Diversity-Multiplexing Tradeoff in MIMO Channels

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Intel Smart Antenna Workshop
Two objectives of the talk:

- Present a new performance metric for evaluating MIMO coding schemes.
- Give some examples of new coding schemes designed to optimize the metric.
Diversity and Freedom

Two fundamental resources of a MIMO fading channel:

diversity

degrees of freedom
Diversity

A channel with more diversity has smaller probability in deep fades.
Diversity

- Additional independent channel paths increase diversity.
- Spatial diversity: receive, transmit or both.
- For a $m \times n$ channel, maximum diversity is $mn$. 

![Fading Channel: $h_1$](image-url)
• Additional independent fading channels increase diversity.
Diversity

- Additional independent fading channels increase diversity.
- Spatial diversity
Additional independent fading channels increase diversity.

Spatial diversity: receive, transmit.
• Additional independent fading channels increase diversity.
• Spatial diversity: receive, transmit or both.
Additional independent fading channels increase diversity.

Spatial diversity: receive, transmit or both.

For a $m$ by $n$ channel, diversity is $mn$. 
Signals arrive in multiple directions provide multiple degrees of freedom for communication. Same effect can be obtained via scattering even when antennas are close together. In a \( m \times n \) channel with rich scattering, there are \( \min\{m, n\} \) degrees of freedom.
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Same effect can be obtained via scattering even when antennas are close together.

In a $m$ by $n$ channel with rich scattering, there are $\min\{m,n\}$ degrees of freedom.
Diversity and Freedom

In a MIMO channel with rich scattering:

maximum diversity = $mn$

degrees of freedom = $\min\{m, n\}$

The name of the game in space-time coding is to design schemes which exploit as much of both these resources as possible.
Space-Time Code Examples: 2 × 1 Channel

Repetition Scheme:

\[
\begin{bmatrix}
    x_1 & 0 \\
    0 & x_1 \\
\end{bmatrix}
\]

diversity: 2
data rate: 1/2 sym/s/Hz

Alamouti Scheme:

\[
\begin{bmatrix}
    x_1 & -x_2^* \\
    -x_1^* & x_2 \\
\end{bmatrix}
\]

diversity: 2
data rate: 1 sym/s/Hz
Performance Summary: $2 \times 1$ Channel

<table>
<thead>
<tr>
<th></th>
<th>Diversity gain</th>
<th>Degrees of freedom utilized /s/Hz</th>
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<tbody>
<tr>
<td>Repetition</td>
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Space-Time Code Examples: $2 \times 2$ Channel

Repetition Scheme:

$X = \begin{bmatrix} x_1 & 0 \\ 0 & x_1 \end{bmatrix}$

diversity gain : 4
data rate: 1/2 sym/s/Hz

Alamouti Scheme:

$X = \begin{bmatrix} x_1 & -x_2^* \\ x_2 & x_1^* \end{bmatrix}$

diversity gain : 4
data rate: 1 sym/s/Hz
Space-Time Code Examples: $2 \times 2$ Channel

Repetition Scheme:

$$X = \begin{bmatrix} x_1 & 0 \\ 0 & x_1 \end{bmatrix}$$

diversity: 4  
data rate: $1/2$ sym/s/Hz

But the $2 \times 2$ channel has 2 degrees of freedom!

Alamouti Scheme:

$$X = \begin{bmatrix} x_1 & -x_2^* \\ x_2 & x_1^* \end{bmatrix}$$

diversity: 4  
data rate: 1 sym/s/Hz
V-BLAST with Nulling

Send two independent uncoded streams over the two transmit antennas. Demodulate each stream by nulling out the other stream.

**Data rate:** 2 sym/s/Hz

**Diversity:** 1

Winters, Salz and Gitlins 93:

Nulling out $k$ interferers using $n$ receive antennas yields a diversity gain of $n - k$. 
## Performance Summary: 2 × 2 Channel

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Questions:

- Alaomuti is clearly better than repetition, but how can it be compared to V-Blast?
### Performance Summary: $2 \times 2$ Channel

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- Alaomuti is clearly better than repetition, but how can it be compared to V-Blast?
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### Questions:

- Alaomuti is clearly better than repetition, but how can it be compared to V-Blast?
- How does one quantify the “optimal” performance achievable by any scheme?
- We need to make the notions of “fiversity gain” and “d.o.f. utilized” precise and enrich them.
Classical Diversity Gain

