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Modeling Term Structures of Defaultable Bonds

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This article presents convenient reduced-form models of the valuation of contingent claims subject to default risk, focusing on applications to the term structure of interest rates for corporate or sovereign bonds. Examples include the valuation of a credit-spread option.

This article presents a new approach to modeling term structures of bonds and other contingent claims that are subject to default risk. As in previous "reduced-form" models, we treat default as an unpredictable event governed by a hazard-rate process.[1] Our approach is distinguished by the parameterization of losses at default in terms of the fractional reduction in market value that occurs at default.

Specifically, we fix some contingent claim that, in the event of no default, pays X at time T. We take as given an arbitrage-free setting in which all securities are priced in terms of some short-rate process r and equivalent martingale measure Q [see Harrison and Kreps (1979) and Harrison and Pliska (1981)]. Under this "risk-neutral" probability measure, we let h_t denote the hazard rate for default at time t and let L_t denote the expected fractional loss in market value if default were to occur at time t, conditional

This article is a revised and extended version of the theoretical results from our earlier article "Econometric Modeling of Term Structures of Defaultable Bonds" (June 1994). The empirical results from that article, also revised and extended, are now found in "An Econometric Model of the Term Structure of Interest Rate Swap Yields" (*Journal of Finance*, October 1997). We are grateful for comments from many, including the anonymous referee, Ravi Jagannathan (the editor), Peter Carr, Ian Cooper, Qiang Dai, Ming Huang, Farshid Jamshidian, Joe Langsam, Francis Longstaff, Amir Sadr, Craig Gustaffson, Michael Boulware, Arthur Mezhlumian, and especially Dilip Madan. We are also grateful for financial support from the Financial Research Initiative at the Graduate School of Business, Stanford University. We are grateful for computational assistance from Arthur Mezhlumian and especially from Michael Boulware and Jun Pan. Address correspondence to Kenneth Singleton, Graduate School of Business, Stanford University, Stanford, CA 94305-5015.

[1] Examples of reduced-form models include those of Pye (1974), Litterman and Iben (1988), Madan and Unal (1993), Fons (1994), Lando (1994, 1997, 1998), Artzner and Delbaen (1995), Das and Tufano (1995), Jarrow and Turnbull (1995), Nielsen and Ronn (1995), Jarrow, Lando, and Turnbull (1997), Martin (1997), Schönbucher (1997). Ramaswamy and Sundaresan (1986) and Cooper and Mello (1996) directly assumed that defaultable bonds can be valued by discounting at an adjusted short rate. Among other results, this article provides a particular kind of reduced-form model that justifies this assumption. Litterman and Iben (1991) arrived at a similar model in a simple discrete time setting by assuming zero recovery at default.

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on the information available up to time t. We show that this claim may be priced as if it were default-free by replacing the usual short-term interest rate process r with the default-adjusted short-rate process $R = r + hL$. That is, under technical conditions, the initial market value of the defaultable claim to X is

$$V_0 = E_0^Q \left[\exp\left(-\int_0^T R_t \, dt \right) X \right], \tag{1}$$

where E_0^Q denotes risk-neutral, conditional expectation at date 0. This is natural, in that $h_t L_t$ is the "risk-neutral mean-loss rate" of the instrument due to default. Discounting at the adjusted short rate R therefore accounts for both the probability and timing of default, as well as for the effect of losses on default. Pye (1974) developed a precursor to this modeling approach in a discrete-time setting in which interest rates, default probabilities, and credit spreads all change only deterministically.

A key feature of the valuation equation [Equation (1)] is that, provided we take the mean-loss rate process hL to be given exogenously,[2] standard term-structure models for default-free debt are directly applicable to defaultable debt by parameterizing R instead of r. After developing the general pricing relation [Equation (1)] with exogenous R in Section 1.3, special cases with Markov diffusion or jump-diffusion state dynamics are presented in Section 1.4.

The assumption that default hazard rates and fractional recovery do not depend on the value V_t of the contingent claim is typical of reduced-form models of defaultable bond yields. There are, however, important cases for which this exogeneity assumption is counterfactual. For instance, as discussed by Duffie and Huang (1996) and Duffie and Singleton (1997), h_t will depend on V_t in the case of swap contracts with asymmetric counterparty credit quality. In Section 1.5, we extend our framework to the case of price-dependent (h_t, L_t). We show that the absence of arbitrage implies that V_t is the solution to a nonlinear partial differential equation. For example, with this nonlinear dependence of the price on the contractual payoffs, the value of a coupon bond in this setting is not simply the sum of the modeled prices of individual claims to the principal and coupons.

Section 2 presents several applications of our framework to the valuation of corporate bonds. First, in Section 2.1, we discuss the practical implications of our "loss-of-market" value assumption, compared to a "loss-of-face" value assumption, for the pricing of noncallable corporate bonds. Calculations with illustrative pricing models suggest that these alternative recovery assumptions generate rather similar par yield spreads, even for the same fractional loss coefficients. This robustness suggests that, for some

[2] By "exogenous," we mean that $h_t L_t$ does not depend on the value of the defaultable claim itself.

pricing problems, one can exploit the analytical tractability of our loss-of-market pricing framework for estimating default hazard rates, even when loss-of-face value is the more appropriate recovery assumption. For deep-discount or high-premium bonds, differences in these formulations can be mitigated by compensating changes in recovery parameters.

Second, we discuss several econometric formulations of models for pricing of noncallable corporate bonds. In pricing corporate debt using Equation (1), one can either parameterize R directly or parameterize the component processes r, h, and L (which implies a model for R). The former approach was pursued in Duffie and Singleton (1997) and Dai and Singleton (1998) in modeling the term structure of interest-rate swap yields. By focusing directly on R, these pricing models combine the effects of changes in the default-free short-rate rate (r) and risk-neutral mean loss rate (hL) on bond prices. In contrast, in applying our framework to the pricing of corporate bonds, Duffee (1997) and Collin-Dufresne and Solnik (1998) parameterize r and hL separately. In this way they are able to "extract" information about mean loss rates from historical information on defaultable bond yields. All of these applications are special cases of the affine family of term-structure models.[3]

In Section 2.2 we explore, along several dimensions, the flexibility of affine models to describe basic features of yields and yield spreads on corporate bonds. First, using the canonical representations of affine term-structure models in Dai and Singleton (1998), we argue that the Cox, Ingersoll, and Ross (CIR)-style models used by Duffee (1999) and Collin-Dufresne and Solnik (1998) are theoretically incapable of capturing the negative correlation between credit spreads and U.S. Treasury yields documented in Duffee (1998), while maintaining nonnegative default hazard rates. Several alternative affine formulations of credit spreads are introduced with the properties that hL is strictly positive and that the conditional correlation between changes in r and hL is unrestricted a priori as to sign.

Second, we develop a defaultable version of the Heath, Jarrow, and Morton (1992) (HJM) model based on the forward-rate process associated with R. In developing this model we derive the counterpart to the usual HJM risk-neutralized drift restriction for defaultable bonds.

Third, we apply our framework to the pricing of callable corporate bonds. We show that, as with noncallable bonds, the hazard rate h_t and fractional default loss L_t cannot be separately identified from data on the term structure of defaultable bond prices alone, because h_t and L_t enter the pricing relation [Equation (1)] only through the mean-loss rate $h_t L_t$.

[3] See, for example, Duffie and Kan (1996) for a characterization of the affine class of term-structure models, and Dai and Singleton (1998) for a complete classification of the admissible affine term-structure models and a specification analysis of three-factor models for the swap yield curve.

The pricing of derivatives on defaultable claims in our framework is explored in Section 3. The underlying could be, for example, a corporate or sovereign bond on which a derivative such as an option is written (by a defaultable or nondefaultable) counterparty. In order to illustrate these ideas we price a credit-spread put option on a defaultable bond, allowing for correlation between the hazard rate h_t and short rate r_t. The nonlinear dependence of the option payoffs on h_t and L_t implies that, in contrast to bonds, the default hazard rate and fractional loss rate are separately identified from option price data. Numerical calculations for the spread put option are used to illustrate this point, as well as several other features of credit derivative pricing.

1. Valuation of Defaultable Claims

In order to motivate our valuation results, we first provide a heuristic derivation of our basic valuation equation [Equation (1)] in a discrete-time setting. Then we formalize this intuition in continuous time. For the case of exogenous default processes, the implied pricing relations are derived for special cases in which (h, L, r) is a Markov diffusion or, more generally, a jump diffusion.

1.1 A discrete-time motivation

Consider a defaultable claim that promises to pay $X_{t+\mathcal{T}}$ at maturity date $t + \mathcal{T}$, and nothing before date $t + \mathcal{T}$. For any time $s \geq t$, let

- h_s be the conditional probability at time s under a risk-neutral probability measure Q of default between s and $s + 1$ given the information available at time s in the event of no default by s.
- φ_s denote the recovery in units of account, say dollars, in the event of default at s.
- r_s be the default-free short rate.

If the asset has not defaulted by time t, its market value V_t would be the present value of receiving φ_{t+1} in the event of default between t and $t + 1$ plus the present value of receiving V_{t+1} in the event of no default, meaning that

$$V_t = h_t e^{-r_t} E_t^Q(\varphi_{t+1}) + (1 - h_t) e^{-r_t} E_t^Q(V_{t+1}), \qquad (2)$$

where $E_t^Q(\cdot)$ denotes expectation under Q, conditional on information available to investors at date t. By recursively solving Equation (2) forward over the life of the bond, V_t can be expressed equivalently as

$$V_t = E_t^Q\left[\sum_{j=0}^{\mathcal{T}-1} h_{t+j} e^{-\sum_{k=0}^{j} r_{t+k}} \varphi_{t+j+1} \prod_{\ell=0}^{j}(1-h_{t+\ell-1})\right] + E_t^Q\left[e^{-\sum_{k=0}^{\mathcal{T}-1} r_{t+k}} X_{t+\mathcal{T}} \prod_{j=1}^{\mathcal{T}}(1-h_{t+j-1})\right]. \qquad (3)$$

Evaluation of the pricing formula [Equation (3)] is complicated in general by the need to deal with the joint probability distribution of φ, r, and h over various horizons. The key observation underlying our pricing model is that Equation (3) can be simplified by taking the risk-neutral expected recovery at time s, in the event of default at time $s+1$, to be a fraction of the risk-neutral expected survival-contingent market value at time $s+1$ ["recovery of market value" (RMV)]. Under this assumption, there is some adapted process L, bounded by 1, such that

RMV: $E_s^Q(\varphi_{s+1}) = (1-L_s)E_s^Q(V_{s+1})$.

