Diversity and Multiplexing:
A Fundamental Tradeoff in Wireless Systems

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UCSB
Wireless Fading Channels

- Fundamental characteristic of wireless channels: multi-path fading.
Wireless Fading Channels

- Fundamental characteristic of wireless channels: multi-path fading.
- Two important resources of a fading channel: diversity and degrees of freedom.
A channel with more diversity has smaller probability in deep fades.
Example: Spatial Diversity

Fading Channel: $h_1$
Example: Spatial Diversity

- Additional independent fading channels increase diversity.
Example: Spatial Diversity

- Additional independent fading channels increase **diversity**.
- Spatial diversity
Additional independent fading channels increase diversity.
Spatial diversity: receive, transmit
Example: Spatial Diversity

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Example: Spatial Diversity

- Additional independent fading channels increase diversity.
- Spatial diversity: receive, transmit or both.
- Repeat and Average: compensate against channel unreliability.
Signals arrive in multiple directions provide multiple degrees of freedom for communication. Same effect can be obtained via scattering even when antennas are close together.
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The right way of looking at the problem is a tradeoff between the two types of gain.

The optimal tradeoff achievable by a coding scheme gives a fundamental performance limit on communication over fading channels.
Talk Outline

- point-to-point MIMO channels (Zheng and Tse 02)
- multiple access MIMO channels (Tse, Viswanath, Zheng 03)
- cooperative relaying systems (Laneman, Tse, Wornell 02)
Point-to-point MIMO Channel

\[ y_t = H_t x_t + w_t, \quad w_t \sim \mathcal{CN}(0, 1) \]

- Rayleigh flat fading i.i.d. across antenna pairs \((h_{ij} \sim \mathcal{CN}(0, 1))\).
- SNR is the average signal-to-noise ratio at each receive antenna.
Coherent Block Fading Model

- Focus on codes over $l$ symbols, where $H$ remains constant.
- $H$ is known to the receiver but not the transmitter.
- Assumption valid as long as
  \[ l \ll \text{coherence time} \times \text{coherence bandwidth}. \]
Space-Time Block Code

\[ Y = HX + W \]

Focus on coding over a single block of length \( l \).
Diversity Gain

Motivation: Binary Detection

\[ y = hx + w \]

\[ P_e \approx P(\|h\| \text{ is small}) \propto \text{SNR}^{-1} \]

\[ \begin{aligned}
  y_1 &= h_1x + w_1 \\
  y_2 &= h_2x + w_2 \\
\end{aligned} \]

\[ P_e \approx P(\|h_1\|, \|h_2\| \text{ are both small}) \propto \text{SNR}^{-2} \]
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General Definition

A space-time coding scheme achieves diversity gain \( d \), if

\[ P_e(\text{SNR}) \sim \text{SNR}^{-d} \]
Spatial Multiplexing Gain

Motivation: Channel capacity (Telatar '95, Foschini'96)

\[ C(\text{SNR}) \approx \min\{m, n\} \log \text{SNR}(\text{bps/Hz}) \]

\( \min\{m, n\} \) degrees of freedom to communicate.
Spatial Multiplexing Gain

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\( \min\{m, n\} \) degrees of freedom to communicate.

**Definition** A space-time coding scheme achieves spatial multiplexing gain \( r \), if

\[ R(\text{SNR}) = r \log \text{SNR}(\text{bps/Hz}) \]
Fundamental Tradeoff

A space-time coding scheme achieves

- **Spatial Multiplexing Gain** \( r \) : \( R = r \log \text{SNR} \) \( (\text{bps/Hz}) \)
- **Diversity Gain** \( d \) : \( P_e \approx \text{SNR}^{-d} \)

Fundamental tradeoff: for any \( r \), the maximum diversity gain achievable:

\( d^*_{m,n}(r) \).
**Fundamental Tradeoff**

A space-time coding scheme achieves

Spatial Multiplexing Gain $r$ : $R = r \log \text{SNR}$ (bps/Hz)

and

Diversity Gain $d$ : $P_e \approx \text{SNR}^{-d}$

Fundamental tradeoff: for any $r$, the maximum diversity gain achievable: $d_{m,n}^*(r)$.

$$r \rightarrow d_{m,n}^*(r)$$
Fundamental Tradeoff

A space-time coding scheme achieves

\textbf{Spatial Multiplexing Gain} \( r \) : \( R = r \log \text{SNR} \) \((\text{bps/Hz})\)

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\( r \rightarrow d_{m,n}^*(r) \)

A tradeoff between data rate and error probability.
Main Result: Optimal Tradeoff

(Zheng and Tse 02)