Motivation: PAM

\[ y = hx + w \quad \Rightarrow \quad P_e \approx P(\|h\| \text{ is small}) \propto SNR^{-1} \]

\[
\begin{cases}
  y_1 = h_1x + w_1 \\
  y_2 = h_2x + w_2
\end{cases}
\]

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Classical Diversity Gain

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\right. \quad \Rightarrow \quad Pe \approx P(\|h_1\|, \|h_2\| \text{ are both small}) \propto \text{SNR}^{-2}
\]

General Definition

A space-time coding scheme achieves (classical) diversity gain \( d_{\text{max}} \), if

\[ Pe(\text{SNR}) \sim \text{SNR}^{-d_{\text{max}}} \]

for a fixed data rate.

i.e. error probability decreases by \( 2^{-d_{\text{max}}} \) for every 3 dB increase in SNR,
by \( 4^{-d_{\text{max}}} \) for every 6dB increase, etc.
Example: PAM vs QAM in 1 by 1 Channel

Every 6 dB increase in SNR doubles the distance between constellation points for a given rate.

\[ P_c \downarrow \frac{1}{4} \]
Example: PAM vs QAM in 1 by 1 Channel

Every 6 dB increase in SNR doubles the distance between constellation points for a given rate.

Both PAM and QAM have the same (classical) diversity gain of 1.
Example: PAM vs QAM in 1 by 1 Channel

Every 6 dB increase in SNR doubles the distance between constellation points for a given rate.

Both PAM and QAM have the same (classical) diversity gain of 1. (classical) diversity gain does not say anything about the d.o.f. utilized by the scheme.
Ask a Dual Question

Every 6 dB doubles the constellation size for a given reliability, for PAM.

PAM

-3a -a +a +3a

+1 bit

-a +a
Every 6 dB doubles the constellation size for a given reliability, for PAM

For QAM, every 6 dB quadruples the constellation size.
Degrees of Freedom Utilized

Definition:
A space-time coding scheme utilizes $r_{\text{max}}$ degrees of freedom/s/Hz if
the data rate scales like

$$R(\text{SNR}) \sim r_{\text{max}} \log_2 \text{SNR} \quad \text{bits/s/Hz}$$

for a fixed error probability (reliability)

In a $1 \times 1$ channel, $r_{\text{max}} = 1/2$ for PAM, $r_{\text{max}} = 1$ for QAM.
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In a $1 \times 1$ channel, $r_{\text{max}} = 1/2$ for PAM, $r_{\text{max}} = 1$ for QAM.

Note: A space-time coding scheme is a family of codes within a certain structure, with varying symbol alphabet as a function of SNR.
Diversity-Multiplexing Tradeoff

Every 3 dB increase in SNR yields

either

a $2^{-d_{\text{max}}}$ decrease in error probability for a fixed rate;

or

$r_{\text{max}}$ additional bits/s/Hz for a fixed reliability.
Diversity-Multiplexing Tradeoff

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But these are two extremes of a rate-reliability tradeoff.
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But these are two extremes of a rate-reliability tradeoff.

More generally, one can increase reliability and the data rate at the same time.
Diversity-Multiplexing Tradeoff of A Scheme

(Zheng and Tse 03)

Definition

A space-time coding scheme achieves a diversity-multiplexing tradeoff curve $d(r)$ if for each multiplexing gain $r$, simultaneously

$$R(\text{SNR}) \sim r \log_2 \text{SNR} \text{ bits/s/Hz}$$

and

$$P_e(\text{SNR}) \sim \text{SNR}^{-d(r)}.$$
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\]

and

\[
P_e(\text{SNR}) \sim \text{SNR}^{-d(r)}.
\]

The largest multiplexing gain is \( r_{\text{max}} \), the d.o.f. utilized by the scheme.

The largest diversity gain is \( d_{\text{max}} = d(0) \), the classical diversity gain.
Diversity-Multiplexing Tradeoff of the Channel

Definition

The diversity-multiplexing tradeoff $d^*(r)$ of a MIMO channel is the best possible diversity-multiplexing tradeoff achievable by any scheme.

$r_{\text{max}}^*$ is the largest multiplexing gain achievable in the channel.