Substituting RMV into Equation (3) leaves

$$\begin{aligned} V_t &= (1-h_t)e^{-r_t}E_t^Q(V_{t+1}) + h_t e^{-r_t}(1-L_t)E_t^Q(V_{t+1}) \\ &= E_t^Q\left(e^{-\sum_{j=0}^{\mathcal{T}-1} R_{t+j}} X_{t+\mathcal{T}}\right), \end{aligned} \qquad (4)$$

where

$$e^{-R_t} = (1-h_t)e^{-r_t} + h_t e^{-r_t}(1-L_t). \qquad (5)$$

For annualized rates but time periods of small length, it can be seen that $R_t \simeq r_t + h_t L_t$, using the approximation of e^c, for small c, given by $1+c$.

Equation (4) says that the price of a defaultable claim can be expressed as the present value of the promised payoff $X_{t+\mathcal{T}}$, treated as if it were default-free, discounted by the default-adjusted short rate R_t. We will show technical conditions under which the approximation $R_t \simeq r_t + h_t L_t$ of the default-adjusted short rate is in fact precise and justified in a continuous-time setting. This implies, under the assumption that h_t and L_t are exogenous processes, that one can proceed as in standard valuation models for default-free securities, using a discount rate that is the default-adjusted rate $R_t = r_t + h_t L_t$ instead of the usual short rate r_t. For instance, R can be parameterized as in a typical single- or multifactor model of the short rate, including the Cox, Ingersoll, and Ross (1985) model and its extensions, or as in the HJM model. The body of results regarding default-free term structure models is immediately applicable to pricing defaultable claims.

The RMV formulation accommodates general state dependence of the hazard rate process h and recovery rates without adding computational com-

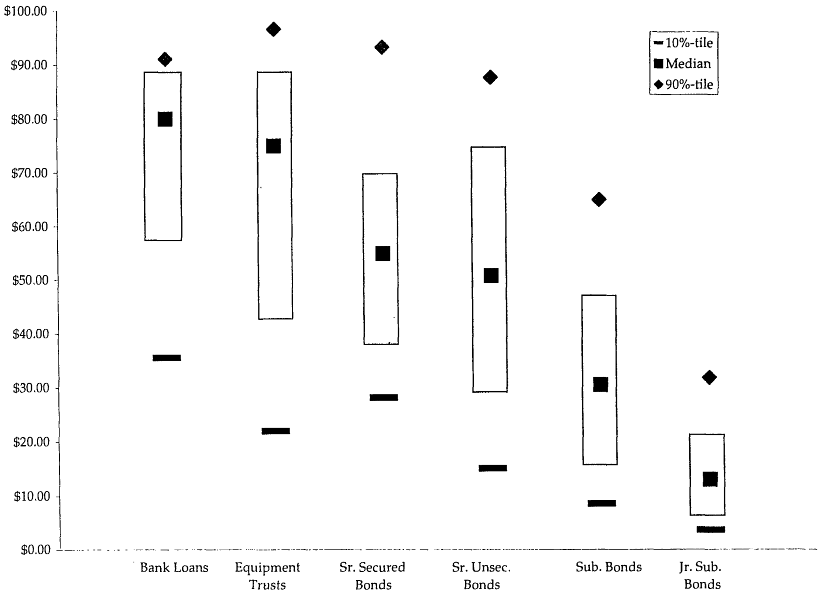

Figure 1
Distributions of recovery by seniority

plexity beyond the usual burden of computing the prices of riskless bonds. Moreover, (h_t, L_t) may depend on or be correlated with the riskless term structure. Some evidence consistent with the state dependence of recovery rates is presented in Figure 1, based on recovery rates compiled by Moody's for the period 1974–1997.[4] The square boxes represent the range between the 25th and 75th percentiles of the recovery distributions. Comparing senior secured and unsecured bonds, for example, one sees that the recovery distribution for the latter is more spread out and has a longer lower tail. However, even for senior secured bonds, there was substantial variation in the actual recovery rates. Although these data are also consistent with cross-sectional variation in recovery that is not associated with stochastic variation in time of expected recovery, Moody's recovery data (not shown in Figure 1) also exhibit a pronounced cyclical component.

There is equally strong evidence that hazard rates for default of corporate bonds vary with the business cycle (as is seen, for example, in Moody's data). Speculative-grade default rates tend to be higher during recessions, when interest rates and recovery rates are typically below their long-run means. Thus allowing for correlation between default hazard-rate processes and

[4] These figures are constructed from revised and updated recovery rates as reported in "Corporate Bond Defaults and Default Rates 1938–1995" (Moody's Investor's Services, January 1996). Moody's measures the recovery rate as the value of a defaulted bond, as a fraction of the $100 face value, recorded in its secondary market subsequent to default.

riskless interest rates also seems desirable. Partly in recognition of these observations, Das and Tufano (1996) allowed recovery to vary over time so as to induce a nonzero correlation between credit spreads and the riskless term structure. However, for computational tractability they maintained the assumption of independence of h_t and r_t.

In allowing for state dependence of h and L, we do not model the default time directly in terms of the issuer's incentives or ability to meet its obligations [in contrast to the corporate debt pricing literature beginning with Black and Scholes (1973) and Merton (1974)]. Our modeling approach and results are nevertheless consistent with a direct analysis of the issuer's balance sheet and incentives to default, as shown by Duffie and Lando (1997), using a version of the models of Fisher, Heinkel, and Zechner (1989) and Leland (1994) that allows for imperfect observation of the assets of the issuer. A general formula can be given for the hazard rate h_t in terms of the default boundary for assets, the volatility of the underlying asset process V at the default boundary, and the risk-neutral conditional distribution of the level of assets given the history of information available to investors. This makes precise one sense in which we are proposing a reduced-form model. While, following our approach, the behavior of the hazard rate process h and fractional loss process L may be fitted to market data and allowed to depend on firm-specific or macroeconomic variables [as in Bijnen and Wijn (1994), McDonald and Van de Gucht (1996), Shumway (1996), and Lundstedt and Hillgeist (1998)], we do not constrain this dependence to match that implied by a formal structural model of default by the issuer.

Our discussion so far presumes the exogeneity of the hazard rate and fractional recovery. There are important circumstances in which these assumptions are counterfactual, and failure to accommodate endogeneity may lead to mispricing. For instance, if the market value of recovery at default is fixed, and does not depend on the predefault price of the defaultable claim itself, then the fractional recovery of market value cannot be exogenous. Alternatively, in the case of some OTC derivatives, the hazard and recovery rates of the counterparties are different and the operative h and L for discounting depends on which counterparty is in the money. [For more details and applications to swap rates, see Duffie and Huang (1996).] While Equation (1) [and Equation (4)] apply with price-dependent hazard and recovery rates, this dependence makes the pricing equation a nonlinear difference equation that must typically be solved by recursive methods. In Section 1.5 we characterize the pricing problem with endogenous hazard and recovery rates and describe methods for pricing in this case.

One can also allow for "liquidity" effects by introducing a stochastic process ℓ as the fractional carrying cost of the defaultable instrument.[5]

[5] Formally, in order to invest in a given bond with price process U, this assumption literally means that one must continually make payments at the rate ℓU.

Then, under mild technical conditions, the valuation model [Equation (1)] applies with the "default and liquidity-adjusted" short-rate process

$$R = r + hL + \ell.$$

In practice, it is common to treat spreads relative to Treasury rates rather than to "pure" default-free rates. In that case, one may treat the "Treasury short rate" r^* as itself defined in terms of a spread (perhaps negative) to a pure default-free short rate r, reflecting (among other effects) repo specials. Then we can also write $R = r^* + hL + \ell^*$, where ℓ^* absorbs the relative effects of repo specials and other determinants of relative carrying costs.

1.2 Continuous-time valuation

This section formalizes the heuristic arguments presented in the preceding section. We fix a probability space $(\Omega, \mathcal{F}, P)$ and a family $\{\mathcal{F}_t \colon t \geq 0\}$ of σ-algebras satisfying the *usual* conditions. [See, for example, Protter (1990) for technical details.] A predictable short-rate process r is also fixed, so that it is possible at any time t to invest one unit of account in default-free deposits and "roll over" the proceeds until a later time s for a market value at that time of $\exp(\int_t^s r_u \, du)$.[6] At this point, we do not specify whether r_t is determined in terms of a Markov state vector, an HJM forward-rate model, or by some other approach.

A contingent claim is a pair (Z, τ) consisting of a random variable Z and a stopping time τ at which Z is paid. We assume that Z is $\mathcal{F}_\tau$ measurable (so that the payment can be made based on currently available information). We take as given an equivalent martingale measure Q relative to the short-rate process r. This means, by definition, that the ex dividend price process U of any given contingent claim (Z, τ) is defined by $U_t = 0$ for $t \geq \tau$ and

$$U_t = E_t^Q \left[\exp\left(- \int_t^\tau r_u \, du \right) Z \right], \quad t < \tau, \tag{6}$$

where E_t^Q denotes expectation under the risk-neutral measure Q, given $\mathcal{F}_t$. Included in the assumption that Q exists is the existence of the expectation in Equation (6) for any traded contingent claim. (Later we extend the definition of a contingent claim to include payments at different times.)

