum_{k=1}^T \frac{1}{1+y_k(g)} \).
The function \( \sum_{k=1}^{T} \frac{1}{1+b_k} \) is schur convex and therefore from the properties of majorization,

\[
\mathcal{F}(g_1) \leq \mathcal{F}(g_2) \text{ if } y(g_1) \preceq y(g_2).
\]

Among all non-negative functions \( g(t) \) that are symmetric about \( \frac{T}{2} \), non-decreasing in \( [0, \frac{T}{2}] \) and a constant area \( \mu \), it can be seen that \( \bar{g}(t) = \frac{\mu}{t} \) has the property,

\[
y(\bar{g}) \preceq y(g) \text{ for all } g
\]

Therefore, \( \bar{g}(t) \) for all \( t \in [0, T] \) minimizes the function \( \mathcal{F}(g) \).

Lemma B.3 plays an important role in attaining the bound on the SIR. If we set \( T = 1 \), then, the above lemma states that between two functions \( g_1(t) \) and \( g_2(t) \) with same area in \( [0, 1] \), \( \mathcal{F}(g) \) is smaller for the one which has more area concentrated in the regions near zero and one. That is,

\[
\frac{\mu}{2} \geq \int_0^x g_1(t)dt \geq \int_0^x g_2(t)dt \quad \forall \quad x \leq \frac{1}{2} \quad \Rightarrow \quad \mathcal{F}(g_1) \leq \mathcal{F}(g_2)
\]

The three lemma stated above provide enough footing to derive the bound from the actual SIR attained. To prove Theorem 3.2, we begin from eqn (7),

\[
w(x) = \frac{P_1}{\sigma^2 + \alpha \mathbb{E}_P \mathbb{E}_\tau \left\{ I \left( P, P_1, \int_0^\tau w(z)dz \right) 1_{\{\tau \geq x\}} + I \left( P, P_1, \int_\tau^1 w(z)dz \right) 1_{\{\tau \leq x\}} \right\}}
\]

Cross multiplying and integrating both sides with respect to \( x \) from 0 to 1,

\[
\sigma^2 \int_0^1 w(x)dx
\]

\[
= P_1 - \alpha \int_0^1 \mathbb{E}_P \mathbb{E}_\tau \left\{ I \left( P, P_1, \int_0^\tau w(z)dz \right) 1_{\{\tau \geq x\}} + I \left( P, P_1, \int_\tau^1 w(z)dz \right) 1_{\{\tau \leq x\}} \right\} w(x)dx
\]

\[
= P_1 - \alpha \mathbb{E}_P \left[ \int_0^1 \int_0^1 \frac{PP_1}{P_1 + P} w(x) \frac{dG(\tau)}{dx} \cdot dx + \int_0^1 \int_0^\tau \frac{PP_1}{P_1 + P} w(x) \frac{dG(\tau)}{dx} \cdot dx + \int_0^1 \int_0^\tau \frac{PP_1}{P_1 + P} w(x) \frac{dG(\tau)}{dx} \cdot \right] dx
\]

\[
= P_1 - \alpha \mathbb{E}_P \left[ \int_0^1 \frac{PP_1}{P_1 + P} \frac{w(x)}{\int_0^\tau w(z)dz} \cdot dx + \int_0^1 \frac{PP_1}{P_1 + P} \frac{w(x)}{\int_0^\tau w(z)dz} \cdot \right] dx
\]

\[
= P_1 - \alpha \mathbb{E}_P \left[ \int_0^1 \frac{PP_1}{P_1 + P} \frac{w(x)}{\int_0^\tau w(z)dz} \cdot dx + \int_0^1 \frac{PP_1}{P_1 + P} \frac{w(x)}{\int_0^\tau w(z)dz} \cdot \right] dx
\]
where the last step follows from the fact that the first integral is over the triangular area \( \tau = [0, x] \) and \( x = [0, 1] \), which is equivalent to the integral over \( x = [\tau, 1] \) and \( \tau = [0, 1] \). The second integral follows from a similar argument over the other half of the square in \( \tau - x \) plane. Therefore,

\[
\sigma^2 \int_0^1 w(x)dx = (1 - 2\alpha)P_1 + \alpha P_1^2 \mathbb{E}_P \mathbb{E}_\tau \left[ \frac{1}{P_1 + P \int_0^\tau w(z)dz} + \frac{1}{P_1 + P \int_\tau^1 w(z)dz} \right]
\]

(41)