$m$: # of Tx. Ant.
$n$: # of Rx. Ant.
$l$: block length
$l \geq m + n - 1$

d: diversity gain
$P_e \approx \text{SNR}^{-d}$

$r$: multiplexing gain
$R = r \log \text{SNR}$

Spatial Multiplexing Gain: $r = \frac{R}{\log \text{SNR}}$

Diversity Gain: $d(r) = \min\{m, n\}$, $0$

For integer $r$, it is as though $r$ transmit and $r$ receive antennas were dedicated for multiplexing and the rest provide diversity.
Main Result: Optimal Tradeoff

(Zheng and Tse 02)

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\( r \): multiplexing gain
\( R = r \log \text{SNR} \)

\( (m-1)(n-1) \)

\( (\min\{m,n\},0) \)
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\(r\): multiplexing gain
\(R = r \log \text{SNR}\)

For integer \(r\), it is as though \(r\) transmit and \(r\) receive antennas were dedicated for multiplexing and the rest provide diversity.
What do I get by adding one more antenna at the transmitter and the receiver?
Adding More Antennas

\[ m: \# \text{ of Tx. Ant.} \]
\[ n: \# \text{ of Rx. Ant.} \]
\[ l: \text{ block length} \]
\[ l \geq m + n - 1 \]

\[ d: \text{ diversity gain} \]
\[ r: \text{ multiplexing gain} \]
Adding More Antennas

\[ l \geq m + n - 1 \]

\( d \): diversity gain

\( r \): multiplexing gain

- **Capacity result**: increasing \( \min\{m, n\} \) by 1 adds 1 more degree of freedom.
Adding More Antennas

\[ m: \text{# of Tx. Ant.} \]
\[ n: \text{# of Rx. Ant.} \]
\[ l: \text{block length} \]
\[ l \geq m + n - 1 \]

\[ d: \text{diversity gain} \]
\[ r: \text{multiplexing gain} \]

- **Capacity result:** increasing \( \min\{m, n\} \) by 1 adds 1 more degree of freedom.
- **Tradeoff curve:** increasing both \( m \) and \( n \) by 1 yields multiplexing gain +1 for any diversity requirement \( d \).
Sketch of Proof

Lemma:

For block length $l \geq m + n - 1$, the error probability of the best code satisfies at high SNR:

$$P_e(SNR) \approx P(\text{Outage}) = P(I(H) < R)$$

where

$$I(H) = \log \det [I + \text{SNR}HH^*]$$

is the mutual information achieved by the i.i.d. Gaussian input.
Outage Analysis

\[ P(\text{Outage}) = P\{\log \det[I + \text{SNR}HH^\dagger] < R\} \]

- In scalar $1 \times 1$ channel, outage occurs when the channel gain $\|h\|^2$ is small.
Outage Analysis

\[ P(\text{Outage}) = P\{ \log \det[I + \text{SNR}HH^\dagger] < R \} \]

- In scalar \(1 \times 1\) channel, outage occurs when the channel gain \(\|h\|^2\) is small.
- In general \(m \times n\) channel, outage occurs when some or all of the singular values of \(H\) are small. There are many ways for this to happen.
Outage Analysis

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- In scalar $1 \times 1$ channel, outage occurs when the channel gain $\|h\|^2$ is small.
- In general $m \times n$ channel, outage occurs when some or all of the singular values of $H$ are small. There are many ways for this to happen.
- Let \( v = \) vector of singular values of $H$:
  Laplace Principle:
  \[ P(\text{Outage}) \approx \min_{v \in \text{Out}} \text{SNR}^{-f(v)} \]
Geometric Picture (integer $r$)

Scalar Channel

\[ \begin{center}
\includegraphics[width=0.5\textwidth]{geometric_picture.png}
\end{center} \]

Result: At rate $R = r \log SNR$, for $r$ integer, outage occurs typically when $H$ is in or close to the set $\{H: \text{rank}(H) \leq r\}$, with $\epsilon^2 = SNR - 1$.

The dimension of the normal space to the sub-manifold of rank $r$ matrices within the set of all $M \times N$ matrices is $(M - r)(N - r)$.

\[ P(\text{Outage}) \approx SNR - (M - r)(N - r) \]
**Geometric Picture (integer \( r \))**

Scalar Channel

\[
\varepsilon \quad \text{Bad H} \quad \text{Good H}
\]

Result: At rate \( R = r \log \text{SNR} \), for \( r \) integer, outage occurs typically when \( H \) is close to the set \( \{ H : \text{rank}(H) \leq r \} \), with \( \varepsilon^2 = \text{SNR} - 1 \).