We define a defaultable claim to be a pair $((X, T), (X', T'))$ of contingent claims. The underlying claim (X, T) is the obligation of the issuer to pay X at date T. The secondary claim (X', T') defines the stopping time T' at which the issuer defaults and claimholders receive the payment X'. This means that the actual claim (Z, τ) generated by a defaultable claim $((X, T), (X', T'))$

[6] We assume that this integral exists.

is defined by

$$\tau = \min(T, T'); \qquad Z = X1_{\{T<T'\}} + X'1_{\{T\geq T'\}}. \tag{7}$$

We can imagine the underlying obligation to be a zero-coupon bond ($X = 1$) maturing at T, or some derivative security based on other market prices, such as an option on an equity index or a government bond, in which case X is random and based on market information at time T. One can apply the notion of a defaultable claim $((X, T), (X', T'))$ to cases in which the underlying obligation (X, T) is itself the actual claim generated by a more primitive defaultable claim, as with an OTC option or credit derivative on an underlying corporate bond. The issuer of the derivative may or may not be the same as that of the underlying bond.

Our objective is to define and characterize the price process U of the defaultable claim $((X, T), (X', T'))$. We suppose that the default time T' has a risk-neutral default hazard rate process h, which means that the process Λ which is 0 before default and 1 afterward (that is, $\Lambda_t = 1_{\{t\geq T'\}}$) can be written in the form

$$d\Lambda_t = (1 - \Lambda_t)h_t\,dt + dM_t, \tag{8}$$

where M is a martingale under Q. One may safely think of h_t as the jump arrival intensity at time t (under Q) of a Poisson process whose first jump occurs at default.[7] Likewise, the risk-neutral conditional probability, given the information $\mathcal{F}_t$ available at time t, of default before $t + \Delta$, in the event of no default by t, is approximately $h_t\Delta$ for small Δ.

We will first characterize and then (under technical conditions) prove the existence of the unique arbitrage-free price process U for the defaultable claim. For this, one additional piece of information is needed: the payoff X' at default. If default occurs at time t, we will suppose that the claim pays

$$X' = (1 - L_t)U_{t-}, \tag{9}$$

where $U_{t-} = \lim_{s\uparrow t} U_s$ is the price of the claim "just before" default,[8] and L_t is the random variable describing the fractional loss of market value of the claim at default. We assume that the fractional loss process L is bounded by 1 and predictable, which means roughly that the information determining L_t is available before time t. Section 1.6 provides an extension to handle a fractional loss in market value that is uncertain even given all information available up to the time of default.

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[7] The process $\{(1 - \Lambda_{t-})h_t\colon t \geq 0\}$ is the intensity process associated with Λ, and is by definition nonnegative and predictable with $\int_0^t h_s\,ds < \infty$ almost surely for all t. See Brémaud (1980). Artzner and Delbaen (1995) showed that, if there exists an intensity process under P, then there exists an intensity process under any equivalent probability measure, such as Q.

[8] We will also show that the left limit U_{t-} exists.

As a preliminary step, it is useful to define a process V with the property that, if there has been no default by time t, then V_t is the market value of the defaultable claim.[9] In particular, $V_T = X$ and $U_t = V_t$ for $t < T'$.

1.3 Exogenous expected loss rate

From the heuristic reasoning used in Section 1.1, we conjecture the continuous-time valuation formula

$$V_t = E_t^Q \left[\exp\left(-\int_t^T R_s \, ds \right) X \right], \tag{10}$$

where

$$R_t = r_t + h_t L_t. \tag{11}$$

In order to confirm this conjecture, we use the fact that the gain process (price plus cumulative dividend), after discounting at the short-rate process r, must be a martingale under Q. This discounted gain process G is defined by

$$\begin{aligned} G_t = {} & \exp\left(-\int_0^t r_s \, ds \right) V_t (1 - \Lambda_t) \\ & + \int_0^t \exp\left(-\int_0^s r_u \, du \right) (1 - L_s) V_{s-} \, d\Lambda_s. \end{aligned} \tag{12}$$

The first term is the discounted price of the claim; the second term is the discounted payout of the claim upon default. The property that G is a Q martingale and the fact that $V_T = X$ together provide a complete characterization of arbitrage-free pricing of the defaultable claim.

Let us suppose that V does not itself jump at the default time T'. From Equation (10), this is a primitive condition on (r, h, X) and the information filtration $\{\mathcal{F}_t \colon t \geq 0\}$. This means essentially that, although there may be "surprise" jumps in the conditional distribution of the market value of the default-free claim (X, T), h, or L, these surprises occur precisely at the default time with probability zero. This is automatically satisfied in the diffusion settings described in Section 1.4.1, since in that case $V_t = J(Y_t, t)$, where J is continuous and Y is a diffusion process. This condition is also satisfied in the jump-diffusion model of Section 1.4.2, provided jumps in the conditional distribution of (h, L, X) do not occur at default.[10]

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[9] Because $V(\omega, t)$ is arbitrary for those ω for which default has occurred before t, the process V need not be uniquely defined. We will show, however, that V is uniquely defined up to the default time, under weak regularity conditions.

[10] Kusuoka (1999) gives an example in which a jump in V at default is induced by a jump in the risk premium. This may be appropriate, for example, if the arrival of default changes risk attitudes. In any case, given (h, L, X), one can always construct a model in which there is a stopping time τ with Q hazard rate process h and with no jump in V at τ. For this, one can take any exponentially distributed random variable z with

Applying Ito's formula [see Protter (1990)] to Equation (12), using Equation (9) and our assumption that V does not jump at T', we can see that for G to be a Q martingale, it is necessary and sufficient that

$$V_t = \int_0^t R_s V_s \, ds + m_t \tag{13}$$

for some Q martingale m. (Since V jumps at most a countable number of times, we can replace V_{s-} in Equation (12) with V_s for purposes of this calculation.) Given the terminal boundary condition $V_T = X$, this implies Equations (10)–(12). The uniqueness of solutions of Equation (13) with $V_T = X$ can be found, for example, in Antonelli (1993). Thus we have shown the following basic result.

Theorem 1. *Given (X, T, T', L, r), suppose the default time T' has a risk-neutral hazard rate process h. Let $R = r + hL$ and suppose that V is well defined by Equation (10) and satisfies $\Delta V(T') = 0$ almost surely. Then there is a unique defaultable claim $((X, T), (X', T'))$ and process U satisfying Equations (6), (7), and (9). Moreover, for $t < T'$, $U_t = V_t$.*

For a defaultable asset, such as a coupon bond, with a series of payments X_k at T_k, assuming no default by T_k, for $1 \leq k \leq K$, the claim to all K payments has a value equal to the sum of the values of each, in this setting in which h and L are exogenously given processes. (It may be appropriate to specify recovery assumptions that distinguish the various claims making up the asset.) The proof is an easy extension of Theorem 1, again using the fact that the total gain process, including the jumps associated with interim payments, is a Q martingale. This linearity property does not hold, however, for the more general case, treated in Section 1.5, in which h or L may depend on the value of the claim itself.

1.4 Special cases with exogenous expected loss

Next, we specialize to the case of valuation with dependence of exogenous r, h, and L on continuous-time Markov state variables.

1.4.1 A continuous-time Markov formulation. In order to present our model in a continuous-time state-space setting that is popular in finance applications, we suppose for this section that there is a state-variable process Y that is Markovian under an equivalent martingale measure Q. We assume that the promised contingent claim is of the form $X = g(Y_T)$, for some

parameter 1, independent under Q of (h, L, X), and let τ be defined by $\int_0^\tau h_s \, ds = z$. (This allows for $\tau = +\infty$ with positive probability.) If necessary, one can redefine the underlying probability space so that there exists such a random variable z and minimally enlarge each information set $\mathcal{F}_t$ so that τ is a stopping time.

function g, and that $R_t = \rho(Y_t)$, for some function $\rho(\cdot)$.[11] Under the conditions of Theorem 1, a defaultable claim to payment of $g(Y_T)$ at time T has a price at time t, assuming that the claim has not defaulted by time t, of

$$J(Y_t, t) = E^Q\left[\exp\left(-\int_t^T \rho(Y_s)\,ds\right) g(Y_T) \,\middle|\, Y_t\right]. \tag{14}$$

Modeling the default-adjusted short rate R_t directly as a function of the state variable Y_t allows one to model defaultable yield curves analogously with the large literature on dynamic models of default-free term structures. For example, suppose $Y_t = (Y_{1t}, \ldots, Y_{nt})'$, for some n, solves a stochastic differential equation of the form

$$dY_t = \mu(Y_t)\,dt + \sigma(Y_t)\,dB_t, \tag{15}$$

where B is an $\{\mathcal{F}_t\}$-standard Brownian motion in $\mathbb{R}^n$ under Q, and where μ and σ are well-behaved functions on $\mathbb{R}^n$ into $\mathbb{R}^n$ and $\mathbb{R}^{n\times n}$, respectively. Then we know from the "Feynman-Kac formula" that, under technical conditions, examples of which are given in Friedman (1975) and Krylov (1980), Equation (14) implies that J solves the backward Kolmogorov partial differential equation

$$\mathcal{D}^{\mu,\sigma} J(y,t) - \rho(y)J(y,t) = 0, \qquad (y,t) \in \mathbb{R}^n \times [0,T], \tag{16}$$

with the boundary condition

$$J(y,T) = g(y), \qquad y \in \mathbb{R}^n, \tag{17}$$

where

$$\begin{aligned} \mathcal{D}^{\mu,\sigma} J(y,t) &= J_t(y,t) + J_y(y,t)\mu(y) \\ &\quad + \frac{1}{2}\,\text{trace}\left[J_{yy}(y,t)\sigma(y)\sigma(y)'\right]. \end{aligned} \tag{18}$$

This is the framework used in models for pricing swaps and corporate bonds discussed in Section 2.