Setting \( T = 1 \) in lemma B.3, we have that the right hand side is minimized when \( w(x) = \int_0^1 w(z)dz \) for all \( x \in [0, 1] \). Using this, and since \( \beta^*_1 = \int_0^1 w(z)dz \), we have,

\[
\sigma^2 \beta^*_1 \geq P_1 - \alpha \beta^*_1 \mathbb{E}_P \mathbb{E}_\tau \{ I(\tau P, P_1, \beta^*_1) + I((1 - \tau)P, P_1, \beta^*_1) \}
\]

Therefore, using the monotonicity property of effective interference, \( \beta^*_1 \) is lower bounded by the unique solution \( \gamma^*_1 \) of the fixed point equation,

\[
\frac{P_1}{\sigma^2 + \alpha \mathbb{E}_P \mathbb{E}_\tau [I(\tau P, P_1, \gamma^*_1) + I((1 - \tau)P, P_1, \gamma^*_1)]}
\]

where \( I(P, P_1, \gamma^*_1) \) is the effective interference from an interferer of power \( P \) at SIR \( \gamma^*_1 \) defined as

\[
I(P, P_1, \gamma^*_1) = \frac{PP_1}{P_1 + P \gamma^*_1}
\]

This proves Theorem 3.2.

In the case of multiple symbol observation window, we observe that the function \( w(x) \) defined in eqn (23) is symmetric about \( \frac{T}{2} \) and is non-decreasing in \([0, \frac{T}{2}]\). If we cross-multiply the two sides of eqn (23) and integrate between 0 and \( T \),

\[
\sigma^2 \int_0^T w(x)dx = P_1T - \alpha \int_0^T \mathbb{E}_{P, \tau} I(P, P_1, \int_0\mathbb{E}_{x, \tau} w(z)dz) w(x)dx
\]

\[
= TP_1 - (T + 1)P_1\alpha + \alpha P_1^2 \mathbb{E}_{P, \tau} \left[ \frac{1}{P_1 + P \int_0^\tau w(z)dz} + \sum_{i=1}^{T-1} \frac{1}{P_1 + P \int_{\tau+i-1}^{\tau+i} w(z)dz} + \frac{1}{P_1 + P \int_{T-1}^T w(z)dz} \right]
\]
From lemma B.3, we have that the right hand side is minimized when \( w(x) = \frac{1}{T} \int_0^T w(z)dz \) for all \( x \in [0, T] \) and hence,

\[
\int_{\frac{T}{2}}^{\frac{3T}{2}} w(x)dx = \frac{1}{T} \int_0^T w(x)dx
\]

Thus, we arrive at the bound as stated in Theorem 5.2.

Let us now give some intuition about how this bound comes about, by tying it more closely to the MMSE estimation problem. For simplicity, we will assume that \( P_k = 1 \) for all \( k \), and focus on the \( T = 1 \) case, with the channel given by eqn (2).

As defined in eqn (5), \( \text{SIR}_1 \) is the SIR achieved by the MMSE receiver in estimating \( x_1 \) from the received vector \( r \). Let \( \text{SIR}_{xk} \) and \( \text{SIR}_{yk} \) be the SIR achieved in the MMSE estimation of symbols \( x_k \) and \( y_k \) of user \( k \) from \( r \). Note that this estimation is from an observation window locked onto symbol \( x_1 \) of user 1, not onto \( x_k \) or \( y_k \). Following similar development as that of eqn (27) in Appendix A of [14], we obtain the following key equation relating the SIR's achieved by all the users from the same observation window:

\[
\sigma^2 \text{Tr}(SS' + \sigma^2 I)^{-1} = N - 2K - 1 + \frac{1}{1 + \text{SIR}_1} + \sum_{k=1}^{K} \left[ \frac{1}{1 + \text{SIR}_{xk}} + \frac{1}{1 + \text{SIR}_{yk}} \right]
\]  

(42)

where \( S \) is the matrix whose columns are \( x_1, u_2, \ldots, u_K, v_1, \ldots, v_K \). Let

\[
C^{(N)} := (SS' + \sigma^2 I)^{-1}
\]

Lemma A.1 tells us that in a large system:

\[
\text{SIR}_1 \approx \frac{1}{N} \text{Tr}(S_1S_1' + \sigma^2 I)^{-1} = \frac{1}{N} \text{Tr}C^{(N)}
\]

Hence, eqn (42) quantifies (approximately) the effect of the virtual interferers on \( \text{SIR}_1 \) in terms of the SIR achieved by the interferers themselves. The larger the SIR achieved by an interfering symbol, the more adverse is its interfering effect.