The dimension of the normal space to the sub-manifold of rank \( r \) matrices within the set of all \( M \times N \) matrices is \((M - r)(N - r)\).
Result: At rate $R = r \log \text{SNR}$, for $r$ integer, outage occurs typically when $H$ is close to the set $\{H : \text{rank}(H) \leq r\}$, with $\epsilon^2 = \text{SNR} - 1$. The co-dimension of the manifold of rank $r$ matrices within the set of all $m \times n$ matrices is $(m - r)(n - r)$. The probability of outage $P(\text{Outage}) \approx \text{SNR} - (M - r)(N - r)$.
Result: At rate $R = r \log \text{SNR}$, for $r$ integer, outage occurs typically when $H$ is close to the set $\{H : \text{rank}(H) \leq r\}$,
Result: At rate $R = r \log \text{SNR}$, for $r$ integer, outage occurs typically when $H$ is close to the set $\{H : \text{rank}(H) \leq r\}$, with $\epsilon^2 = \text{SNR}^{-1}$. 
Result: At rate $R = r \log \text{SNR}$, for $r$ integer, outage occurs typically when $\mathbf{H}$ is close to the set $\{\mathbf{H} : \text{rank} (\mathbf{H}) \leq r\}$, with $\epsilon^2 = \text{SNR}^{-1}$.

The co-dimension of the manifold of rank $r$ matrices within the set of all $m \times n$ matrices is $(m - r)(n - r)$.

$$P(\text{Outage}) \approx \text{SNR}^{-(m-r)(n-r)}$$
For non-integer $r$, qualitatively same outage behavior as $\lfloor r \rfloor$ but with larger $\epsilon$.

Scalar channel: qualitatively same outage behavior for all $r$.

Vector channel: qualitatively different outage behavior in different segments of the tradeoff curve.
Tradeoff Analysis of Specific Designs

Focus on two transmit antennas.

\[ Y = HX + W \]

Repetition Scheme:

\[
X = \begin{bmatrix} x_1 & 0 \\ 0 & x_1 \end{bmatrix}
\]

\[ y_1 = \|H\|x_1 + w_1 \]

Alamouti Scheme:

\[
X = \begin{bmatrix} x_1 & -x_2^* \\ x_2 & x_1^* \end{bmatrix}
\]

\[ [y_1y_2] = \|H\|[x_1x_2] + [w_1w_2] \]
Comparison: $2 \times 1$ System

Repetition: $y_1 = \|H\| x_1 + w$

Alamouti: $[y_1 y_2] = \|H\| [x_1 x_2] + [w_1 w_2]$

Spatial Multiplexing Gain: $r = R / \log \text{SNR}$

Diversity Gain: $d(r)$

$(1/2,0)$

$(0,2)$
Comparison: $2 \times 1$ System

Repetition: $y_1 = \|H\|x_1 + w$

Alamouti: $[y_1 y_2] = \|H\|[x_1 x_2] + [w_1 \, w_2]$
Comparison: $2 \times 1$ System

Repetition: \[ y_1 = \|H\| x_1 + w \]

Alamouti: \[
\begin{bmatrix}
y_1 & y_2 \\
\end{bmatrix}
= \|H\| \begin{bmatrix}
x_1 \\
x_2 \\
\end{bmatrix} + \begin{bmatrix}
w_1 \\
w_2 \\
\end{bmatrix}
\]
Comparison: $2 \times 2$ System

Repetition: \( y_1 = \|H\|\mathbf{x}_1 + \mathbf{w} \)

Alamouti: \( [y_1 y_2] = \|H\|[\mathbf{x}_1 \mathbf{x}_2] + [\mathbf{w}_1 \mathbf{w}_2] \)

Spatial Multiplexing Gain: \( r = R / \log \text{SNR} \)

Diversity Gain: \( d^* (r) \)

(1/2,0) (0,4)
Comparison: $2 \times 2$ System

Repetition: $y_1 = \|H\|x_1 + w$

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Spatial Multiplexing Gain: $r = R/\log \text{SNR}$

Diversity Gain: $d^*(r) = (1/2,0), (1,0), (0,4)$
Comparison: $2 \times 2$ System

Repetition: $y_1 = \|H\| x_1 + w$

Alamouti: $[y_1 y_2] = \|H\| [x_1 x_2] + [w_1 w_2]$

Spatial Multiplexing Gain: $r = R / \log \text{SNR}$

Diversity Gain: $d^*(r)$

Optimal Tradeoff
Talk Outline

- point-to-point MIMO channels
- multiple access MIMO channels
- cooperative relaying systems
In a point-to-point link, multiple antennas provide diversity and multiplexing gain.

In a system with $K$ users, multiple antennas can be used to discriminate signals from different users too.

Continue assuming i.i.d. Rayleigh fading, $n$ receive antennas, $m$ transmit antennas per user.
Multiuser Diversity-Multiplexing Tradeoff

Suppose we want every user to achieve an error probability:

\[ P_e \sim \text{SNR}^{-d} \]

and a data rate

\[ R = r \log \text{SNR} \text{ bits/s/Hz.} \]

What is the optimal tradeoff between the diversity gain \( d \) and the multiplexing gain \( r \)?