1.4.2 Jump-diffusion state process. Because of the possibility of sudden changes in perceptions of credit quality, particularly among low-quality issues such as Brady bonds, one may wish to allow for "surprise" jumps in Y. For example, one can specify a standard jump-diffusion model for the

[11] For notational reasons, we have not shown any dependence of ρ on time t, which could be captured by including time as one of the state variables. Of course, we assume that ρ and g are measurable real-valued functions on the state space of Y, and that Equation (14) is well defined.

risk-neutral behavior of Y, replacing $\mathcal{D}^{\mu,\sigma}$ in Equation (16), under technical regularity, with the jump-diffusion operator $\mathcal{D}$ given by

$$\mathcal{D}J(y,t)=\mathcal{D}^{\mu,\sigma}J(y,t)+\lambda(y)\int_{\mathbb{R}^n}[J(y+z,t)-J(y,t)]d\nu_y(z), \tag{19}$$

where $\lambda\colon \mathbb{R}^n \to [0,\infty)$ is a given function determining the arrival intensity $\lambda(Y_t)$ of jumps in Y at time t, under Q, and where, for each y, ν_y is a probability distribution for the jump size (z) of the state variable. Examples of affine, defaultable term-structure models with jumps are presented in Section 2.

1.5 Price-dependent expected loss rate

If the risk-neutral expected loss rate $h_t L_t$ is price dependent, then the valuation model is nonlinear in the promised cash flows. We can accommodate this in a model in which default at time t implies a fractional loss $L_t = \hat{L}(Y_t, U_t)$ of market value and hazard rate $h_t = H(Y_t, U_t)$ that may depend on the current price U_t of the defaultable claim.[12] This would allow, for example, for recovery of an exogenously specified fraction of face value at default.

In our Markov setting, we can now write $R_t = \rho(Y_t, U_t)$, where $\rho(y,u) = H(y,u)L(y,u)+\hat{r}(y)$, where $\{\hat{r}(Y_t)\colon t \geq 0\}$ is the state-dependent default-free short-rate process. By the same reasoning used in Section 1.3, and under technical regularity conditions, the price U_t of the defaultable claim at any time t before default is, in our general Markov setting, given by

$$J(Y_t,t)=E^Q\left[\exp\left(-\int_t^T \rho(Y_s, J(Y_s,s))ds\right)g(Y_T)\;\middle|\; Y_t\right]. \tag{20}$$

With the diffusion or jump-diffusion assumption for Y, and under additional technical regularity conditions [as, for example, in Ivanov (1984)], J solves the quasi-linear equation

$$\mathcal{D}J(y,t) + \rho(y, J(y,t))J(y,t) = 0, \quad y \in \mathbb{R}^n, \tag{21}$$

where $\mathcal{D}J(y,t)$ is defined by Equation (19), with the boundary condition of Equation (17). This PDE can be treated numerically, essentially as with the linear case [Equation (16)]. Duffie and Huang (1996) and Huge and Lando (1999) have several numerical examples of an application of this framework to defaultable swap rates.

[12] We suppose for this section that the price process is left-continuous, so that if default occurs at time t, U_t is the price of the claim just before it defaults, and $(1 - L_t)U_t$ is the market value just after default. This is simply for notational convenience. We could also allow L to depend on U and other state information. For example, for a model of a bond collateralized by an asset with price process U, we could let $(1 - L_t)U_t = q_t \min(U(t), K)$, where K is the maximal effective legal claim at default, say par, and q_t is the conditional expected fraction recovered at default of the effective legal claim.

For cases of endogenous dependence of the risk-neutral mean loss rate hL on the price of the claim, not necessarily based on a Markovian state space, Duffie, Schroder, and Skiadas (1996) provide technical conditions for the existence and uniqueness of pricing, and explore the pricing implications of advancing in time the resolution of information.

1.6 Uncertainty about recovery

We have been assuming that the fractional loss in market value due to default at time t is determined by the information available up to time t. An extension of our model to allow for conditionally uncertain jumps in market value at default is due to Schönbucher (1997). A simple version of this extension is provided below for completeness.

Suppose that at default, instead of Equation (9), the claim pays

$$X' = (1 - \ell)U(T'-), \tag{22}$$

where ℓ is a bounded ($\mathcal{F}_{T'}$ measurable) random variable describing the fractional loss of market value of the claim at default. It would not be natural to require that $\ell \geq 0$, as the onset of default could actually reveal, with non zero probability, "good" news about the financial condition of the issuer. Given limited liability, we require that $\ell \leq 1$.

It can be shown that there exists a process L such that L_t is the expectation of the fractional default loss ℓ given all current information up to, *but not including*, time t. To be precise, L is a predictable process, and $L_{T'} = E(\ell \mid \mathcal{F}_{T'-})$.

With this change in the definitions of X' and L_t, the pricing formula of Equation (10) applies as written, with $R = r + hL$, under the conditions of Theorem 1. The proof is almost identical to that of Theorem 1.

2. Valuation of Defaultable Bonds

An important application of the basic valuation equation [Equation (10)] with exogenous default risk is the valuation of defaultable corporate bonds. We discuss various aspects of this pricing problem in this section, beginning with the sensitivity of bond prices to the nature of the default recovery assumption. We argue that the tractability of assumption RMV may come at a low cost in terms of pricing errors for bonds trading near par even if, in truth, bonds are priced in the markets assuming a given fractional recovery of face value. Then, maintaining our assumption RMV, we present several "affine" models for pricing defaultable, noncallable bonds, giving particular attention to parameterizations that allow for flexible correlations among the riskless rate r and the default hazard rate h. In addition, we derive the default-environment counterparts to the HJM no-arbitrage conditions for term structure models based on forward rates. Finally, we discuss the valuation of callable corporate bonds.

2.1 Recovery and valuation of bonds

The determination of recoveries to creditors during bankruptcy proceedings is a complex process that typically involves substantial negotiation and litigation. No tractable, parsimonious model captures all aspects of this process so, in practice, all models involve trade-offs regarding how various aspects of default (hazard and recovery rates) are captured. To help motivate our RMV convention, consider the following alternative recovery of face value (RFV) and recovery of treasury (RT) formulations of φ_t:

RT: $\varphi_t = (1 - L_t)P_t$, where L is an exogenously specified fractional recovery process and P_t is the price at time t of an otherwise equivalent, default-free bond [Jarrow and Turnbull (1995)].

RFV: $\varphi_t = (1 - L_t)$; the creditor receives a (possibly random) fraction $(1 - L_t)$ of face ($1) value immediately upon default [Brennan and Schwartz (1980) and Duffee (1998)].

Under RT, the computational burden of directly computing V_t from Equation (3), for a given fractional recovery process $(1 - L_t)$, can be substantial. Largely for this reason, various simplifying assumptions have been made in previous studies. Jarrow and Turnbull (1995), for example, assumed that the risk-neutral default hazard rate process h is independent (under Q) of the short rate r and, for computational examples, that the fractional loss process L is constant. Lando (1998) relaxes the Jarrow and Turnbull model within the RT setting by allowing a random hazard rate process that need not be independent of the short rate r, but at the cost of added computational complexity. With $L_t = \bar{L}$ a constant, the payoff at maturity in the event of default is $(1 - \bar{L})$, *regardless of when the default occurred.* This simplification is lost when L_t is time varying, since the payoff at maturity will be indexed by the period in which default occurred. Not only is the time of default relevant, but the joint $\mathcal{F}_t$-conditional distributions of L_v, h_s, and r_u, for all v, s, and u between t and T, play a computationally challenging role in determining V_t.

Turning to RMV and RFV, one basis for choosing between these two assumptions is the legal structure of the instrument to be priced. For instance, Duffie and Singleton (1997) and Dai and Singleton (1998) apply the model in this article (assumption RMV) to the determination of at-market, U.S. dollar, fixed-for-variable swap rates. These authors assume exogenous (h_t, L_t, r_t) and parameterize the default-adjusted discount rate R directly as an affine function of a Markov-state vector Y. In this manner they were able to apply frameworks for valuing default-free bonds without modification to the problem of determining at-market swap rates. The RMV assumption is well matched to the legal structure of swap contacts in that standard agreements typically call for settlement upon default based on an obligation represented by an otherwise equivalent, nondefaulted swap.

For the case of corporate bonds, on the other hand, we see the choice of recovery assumptions as involving both conceptual and computational trade-offs. The RMV model is easier to use, because standard default-free term-structure modeling techniques can be applied. If, however, one assumes liquidation at default and that absolute priority applies, then assumption RFV is more realistic as it implies equal recovery for bonds of equal seniority of the same issuer. Absolute priority, however, is not always maintained by bankruptcy courts and liquidation at default is often avoided.

In the end, is there a significant difference between the pricing implications of models under RMV and RFV? In order to address this question, we proceed under the assumption of exogenous (h_t, L_t) (as in Section 1.3) and, for simplicity, take $L_t = \bar{L}$, a constant. We adopt for this illustration a "four-factor" model of r and h given by

$$r_t = \rho_0 + Y_t^1 + Y_t^2 - Y_t^0, \tag{23}$$

$$h_t = bY_t^0 + Y_t^3, \tag{24}$$

where Y_1, Y_2, Y_3, and Y_0 are independent "square-root diffusions" under Q, and ρ_0 and b are constant coefficients. The degree of *negative* correlation between r_t and h_t [consistent with Duffee (1998)] is controlled by choice of b.[13]

Under assumption RMV, the price V_t at any time t before default of an n-year bond with semiannual coupon payments of c is, under the regularity conditions of Theorem 1, given by

$$V_{nt}^{RMV} = cE_t^Q\left(\sum_{j=1}^{2n} e^{-\int_t^{t+.5j} R_s\,ds}\right) + E_t^Q\left(e^{-\int_t^{t+n} R_s\,ds}\right), \tag{25}$$

where $R_t = r_t + h_t\bar{L}$. For the case of assumption RFV (zero recovery of coupon payments and recovery of the fraction $(1 - \bar{L})$ of face value) the results of Lando (1998) imply that the market value of the same bond at any time t before default is given by

$$V_{nt}^{RFV} = cE_t^Q\left(\sum_{j=1}^{2n} e^{-\int_t^{t+0.5j} (r_s+h_s)\,ds}\right) + E_t^Q\left(e^{-\int_t^{t+n} (r_s+h_s)\,ds}\right) + \int_t^{t+n} (1 - \bar{L})\gamma(Y_t, t, s)\,ds, \tag{26}$$

[13] The parameters for default-free rates were chosen to match the level and volatilities of the riskless zero-coupon yield curve, out to 10 years, implied by the Duffie and Singleton (1997) estimated two-factor LIBOR swap model. For details on the parameterization of this example, one may consult the more extensive working paper version of this article, available from the web pages of the authors.