We can apply similar type of approximation to \( \text{SIR}_{xk}, \text{SIR}_{yk} \), the only difference being that the signature sequences of these symbols do not lie entirely in the observation window.

\[
\text{SIR}_{xk} \approx \frac{1}{N} \sum_{i=1}^{d_k} C^{(N)}_{ii}
\]

\[
\text{SIR}_{yk} \approx \frac{1}{N} \sum_{i=d_k+1}^{N} C^{(N)}_{ii}
\]
Substituting all these approximations into (42) and dividing by \( N \), it holds that for large \( N \),

\[
\sigma^2 \frac{1}{N} \text{Tr} C^{(N)} \approx 1 - 2\alpha + \frac{1}{N} \sum_{i=1}^{K} \left[ \frac{1}{1 + \frac{1}{N} \sum_{k=1}^{d_k} C^{(N)}_{ii}} + \frac{1}{1 + \frac{1}{N} \sum_{k=d_k+1}^{N} C^{(N)}_{ii}} \right]
\]  

(43)

In the synchronous case, all delays are 0, and (43) yields a fixed point equation for \( \text{SIR}_1 \approx \frac{1}{N} \text{Tr} C^{(N)} \). In the asynchronous case, however, this equation is not enough to characterize \( \text{SIR}_1 \).

A closer inspection of the proof of Theorem A.2 reveals that the function \( w(\cdot) \) in fact represents the limiting value of the diagonal elements of \( \frac{1}{N} C^{(N)} \), such that

\[
\frac{1}{N} \sum_{i=1}^{d_k} C^{(N)}_{ii} \approx \int_0^{d_k} w(z)dz
\]

\[
\frac{1}{N} \sum_{i=d_k+1}^{N} C^{(N)}_{ii} \approx \int_{d_k+1}^{1} w(z)dz
\]

This establishes the correspondence between eqn (41) and (43) (for the case of \( P_k = 1 \) for all \( k \)). Through this correspondence, we see that the effect on the demodulation of user 1 from the interfering symbol \( x_k \), at a normalized delay \( \tau_k := d_k/N \), is through the term

\[
\frac{1}{1 + \int_0^{\tau_k} w(z)dz}
\]

where \( \int_0^{\tau_k} w(z)dz \) is the limiting value of \( \text{SIR}_{ck} \). The fact that \( w(\cdot) \) decreases towards the edge of the observation window can be interpreted as saying that the MMSE receiver gets poorer quality of information from the chips on the edge of the window than the chips near the center. But this also means that the interference caused by the received sequence of a symbol is lower at the edge of the window than at the center. The majorization argument behind Lemma B.3 says that this interference effect can be bounded by assuming the interfering effect is constant throughout the window, giving rise to a bound in which the effective interference of a symbol depends only on the amount of overlap of that symbol with the observation window.

### C Proof of Theorem 3.3

To prove this theorem, we shall rely on a geometric interpretation of the decorrelator. The decorrelator can be defined as a linear function of the received vector, \( \mathbf{r} \), which maximizes the SIR, subject to the constraint that the estimate is independent of the other interfering symbols. So, if
\( \hat{x}_{\text{dec}}(r) \) denotes the estimate of user 1’s information symbol, then,

\[
\hat{x}_{\text{dec}}(r) = c^t r \\
= (c^t s_1) x_1 + \sum_{k=2}^{K} (c^t u_k + y_k c^t v_k) + c^t n
\]

Since the estimate is independent of all the interfering symbols, \( x_2, \ldots, x_K, y_2, \ldots, y_K \), we have \( c^t u_k = c^t v_k = 0 \) for all \( i = 2, \ldots, K \). Therefore, if we define \( C = \text{span}\{u_2, \ldots, u_K, v_2, \ldots, v_K\}^\perp \), then, the decorrelator can be obtained by constraining the vector \( c \in C \) and maximizing the SIR achieved. From the fact that \( ||s_1 - c||^2 = ||s_1||^2 + ||c||^2 - 2c^t s_1 \), we have that the maximum is achieved when \( c \) is the projection of \( s_1 \) on to the subspace \( C \) and has a norm \( ||c|| \). If we denote this optimal \( c \) by \( h \), then the SIR of the decorrelator is given by,

\[
\beta_1 = \frac{P_1 (h^t s_1)^2}{\sigma^2 h^t h} = \frac{P_1}{\sigma^2} h^t h
\]  \hspace{1cm} (44)

When the spreading sequences are independently and randomly chosen, \( C \) (comprising of \( 2(K-1) \) signature sequences) will have a dimension of \( \max\{N-2(K-1), 0\} \), with high probability, as \( N \to \infty \). Moreover from the independence of the signature sequences, we have that \( s_1 \) is independent of the subspace \( C \). Hence, from lemma A.1, we have that \( h^t h \to 1 - 2\alpha \) in probability as \( N, K \to \infty, \frac{K}{N} \to \alpha \) and \( \alpha < 1 \). If \( \alpha \geq 1 \), then \( h^t h \to 0 \) in probability, thus proving the theorem. \( \square \)