$d_{\text{max}}^* = d^*(0)$ is the largest diversity gain achievable.
Diversity-Multiplexing Tradeoff of the Channel

Definition

The diversity-multiplexing tradeoff $d^*(r)$ of a MIMO channel is the best possible diversity-multiplexing tradeoff achievable by any scheme.

$r_{\text{max}}^*$ is the largest multiplexing gain achievable in the channel.

$d_{\text{max}}^*$ is the largest diversity gain achievable.

For a $m \times n$ MIMO channel, it is not difficult to show:

$$r_{\text{max}}^* = \min\{m, n\}$$

$$d_{\text{max}}^* = mn$$

What is more interesting is how the entire curve looks like.
Example: $1 \times 1$ Channel

Spatial Multiplexing Gain: $r = R / \log \text{SNR}$

Diversity Gain: $d \cdot (r)$

- $(0,1)$
- $(1/2,0)$
- $(1,0)$

Fixed Rate

PAM

QAM

Fixed Reliability
Example: $2 \times 1$ Channel

Spatial Multiplexing Gain: $r = R / \log \text{SNR}$

Diversity Gain: $d(r)$

- Repetition
  - $(0, 2)$
  - $(1/2, 0)$

Spatial Multiplexing Gain: $r = R / \log \text{SNR}$
Example: $2 \times 1$ Channel

Spatial Multiplexing Gain: $r = R / \log \text{SNR}$

Diversity Gain: $d(r)$

- $(0,2)$
- $(1/2,0)$
- $(1,0)$

Alamouti Repetition

Spatial Multiplexing Gain: $r = R / \log \text{SNR}$
Example: $2 \times 1$ Channel

Spatial Multiplexing Gain: $r = R / \log \text{SNR}$

Diversity Gain: $d^*(r)$

Optimal Tradeoff

- Alamouti
- Repetition

Points:
- $(0, 2)$
- $(1/2, 0)$
- $(1, 0)$

$D = R / \log \text{SNR}$

$S = R / \log \text{SNR}$
Example: $2 \times 2$ Channel

Spatial Multiplexing Gain: $r = R / \log \text{SNR}$

Diversity Gain: $d(r)$

Repetition

(0,4)

(1/2,0)
Example: $2 \times 2$ Channel

Spatial Multiplexing Gain: $r = R / \log \text{SNR}$

Diversity Gain: $d(\cdot)$

- $r = 0$ (0,4)
- $r = 1/2$ (1/2,0)
- $r = 1$ (1,0)

Alamouti Repetition
Example: $2 \times 2$ Channel

Spatial Multiplexing Gain: $r = R / \log\text{SNR}$

Diversity Gain: $d(r)$

- $(1/2,0)$
- $(1,0)$
- $(0,4)$

Alamouti

Repetition

V-BLAST (Nulling)

$(0,1)$

$(2,0)$
Example: 2 × 2 Channel

Spatial Multiplexing Gain: \( r = R / \log \text{SNR} \)

Diversity Gain: \( d \)

Optimal Tradeoff

Alamouti

Repetition

V-BLAST (Nulling)
Example: $2 \times 2$ Channel
Winters, Salz and Gitlins 93:

Nulling out $k$ interferers using $n$ receive antennas provides a diversity gain of $n - k$. 

\[ \text{Spatial Multiplexing Gain: } r = \frac{R}{\log \text{SNR}} \]

\[ \text{Diversity Gain: } d(t) \]

\[ (0, 2) \]

\[ (0, 1) \]

\[ (2, 0) \]

\[ \text{V-BLAST(Nulling)} \]

\[ \text{V-BLAST(ML)} \]
Winters, Salz and Gitlins 93:

Nulling out $k$ interferers using $n$ receive antennas provides a diversity gain of $n - k$.

Tse, Viswanath and Zheng 03:

Jointly detecting all users provides a diversity gain of $n$ to each.
Winters, Salz and Gitlins 93:
Nulling out $k$ interferers using $n$ receive antennas provides a diversity gain of $n - k$.