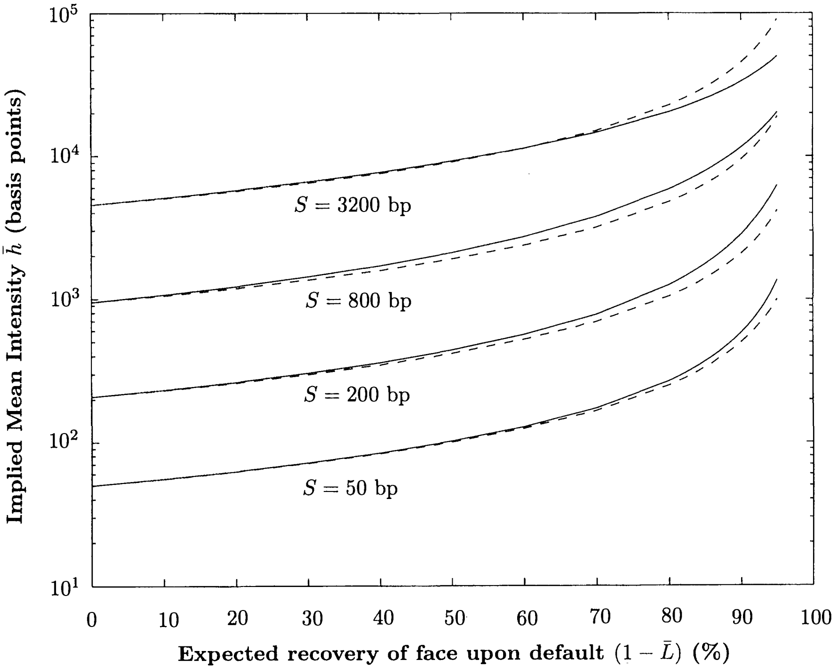

Figure 2
For fixed ten-year par-coupon spreads, S, this figure shows the dependence of the mean hazard rate $\bar{h}$ on the assumed fractional recovery $1 - \bar{L}$. The solid lines correspond to the model *RFV*, and the dashed lines correspond to the model *RMV*.

where

$$\gamma(Y_t, t, s) = E_t^Q\left(h_s e^{-\int_t^s (r_u + h_u)\,du}\right).$$

For this multifactor CIR setting, $\gamma(Y_t, t, s)$ can be computed explicitly, so the computation of V_{nt} from Equation (26) calls for one numerical integration. With $\bar{L} = 1$ (zero recovery), all bond prices are clearly identical under the two models.

In calculating par-bond spreads, or the risk-neutral default hazard rates implied by par spreads, for the case of $\bar{L} < 1$, we find rather little difference between the RFV and RMV formulations. This is true even without making compensating adjustments to $\bar{L}$ across the two models in order to calibrate one to the other. For example, Figure 2 shows the initial (equal to long-run mean) default hazard rate for both models, implied by a given 10-year spread and a given fractional recovery coefficient $(1 - \bar{L})$. The implied risk-neutral hazard rates are obviously rather close.

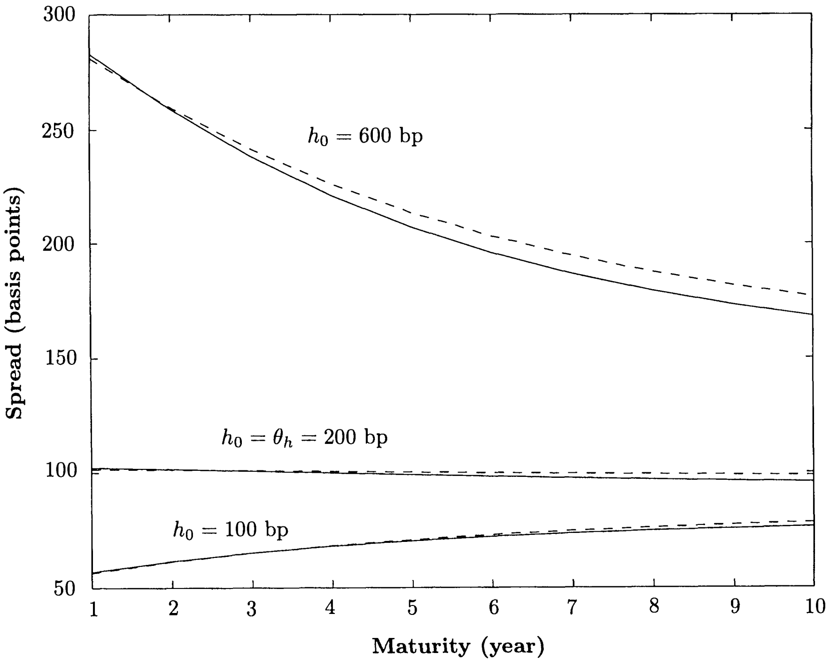

Figure 3
Term structures of par-coupon yield spreads for *RMV* (dashed lines) and *RFV* (solid lines), with 50% recovery upon default, a long-run mean hazard rate of $\theta_h = 200$ bp, a mean reversion rate of $\kappa = 0.25$, and an initial hazard-rate volatility of 100%.

The assumption that the initial and long-run mean intensities are equal makes for a rather gentle stress test of the distinction between the RMV and RFV formulations, as it implies that the term structure of risk-neutral forward[14] default probabilities is rather flat, and therefore that the risk-neutral expected market value given survival to a given time is close to face value. Fixing $\overline{L}$, the present value of recoveries for the RMV and RFV models would therefore be rather similar.

In order to show the impact of upward- or downward-sloping term structures of forward default probabilities, we provide in Figure 3 the term structures of par-coupon yield spreads (semiannual, bond equivalent) for cases in which the initial risk-neutral default hazard rate h_0 is much higher or lower than its long-run mean, $\theta = 200$ basis points. Under both recovery assumptions, $\bar{L}$ was set at 50%. With an increasing term structure of default risk ($h_0 \leq \theta$) bond prices under the two recovery assumptions are again

[14] The forward default probability at a future time t, a definition due to Litterman and Iben (1991), means the probability of default between t and one "short" unit of time after t, conditional on survival to t.

rather similar. On the other hand, for a steeply declining term structure of default risk, the implied credit spreads are larger under RMV, with the maximum difference (at 10 years) of 8.4 basis points, for actual spreads of 168.2 basis points for RFV and 176.6 basis points for RMV. From Equations (25) and (26) we see that a higher value of h_0 tends to depress V^{RFV} more than V^{RMV} through the effect of the present values of the coupons. A higher h_0 also implies a higher contribution from the accelerated fractional recovery of par [the last term in Equation (26)] under RFV. The latter effect dominates, giving a larger spread in the case of assumption RMV.

For bonds with a significant premium or discount, or with steeply upward or downward sloping term structures of interest rates, the RMV and RFV assumptions may have more markedly differing spread implications for a given exogenous loss fraction. For example, a given fractional loss, say 50%, of a premium bond's market value represents a greater loss in market value than does a 50% loss of the same bond's face value. In such cases, much of the distinction between the two model assumptions can be compensated for with different fractional loss processes.

2.2 Valuation of noncallable corporate bonds

The valuation framework set forth in Section 1.3 with exogenous hazard and recovery rates has been applied by Duffee (1999), Duffie and Liu (1997), and Collin-Dufresne and Solnik (1998) to value noncallable corporate bonds. These studies focus on special cases in which Y is a vector of independent square-root diffusions. In this section we nest these RMV specifications within a more general affine diffusion model and argue that square-root diffusions are limited theoretically in their flexibility to explain term structures of corporate yield spreads. We then introduce an alternative affine formulation, motivated in part by the empirical analysis in Dai and Singleton (1998), that offers greater flexibility in capturing nonzero correlations among the variables (h_t, L_t, r_t) while preserving positivity of the hazard rate.

At the outset, it is important to note that the hazard rate process h and fractional loss at default process L enter the adjustment for default in the discount rate $R = r + hL$ in the product form hL. Furthermore, under the assumption of exogenous (h, L, r), the value of a noncallable corporate bond is simply the sum of the present values of the promised coupon payments. It follows that knowledge of defaultable bond prices (before default) alone is not sufficient to separately identify h and L. At most, we can extract information about the risk-neutral mean loss rate $h_t L_t$. In order to learn more about the hazard and recovery rates implicit in market prices (within our RMV pricing framework), it is necessary to examine either a collection of bonds that share some but not all of the same default characteristics, or derivative securities with payoffs that depend in different ways on h and L (see Section 3).

As an illustration of the former strategy, suppose that one has prices of undefaulted junior (price V_t^J) and senior (price V_t^S) bonds of the same issuer, along with the prices of one or more default-free (Treasury) bonds. In this case, it seems reasonable to assume that the corporates share a hazard rate process h, but have different conditional expected fractional losses at default, L_t^J and L_t^S, respectively, consistent with the evidence in Figure 1. Using models of the type discussed subsequently, it will often be possible (for given parameters determining the dynamics of r and hL) to extract observations on $h_t L_t^J$ and $h_t L_t^S$ from these prices. In this case, we can infer relative recovery rates, in terms of L_t^J / L_t^S, but cannot extract the hazard rate h_t or the individual levels of recovery rates. Of course, if the hazard rate or either recovery rate were observed, or known functions of observable variables, then the identification problem would be solved. Having prices on both junior and senior debt would then serve to provide more market information for the estimation of h.