Assume a coding block length \( l \geq Km + n - 1 \).
Optimal Multiuser D-M Tradeoff: \( m \leq n/(K + 1) \)

(Tse, Viswanath and Zheng 02)

In this regime, diversity-multiplexing tradeoff of each user is as though it is the only user in the system, i.e. \( d_{m,n}^*(r) \)
**Multiuser Tradeoff:** $m > n/(K + 1)$

Single-user diversity-multiplexing tradeoff up to $r^* = n/(K + 1)$. 
Multiuser Tradeoff: \( m > n/(K + 1) \)

Single-user diversity-multiplexing tradeoff up to \( r^* = m/(K + 1) \).

For \( r \) from \( n/(K + 1) \) to \( \min\{n/K, m\} \), tradeoff is as though the \( K \) users are pooled together into a single user with \( Km \) antennas and rate \( Kr \), i.e. \( d^*_{Km,n}(Kr) \).
Question: what does adding one more antenna at each mobile buy me?

Assume there are more users than receive antennas.
Question: what does adding one more antenna at each mobile buy me?
Assume there are more users than receive antennas.
Answer

Spatial Multiplexing Gain: \( r = \frac{R}{\log \text{SNR}} \)

Diversity Gain: \( d(r) \)

Optimal tradeoff

1 Tx antenna

Adding one more transmit antenna does not increase the number of degrees of freedom for each user. However, it increases the maximum diversity gain from \( N \) to \( 2N \).

More generally, it improves the diversity gain \( d(r) \) for every \( r \).
Adding one more transmit antenna does not increase the number of degrees of freedom for each user.

However, it increases the maximum diversity gain from $n$ to $2n$.

More generally, it improves the diversity gain $d(r)$ for every $r$. 
Consider only the case of $m = 1$ transmit antenna for each user and number of users $K < n$. 

**Suboptimal Receiver: the Decorrelator/Nuller**
Tradeoff for the Decorrelator

Maximum diversity gain is $n - K + 1$: “costs $K - 1$ diversity gain to null out $K - 1$ interferers.” (Winters, Salz and Gitlin 93)
Tradeoff for the Decorrelator

Maximum diversity gain is \( n - K + 1 \): “costs \( K - 1 \) diversity gain to null out \( K - 1 \) interferers.” (Winters, Salz and Gitlin 93)

Adding one receive antenna provides either more reliability per user or accommodate 1 more user at the same reliability.
Maximum diversity gain is $n - K + 1$: “costs $K - 1$ diversity gain to null out $K - 1$ interferers.” (Winters, Salz and Gitlin 93)

Adding one receive antenna provides either more reliability per user or accommodate 1 more user at the same reliability.

Optimal tradeoff curve is also a straight line but with a maximum diversity gain of $n$.

Adding one receive antenna provides more reliability per user and accommodate 1 more user.
Talk Outline

- point-to-point MIMO channels
- multiple access MIMO channels
- cooperative relaying systems
Cooperative relaying protocols can be designed via a diversity-multiplexing tradeoff analysis. (Laneman, Tse, Wornell 01)
Cooperative relaying protocols can be designed via a 
diversity-multiplexing tradeoff analysis.

(Laneman, Tse and Wornell 01)
Tradeoff Curves of Relaying Strategies

Diversity gain

Multiplexing gain

1

½

direct transmission

1 Multiplexing gain
Cooperative Relaying

Tx 1

Channel 1

Rx

Channel 2

Tx 2

Cooperation
Tradeoff Curves of Relaying Strategies

Diversity gain

Multiplexing gain

1 1/2 2

direct transmission

1 Multiplexing gain
Tradeoff Curves of Relaying Strategies

- Direct transmission
- Amplify + forward

Diversity gain

Multiplexing gain
Tradeoff Curves of Relaying Strategies

Diversity gain

Multiplexing gain

direct transmission

amplify + forward

1/2

1

2

Multiplexing gain
Cooperative Relaying

Channel 1

Tx 1

Cooperation

Channel 2

Tx 2

Rx
Tradeoff Curves of Relaying Strategies

![Graph showing tradeoff curves of relaying strategies.]
Tradeoff Curves of Relaying Strategies

- Direct transmission
- Amplify + forward
- Amplify + forward + ack

Diversity gain

Multiplexing gain

Gain

1/2

1

2
Conclusion

Diversity-multiplexing tradeoff is a unified way to look at performance over wireless channels.

Future work:

- Code design.
- Application to other wireless scenarios.
- Extension to channel-uncertainty-limited rather than noise-limited regime.