Tse, Viswanath and Zheng 03:
Jointly detecting all users provides a diversity gain of $n$ to each.
There is free lunch. (?)
Optimal D-M Tradeoff for General $m \times n$ Channel

(Zheng and Tse 03)

As long as block length $l \geq m + n - 1$:

- **Spatial Multiplexing Gain:** $r = R / \log \text{SNR}$
- **Diversity Gain:** $d^*(r) = \min\{m, n\}, 0$ for $(0, mn)$

For integer $r$, it is as though $r$ transmit and $r$ receive antennas were dedicated for multiplexing and the rest provide diversity.
Optimal D-M Tradeoff for General $m \times n$ Channel

(Zheng and Tse 03)

As long as block length $l \geq m + n - 1$:

Spatial Multiplexing Gain: $r = R/\log \text{SNR}$

Diversity Gain: $d^*(r) = \min\{m, n\}, 0, mn, (m-1)(n-1)$

For integer $r$, it is as though $r$ transmit and $r$ receive antennas were dedicated for multiplexing and the rest provide diversity.
Optimal D-M Tradeoff for General $m \times n$ Channel

(Zheng and Tse 03)

As long as block length $l \geq m + n - 1$:

$$\begin{align*}
\text{Spatial Multiplexing Gain: } & r = R / \log \text{SNR} \\
\text{Diversity Gain: } & d^*(r) \\
& (\min\{m,n\}, 0) \\
& (0, mn) \\
& (1, (m-1)(n-1)) \\
& (2, (m-2)(n-2)) \\
& (\min\{m,n\}, 0)
\end{align*}$$

For integer $r$, it is as though $r$ transmit and $r$ receive antennas were dedicated for multiplexing and the rest provide diversity.
Optimal D-M Tradeoff for General $m \times n$ Channel

(Zheng and Tse 03)

As long as block length $l \geq m + n - 1$:

\[
\text{Spatial Multiplexing Gain: } r = \frac{R}{\log \text{SNR}} \\
\text{Diversity Gain: } d^*(r) = \begin{cases} 
(\min\{m,n\}, 0) \\
(0, mn) \\
(r, (m-r)(n-r)) \\
(1, (m-1)(n-1)) \\
(2, (m-2)(n-2)) \\
(m-n, 0) 
\end{cases}
\]

For integer $r$, it is as though $r$ transmit and $r$ receive antennas were dedicated for multiplexing and the rest provide diversity.
Optimal D-M Tradeoff for General $m \times n$ Channel

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For integer $r$, it is as though $r$ transmit and $r$ receive antennas were dedicated for multiplexing and the rest provide diversity.
Achieving Optimal Diversity-Multiplexing Tradeoff

- Hao and Wornell 03: MIMO rotation code (2 × 2 channel only).
- Tavildar and Viswanath 04: D-Blast plus permutation code.
- El Gamal, Caire and Damen 03: Lattice codes.
Alamouti scheme:

\[
\begin{bmatrix}
  x_1 & -x_2^* \\
  x_2 & x_1^*
\end{bmatrix}
\]

Hao and Wornell's scheme:

\[
\begin{bmatrix}
  x_1 & x_2 \\
  x_3 & x_4
\end{bmatrix}
\]

where

\[
\begin{bmatrix}
  x_1 \\
  x_4
\end{bmatrix} = \text{Rotate}(\theta_1^*) \begin{bmatrix}
  u_1 \\
  u_4
\end{bmatrix}
\]

\[
\begin{bmatrix}
  x_2 \\
  x_3
\end{bmatrix} = \text{Rotate}(\theta_2^*) \begin{bmatrix}
  u_2 \\
  u_3
\end{bmatrix}
\]

and \( u_1, u_2, u_3, u_4 \) are independent QAM symbols.
• First use D-Blast to convert the MIMO channel into a parallel channel.

• Then design permutation codes to achieve the optimal diversity-multiplexing tradeoff on the parallel channel.
Antenna 1: 
Antenna 2: 

Receive

D-BLAST
D-BLAST

Antenna 1:

Receive

Antenna 2:

Null
D-BLAST

Antenna 1:  
Antenna 2:  

-
D-BLAST

Antenna 1:
Antenna 2:

Cancel
Receive
Original D-Blast is sub-optimal.

D-Blast with MMSE suppression is information lossless
Permutation Coding for Parallel Channel

The channel is parallel but the fading at the different sub-channels are correlated.

Nevertheless it is shown that the permutation codes can achieve the optimal diversity-multiplexing tradeoff of the parallel channel.
Conclusion

Diversity-multiplexing tradeoff is a unified way to look at space-time code design for MIMO channels.

It puts diversity and multiplexing on an equal footing.

It provides a framework to compare existing schemes as well as stimulates the design of new schemes.