With this identification problem in mind, suppose that one has data on the prices of a collection of defaultable bonds with different maturities and the same associated hazard and fractional loss rates. We also suppose that the objective of the empirical analysis is to model jointly the dynamic properties of r_t and the "short spread" $s_t \equiv h_t L_t$.

Case 1: Square-root diffusion model of Y. Consider the case of a three-factor model in which the instantaneous, default-free short-rate process r is given by

$$r_t = \delta_0 + \delta_1 Y_{1t} + \delta_2 Y_{2t} + \delta_3 Y_{3t}, \tag{27}$$

for state variables Y_1, Y_2, and Y_3 that are "square-root diffusions," in the sense that the conditional volatility of the ith state variable is proportional to $\sqrt{Y_{it}}$. Also, suppose that

$$s_t = \gamma_0 + \gamma_1 Y_{1t} + \gamma_2 Y_{2t} + \gamma_3 Y_{3t}. \tag{28}$$

Dai and Singleton (1998) provide a sense in which the "most flexible" affine term-structure model with this volatility structure and well-defined bond prices has

$$dY_t = \mathcal{K}(\Theta - Y_t)\, dt + \sqrt{S_t}\, dB_t, \tag{29}$$

where $\mathcal{K}$ is a 3×3 matrix with positive diagonal and nonpositive off-diagonal elements; Θ is a vector in $\mathbb{R}^3_+$; S_t is the 3×3 diagonal matrix with diagonal elements Y_{1t}, Y_{2t}, and Y_{3t}; and B is a standard Brownian motion in $\mathbb{R}^3$ under Q. [In the notation of Dai and Singleton (1998), this is the $\mathbb{A}_3(3)$ family of models.]

Duffee (1999) considered the special case of Equations (27) and (28) in which $\delta_0 = -1$ and $\delta_3 = 0$, so that r_t could take on negative values and depends only on the first two state variables. He assumed, moreover, that $\mathcal{K}$ is diagonal (so that Y_1, Y_2, and Y_3 are Q-independent square-root diffusions, as commonly assumed in CIR-style models).

A potential drawback of imposing these overidentifying restrictions is that they unnecessarily constrain the joint conditional distribution of r_t and s_t. A key benefit is that they allowed Duffee to estimate the parameters of (Y_{1t}, Y_{2t}) governing the default-free Treasury using data on Treasury prices alone, while still allowing for a nontrival idiosyncratic factor driving s_t and correlation between r_t and s_t through nonzero γ_i in Equation (28). This two-step estimation strategy would nevertheless have been feasible with nonzero $(\kappa_{12}, \kappa_{21})$ and thus a more flexible correlation structure could have been introduced. Given the independence of the state variables and Duffee's normalizations of the δ_i coefficients to unity, the only means of introducing negative correlation among r_t and s_t in this model is to allow for negative γ_i's. His estimates of some of the γ_i's were in fact negative, implying that the default hazard rates may take on negative values, a technical impossibility. With his formulation, the average error in fitting noncallable corporate bond yields was less than 10 basis points.

The possibility of negative hazard rates in Duffee's model is not a consequence of the particular restricted version of Equations (27) and (28) that he chose to study. More generally, within this correlated square-root model of (r_t, s_t), one cannot simultaneously have a nonnegative hazard rate process and negatively correlated increments of r and h. This is an immediate implication of the observation in Dai and Singleton (1998) that well-defined correlated square-root models do not allow for negative correlation among any of the state variables, because the off-diagonal elements of $\mathcal{K}$ must be nonpositive for the model to be well defined. Therefore r and s cannot have negatively correlated increments (in the usual "instantaneous" sense) in this model unless one or more of the δ_i's or γ_i's is negative.

Case 2: Models with more flexible correlation structures for (r_t, s_t). In an important respect, the limitations of the correlated square-root model are due to the assumed structure of the stochastic volatility in Y. Dai and Singleton (1998) show that, within the affine family of term-structure models, more flexibility in specifying the correlations among the state variables is gained by restricting the dependence of the conditional variances of the state variables on Y. [Duffie and Liu (1997) use the framework in this article to study the pricing of floating rate corporate debt with r_t assumed to be an affine function of squared Gaussian variables, which offers an alternative way of introducing negative correlation among the state variables.]

Suppose, for instance that, instead of Equation (29), we assume that

$$dY_t = \mathcal{K}(\Theta - Y_t)\,dt + \Sigma\sqrt{S_t}\,dB_t, \tag{30}$$

where $\mathcal{K}$ and Θ are as in Equation (29), Σ is a 3×3 matrix, and

$$S_{11}(t) = Y_1(t), \tag{31}$$

$$S_{22}(t) = [\beta_2]_2 Y_2(t), \tag{32}$$

$$S_{33}(t) = \alpha_3 + [\beta_3]_1 Y_1(t) + [\beta_3]_2 Y_2(t), \tag{33}$$

with strictly positive coefficients, $[\beta_i]_j$. Also suppose that

$$r_t = \delta_0 + \delta_1 Y_{1t} + Y_{2t} + Y_{3t}, \tag{34}$$

$$s_t = \gamma_0 + \gamma_1 Y_{1t} + \gamma_2 Y_{2t}, \tag{35}$$

with all of δ_0, δ_1, γ_0, γ_1, and γ_2 strictly positive. Then Dai and Singleton (1998) show a sense in which the most flexible, admissible affine term structure model based on Equation (30) with Equations (31)–(33) has

$$\mathcal{K} = \begin{bmatrix} \kappa_{11} & \kappa_{12} & 0 \\ \kappa_{21} & \kappa_{22} & 0 \\ 0 & 0 & \kappa_{33}, \end{bmatrix} \quad \Sigma = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \sigma_{31} & \sigma_{32} & 1, \end{bmatrix}. \tag{36}$$

with the off-diagonal elements of $\mathcal{K}$ being nonpositive.

The short-spread rate s_t is strictly positive in this model, because it is a positive affine function of a correlated square-root diffusion. At the same time, the signs of σ_{31} and σ_{32} are unconstrained, so the third state variable may have increments that are negatively correlated with those of the first two. This may induce negative correlation between the increments of r and s. Given these interdependencies among the state variables, the parameters must be estimated using corporate and Treasury price data simultaneously.

With the imposition of overidentifying restrictions, we can specialize this model to one in which the riskless term-structure can be estimated independently of the corporate-spread component s_t. Specifically, suppose that $\delta_1 = 0$, so that Y_3 and Y_1 are idiosyncratic risk factors for r and s, respectively. Also, set $\kappa_{21} = 0$, $[\beta_3]_1 = 0$, and $\sigma_{31} = 0$. Then $r_t = \delta_0 + Y_{2t} + Y_{3t}$, $S_{33}(t) = \alpha_3 + [\beta_3]_2 Y_{2t}$, and

$$\mathcal{K} = \begin{bmatrix} \kappa_{11} & \kappa_{12} & 0 \\ 0 & \kappa_{22} & 0 \\ 0 & 0 & \kappa_{33}, \end{bmatrix} \quad \Sigma = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \sigma_{32} & 1. \end{bmatrix}. \tag{37}$$

Under this parameterization, the model of the riskless term structure is a two-factor affine model with r determined by (Y_2, Y_3). All of the parameters of this two-factor Treasury model can be estimated without using corporate bond data.

Corporate bond price data is necessary to estimate the parameters of the diffusion representation of Y_1, as well as the parameters of Equation (35). A nonzero κ_{12} induces positive correlation between the increments of r and s (recall that κ_{12} cannot be positive). On the other hand, negative correlation in the increments of r and s is induced if $\sigma_{32} < 0$. By construction, this model also has the property that hazard rates are strictly positive. We do stress, however, that the restrictions leading to this special case are testable and may restrict the joint distribution of (r_t, s_t) in ways that are not supported by the data. In particular, compared to the preceding model, the degree of negative correlation between the increments of r and s is limited by the restriction that $\sigma_{31} = 0$.

Neither of these cases considers the possibility of jumps. Duffie and Kan (1996) showed that introducing jumps into an affine term structure model preserves the affine dependence of yields on state variables provided the jump-arrival intensity is an affine function of the state vector and the distribution of the jump sizes depends only on time. Thus we could easily extend these parameterizations to incorporate jumps, provided the jump-size distributions respect the positivity of the hazard rate h_t. Duffie, Pan, and Singleton (1998) extend this model and further discuss parameterizations of affine jump diffusions with jumps that preserve the positivity of a subset of the state variables. Their examples could be adapted to our defaultable bond pricing problem.

2.3 A defaultable HJM model

Given an exogenous risk-neutral mean loss rate process hL, one can treat the dynamics of the term structure of interest rates on defaultable debt using the same model developed for default-free forward rates by Heath, Jarrow, and Morton [HJM (1992)]. This section formalizes this idea by deriving counterparts to the HJM no-arbitrage risk-neutral drift restriction on forward rates.

Suppose the defaultable discount function is modeled by taking the price at a given time t before default of a zero-coupon defaultable bond (of a given homogeneous class of defaultable debt) maturing at T to be

$$p_{t,T} = \exp\left(-\int_t^T F(t,u)\,du\right), \tag{38}$$

where

$$F(t,T) = F(0,T) + \int_0^t \mu(s,T)\,ds + \int_0^t \sigma(s,T)\,dB_s, \tag{39}$$

where B is a standard Brownian motion in $\mathbb{R}^n$ under Q and, for each fixed maturity T, the real-valued process $\mu(\cdot, T)$ and the $\mathbb{R}^n$-valued process $\sigma(\cdot, T)$ satisfy the technical regularity conditions imposed by Heath, Jarrow, and Morton (1992) and by Carverhill (1995). We note that $F(t,T)$ does not literally correspond to the interest rate on forward bond contracts unless one allows for special language in the forward rate agreement regarding the obligations of the counterparties in the event of default of the underlying bond before T. We can nevertheless take F as a process that describes, through Equations (38) and (39), the behavior prior to default of the discount function of a given class of defaultable debt.

We first show that, with given processes h and L for the risk-neutral hazard rate and fractional loss of market value, respectively, and under technical regularity conditions, we have the usual HJM risk-neutralized drift restriction

$$\mu(t, T) = \sigma(t, T) \cdot \int_t^T \sigma(t, u)\, du. \tag{40}$$

This restriction is derived, under the conditions of Theorem 1, as follows.

As in Section 1.3, the discounted gain process G for the zero-coupon corporate bond maturing at T is

$$G_t = (1 - \Lambda_t) D_t p_{t,T} + \int_0^t (1 - L_s) D_s p_{s-,T}\, d\Lambda_s, \tag{41}$$

where $D_t = \exp\left(-\int_0^t r_s\, ds\right)$. Since G is a Q martingale, its drift is zero, or, using Ito's formula, and taking $t < T'$, we have (almost surely)

$$0 = \int_0^t D_s p_{s,T} \alpha_{s,T}\, ds, \tag{42}$$

where, after an application of Fubini's theorem for stochastic integrals, as in Protter (1990), we have

$$\begin{aligned} \alpha_{t,T} &= F(t, t) - r_t - \int_t^T \mu(t, u)\, du \\ &\quad + \frac{1}{2} \left(\int_t^T \sigma(t, u)\, du \right) \cdot \left(\int_t^T \sigma(t, u)\, du \right) - h_t L_t. \end{aligned} \tag{43}$$

(Once again, we can ignore the distinction between $p_{s-,T}$ and $p_{s,T}$ for purposes of this calculation.) From Equation (42), we have $\alpha_{t,T} = 0$ (almost everywhere). Taking partial derivatives of $\alpha_{t,T}$ with respect to T leaves Equation (40).

From Equation (43), we can also see that the risk-neutral hazard rate process h implied by the models for F and r is given, under regularity, by

$$h(t) = \frac{F(t, t) - r_t}{L_t}. \tag{44}$$

Suppose, alternatively, that one specifies a model of the Jarrow and Turnbull (1995) variety, in which default of a corporate zero coupon bond at time

t implies recovery of an exogenously specified fraction δ_t of a default-free zero coupon bond of the same maturity, where δ is a nonnegative stochastic process satisfying regularity conditions. This is a version of the recovery formulation RT discussed in Section 2.1. In this case, lack of homogeneity implies a correction term to the usual HJM drift restriction, which is given instead through an analogous calculation by

$$\begin{aligned}\mu^*(t,T) &= \sigma(t,T)\cdot\int_t^T \sigma(t,u)du \\ &\quad + h_t\delta_t\frac{q_{t,T}}{p_{t,T}}[F(t,T)-f(t,T)],\end{aligned} \tag{45}$$

where, for any t and s, $q_{t,s} = \exp\left(-\int_t^s f(t,u)\,du\right)$ is the price at time t of a default-free zero coupon bond maturing at T, and $f(t,T)$ is the associated default-free forward rate.

2.4 Valuation of defaultable callable bonds

The majority of dollar-denominated corporate bonds are callable. In this section we extend our pricing results to the case of defaultable bonds with embedded call options. This extension requires an assumption about the call policy of the issuer. In order to minimize the total market value of a portfolio of corporate liabilities, it may not be optimal for the issuer of the liabilities to call in a particular bond so as to minimize the market value of that particular bond. For simplicity, however, we will assume a callable bond is called so as to minimize its market value. The resulting pricing model is easily extended to the case in which any issuer options embedded in a portfolio of liabilities issued by a given corporation are exercised so as to minimize the total market value of the portfolio.

At each time t, the issuer minimizes the market value of the liability represented by a corporate bond by exercising the option to call in the bond if and only if its market price, if not called, is higher than the strike price on the call, as implied by Bellman's principle of optimality. We will take the simple discrete-time setting of Section 1.1. During the time window of "callability," we thus have the recursive pricing formula

$$V_t = \min\left[\overline{V}_t, e^{-R_t}E_t^Q(V_{t+1}+d_{t+1})\right], \tag{46}$$

where d_t is the coupon on the bond at time t; V_t is the bond price at time t, after the coupon is paid, assuming that the bond has not defaulted by t; $\overline{V}_t$ is the exercise price at time t (often par); and R_t is the discrete default-adjusted short rate at time t, defined by Equation (5). Outside the callability window,

$$V_t = e^{-R_t}E_t^Q(V_{t+1}+d_{t+1}). \tag{47}$$

The "boundary condition" at maturity T is that V_T is the face value of the bond.

In a more general continuous-time context, suppose that a callable bond maturing at time T with first call at $\underline{T}$ has a coupon of size c_i (as a fraction of face value) at time $T(i)$, for $T(1) \leq T(2) \leq \cdots \leq T$. At time t, we let $\mathcal{T}(t, T)$ denote the set of feasible call policies. (These are the stopping times that are bounded above by T and below by $\max(t, \underline{T})$.) By standard arguments for nondefaultable securities, provided default has not occurred by time t, the market price at t, as a fraction of face value, is

$$V_t = \min_{\tau \in \mathcal{T}(t,T)} E_t^Q \left[\sum_{t < T(i) \leq \tau} \gamma_{t,T(i)} c_i + \gamma_{t,\tau} \right], \tag{48}$$

where

$$\gamma_{t,s} = \exp\left(- \int_t^s R_u \, du \right), \tag{49}$$

and where R is the default-adjusted (and perhaps liquidity-adjusted) short-rate process associated with the bond. The pricing relation of Equation (48) applies under the technical regularity conditions given in Section 1, assuming that the minimum in Equation (48) is attained by some stopping time. [Technical regularity conditions can be found, for example, in Karatzas (1988).] This approach, minimization of value over stopping times, was proposed by Merton (1974) in a structural model for the pricing of callable defaultable corporate bonds. In practice, Equation (48) could be solved by a discrete algorithm such as Equation (46) for sufficiently small time periods.

Convertible bonds, whose option to convert is exercised by the investor when the relative value of equity is high, are more common among issuers of low credit quality. For simplicity, one could model the risk-neutral default hazard rate h_t and fractional loss rate L_t so that they depend in part on the equity price process underlying conversion. This captures the idea that the bond is of lower credit quality when the equity price is lower. See Brennan and Schwartz (1980) for a full convertible pricing model in a defaultable setting.

Notable in our setting is the absence of a role for the default variables h and L in callable bond pricing other than through the definition of the default-adjusted short rate R. This is evident from Equations (10) and (48). It follows that h and L cannot be identified directly from the market prices for undefaulted corporate bonds, callable or not. Changes in credit spreads of undefaulted bonds reflect changes in the joint distribution of the risk-neutral mean loss rate process hL and short-rate process r.

3. Pricing Bond and Credit Derivatives

The inability to separately identify h_t and L_t using defaultable bond yields is not an issue for some derivative securities. For example, the payoff on a corporate bond option depends directly on the value of the underlying corporate debt. Due to the nonlinearity of the option payoff, the default characteristics (h, L) of the issuer of debt (as contrasted to those for the writer of the option) may in some cases be identified.

For illustration, suppose that there is a constant treasury yield of 6% and a constant risk-neutral mean loss rate of $s = hL = 2\%$, with $h = 0.08$ and $L = 0.25$. The underlying defaultable bond is therefore priced at a constant yield of 8% until maturity or default. Consider a default-free put option on the bond struck at a yield of 11%. Because the option is initially out of the money, and all yields are constant except at the bond's default, the option will pay off only if the bond defaults. If default occurs at τ, before the expiration of the option, the option pays $\pi(L, \tau) \equiv p(0.11, \tau) - (1 - L)p(0.08, \tau)$, where $p(y, t)$ is the price of the corporate at time t at a bond equivalent yield of y. Because $\pi(L, \tau)$ is not linear in L, whereas the risk-neutral probability of default is approximately linear in the hazard rate h (for small h), the option price depends in distinct ways on h and L, which can therefore be identified from the option price, the corporate bond yield, and the Treasury yield in this simple example.

For example, suppose we halve the default hazard rate h in this example from 0.08 to 0.04, and we double the fractional loss L to 0.5, leaving $s = hL$ unchanged at 2%. The bond price behavior is identical prior to default. With this change, however, the option pays off with half the expected frequency (risk neutralized), but because the option is out of the money before default, it pays more than double the amount when it does pay off. That is, at the default time τ, if before expiration, the option pays $\pi(0.5, \tau) > 2\pi(0.25, \tau)$. The initial option price is therefore higher with $(h = 0.04, L = 0.50)$ than with $(h = 0.08, L = 0.25)$. Thus, in principle, one can identify h and L separately from yield spreads and from bond option prices. This is a simple consequence of the nonlinearity of the option payoff as a function of the underlying bond price.

3.1 Pricing a credit-spread put option

As an illustration of how our framework can be used to value credit derivatives, we price a yield-spread put option with stochastic (h, r). We suppose that the underlying bond pays semiannual coupons, is noncallable, and matures at T_m. Its price is quoted in terms of its yield spread over a Treasury note, of semiannual coupons, and of the same maturity.[15] This credit-spread

[15] In our calculations, we ignore the differences in day-count conventions between Treasury and corporate bonds. They would in any case have a negligible effect for our purposes.

derivative is an option to sell the defaultable bond at a spread $\underline{S}$ over Treasury at an exercise date T. If the actual market spread S_T at T is less than the strike spread $\underline{S}$, then the spread option expires worthless. If the actual market spread S_T at T exceeds $\underline{S}$, then the holder of the credit derivative receives $p(\underline{S}+Z_T, T) - p(S_T + Z_T, T)$, where Z_t is the yield of the Treasury note at t. Such options have been sold, for example, on Argentinian Brady bonds.

As credit spreads are relatively volatile, or may jump, the issuer of the credit derivative may wish to incorporate a feature of the following type: If, at any time t before expiration, the market spread S_t is greater than or equal to a given spread cap $\overline{S} > \underline{S}$, then the credit derivative immediately pays $p(\underline{S}+Z_t, t) - p(\overline{S}+Z_t, t)$. In effect, then the credit derivative insures an owner of the corporate bond against increases in spread above the strike rate $\underline{S}$ up to $\overline{S}$.

In summary, the credit derivative pays

$$X = \max\left[p(\underline{S} + Z_\tau, \tau) - p(\min(S_\tau, \overline{S}) + Z_\tau, \tau), 0\right]$$

at the stopping time $\tau = \min(T, \inf\{t\colon\ S_t \geq \overline{S}\})$. We will assume that the issuer of the credit spread option is default-free. The price at time 0 of the spread option is therefore $E_0^Q[\exp(-\int_0^\tau r_s\, ds)X_\tau]$. The default characteristics (h, L) of the underlying bond play a role in determining both the payoff X (through the price process U of the underlying bond) and the payment time τ.

In order to value this credit derivative, one requires as inputs to a pricing model the coupon and maturity structure of the Treasury and defaultable bonds; the parameters $(\underline{S}, \overline{S}, T)$ of the credit derivative; the risk-neutral behavior of the Treasury short-rate process r, the risk-neutral hazard rate process h and the fractional loss process L of the underlying bond; and the price behavior of the defaultable bond *after* default. The post-default bond price behavior is only relevant if, upon default, the "ceiling" spread $\overline{S}$ is not exceeded. At moderate parameters such as those chosen for our numerical examples to follow, the spread passes through $\overline{S}$ with certainty at default, and post-default price behavior plays no role.

For our numerical example, we take a two-state CIR state process $Y = (Y_1, Y_2)'$ satisfying

$$dY_{it} = \kappa_{it}(\theta_{it} - Y_{it})\, dt + \sigma_{it}\sqrt{Y_{it}}\, dB_t^{(i)}, \tag{50}$$

where κ_{it}, θ_{it}, and σ_{it} are deterministic and continuous in t, while $B^{(1)}$ and $B^{(2)}$ are independent standard Brownian motions under an equivalent martingale measure Q. We take the Treasury short-rate process $r = Y_1$ and the short-spread process $s = hL = Y_1 + Y_2$.

The initial condition Y_0 and the time-dependent parameters κ, θ, and σ are chosen to match given discount functions for Treasury and corporate debt, initial Treasury and defaultable yield vol curves, and initial correlation between the yield on the reference Treasury bond and yield spread of the defaultable bond. By "initial" vol curves and correlation, we mean the instantaneous volatility at time zero of forward rates in the HJM sense, by maturity, and the instantaneous correlation at time zero between the yield Z on the reference Treasury note and the yield spread S on the defaultable bond.

We will consider variations from the following base case assumptions:

- The defaultable and Treasury notes are 5-year semiannual coupon noncallable bonds, with coupon rates of 9% and 7%, respectively.
- The credit derivative has a strike of $\underline{S} = 200$ basis points (that is, at the money), with maximum protection determined by a spread cap of $\overline{S} = 500$ basis points, and an expiration time of $T = 1$ year.
- The default recovery rate $1 - L$ is a constant 50%.

The parameter functions κ, θ, and σ, and the initial state vector $Y(0)$ are set so that the Treasury and defaultable bonds are priced at par, and both forward rate curves are horizontal; the initial yield volatility on the reference Treasury note is 15%; the initial instantaneous correlation between the defaultable bond's yield spread S and the Treasury yield Y is zero;[16] the treasury forward rate vol curve is as illustrated in Figure 4 with the label "Price = 2.10." The forward rate spread volatility is a scaling of this same vol curve, chosen so as to attain an initial yield spread volatility of the defaultable bond of 40%.

Given the number of parameter functions and initial conditions, there are more degrees of freedom than necessary to meet all of these criteria. We have verified that our pricing of the credit derivative is not particularly sensitive to reasonable variation within the class of parameters that meet these criteria.

At base case, the credit derivative has a market value of approximately 2.1% of face value of the defaultable bond. Figure 5 shows the dependence of the credit derivative price, per \$100 face value of the corporate bond, on the recovery rate $1 - L$. With higher recovery at default, the credit derivative is more expensive, for we have fixed the yield spread at the base case assumption of 200 basis points, implying that the risk-neutral probability of default increases with the recovery rate. Indeed, then, one can identify the separate roles of the default hazard rate h and the fractional loss L with price information on derivatives that depend nonlinearly on the underlying defaultable bond.

[16] This does not imply that the Treasury short rate r and the mean loss rate c are independent, and typically requires that they will not be independent.

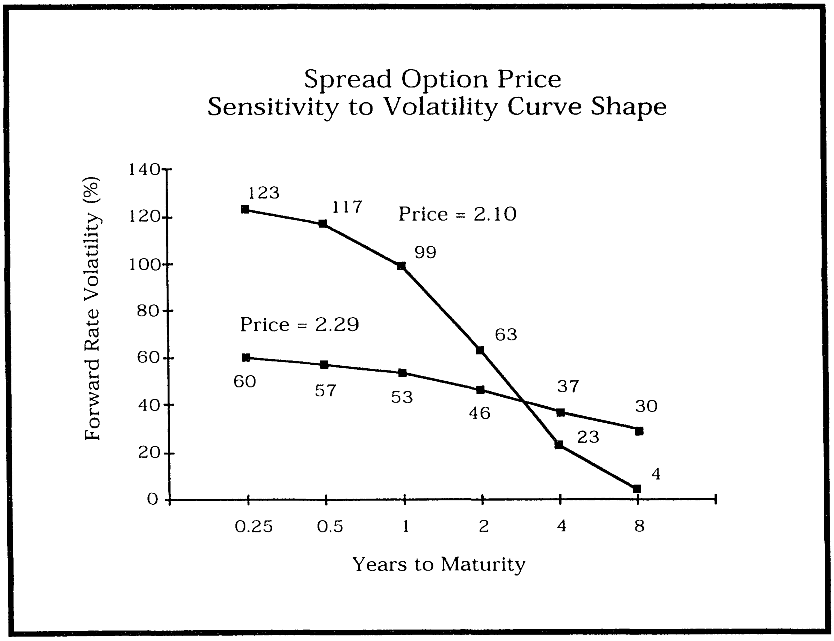

Figure 4
Impact of shape of volatility curve on credit derivative price

Figure 4 shows the base case vol curve for Treasury forward rates and a variation with a significantly flatter term structure of volatility, maintaining the given base case Treasury yield and yield-spread volatilities and correlation. Flattening the shape of the vol curves increases the price of the credit derivative, although not markedly.

Additional comparative statics show that

- Increasing the spread cap $\overline{S}$ from the base case of 500 basis points up to 900 basis points, holding all else the same, increases the price of the yield-spread option to 2.7% of face value.
- Increasing the strike spread $\underline{S}$ from 200 basis points to 300 basis points reduces the price to 0.7% of face value.
- Increasing the yield-spread volatility from the base case of 40% up to a volatility of 65% increases the price to 2.9% of face value.
- Increasing from 0 to 0.4 the initial correlation of yield spread changes with Treasury note yield changes reduces the yield-spread option price, but only slightly, to 2.0% of face value.
- Increasing the expiration date T from 1 to 4 years increases the yield spread option price to 3.8% of face value.

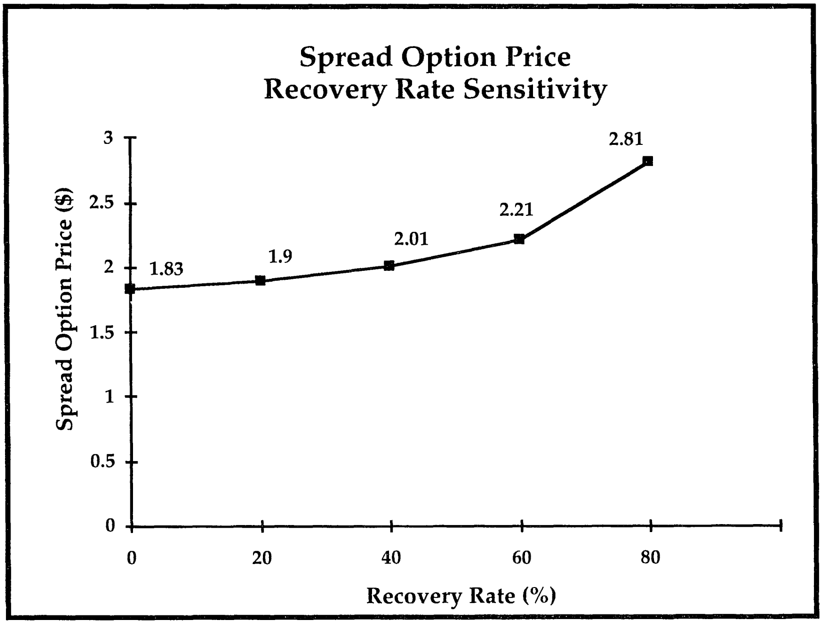

Figure 5
Impact of recovery rate on credit derivative price

- Within conventional ranges, the yield-spread option price is relatively insensitive to the Treasury yield volatility. For example, as one changes the initial yield volatility of the underlying Treasury note from 15 to 25%, by scaling the base-case vol curve shown in Figure 4, the price of the yield-spread option is not affected, up to two significant figures.

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