As Peter Sarnak is fond of saying, the only infinite sum that we really know how to calculate is the geometric sum. Multiplying the summand by a polynomial yields also an explicitly summable result, albeit a combinatorially more difficult one:

**Theorem.** Let \( P \in \mathbb{C}[x_1, \ldots, x_n] \) be given by

\[
P(x_1, \ldots, x_n) = \sum_{\alpha_1, \ldots, \alpha_n=0}^{N} c_{\alpha_1, \ldots, \alpha_n} x_1^{\alpha_1} \ldots x_n^{\alpha_n}
\]

and \( R_1, \ldots, R_n \in \mathbb{C} \) all have absolute value less than one. Then

\[
\sum_{i_1, \ldots, i_n=0}^{\infty} \prod_{\ell=1}^{n} R_\ell^{i_\ell} \cdot P(i_1, \ldots, i_n) = \sum_{\alpha_1, \ldots, \alpha_n=0}^{N} c_{\alpha_1, \ldots, \alpha_n} \prod_{\ell=1}^{n} \sum_{i_\ell=0}^{\alpha_\ell} A(\alpha_\ell, i_\ell-1) R_\ell^{i_\ell} \frac{(1-R_\ell)^{\alpha_\ell+1}}{(1-R_\ell)^{\alpha_\ell+1}},
\]

where \( A(\alpha, k) \) is the number of permutations of the numbers 1 to \( \alpha \) in which exactly \( k \) elements are greater than the previous element (the so-called Eulerian numbers), provided we define \( A(\alpha, -1) \) to be equal to zero if \( \alpha > 0 \) and equal to one if \( \alpha = 0 \).

**Proof.** We achieve immediate simplification by linearity, so we only have to deal with one monomial at a time, and in the case of a monomial the multi-parameter sum becomes a product of one-parameter sums (the theorem is written in the above way simply to emphasize the generality). We are left to evaluate sums of the type

\[
Q(\alpha) = \sum_{i=0}^{\infty} R^i \cdot i^\alpha,
\]

where \( |R| < 1 \) and \( \alpha \) is a nonnegative integer. We have the following formal calculation, whose correctness is justified by the geometric convergence of the \( R^i \) term:

\[
\frac{\partial Q(\alpha)}{\partial R} = \sum_{i=1}^{\infty} R^{i-1} i^{\alpha+1} = \frac{1}{R} \sum_{i=0}^{\infty} R^i \cdot i^{\alpha+1} = \frac{1}{R} Q(\alpha + 1).
\]

Additionally, the ordinary geometric series yields

\[
Q(0) = \frac{1}{1 - R}.
\]

If we let

\[
Q(\alpha) = \frac{T_\alpha(R)}{(1-R)^{\alpha+1}},
\]

then the quotient rule together with (1) gives

\[
Q(\alpha + 1) = R \cdot \frac{\partial Q(\alpha)}{\partial R} = R \cdot \left( \frac{T_\alpha'(R) \cdot (1-R) + (\alpha + 1)T_\alpha(R)}{(1-R)^{\alpha+2}} \right).
\]
Therefore we have the recurrence relation
\begin{align*}
T_{\alpha+1}(R) &= R \cdot (T_\alpha'(R) \cdot (1 - R) + (\alpha - 1)T_\alpha(R)).
\end{align*}

By induction it is clear that \( \deg(T_\alpha) \leq \alpha \). Let
\[
T_\alpha(R) = \sum_{i=0}^\alpha c_{\alpha,i}R^i.
\]

Then (2) yields
\[
\sum_{i=0}^{\alpha+1} c_{\alpha+1,i}R^i = R \cdot \left( \sum_{i=0}^\alpha (i + 1)c_{\alpha,i+1}R^i - \sum_{i=0}^\alpha ic_{\alpha,i}R^i + (\alpha + 1) \sum_{i=0}^\alpha c_{\alpha,i}R^i \right)
\]
\[
= \sum_{i=1}^\alpha (i \cdot c_{\alpha,i} + (\alpha - i + 2) \cdot c_{\alpha,i-1})R^i,
\]
which implies that \( c_{\alpha,i} \) satisfies the recurrence relation
\begin{align*}
(3) \quad c_{\alpha+1,i} &= i \cdot c_{\alpha,i} + (\alpha - i + 2) \cdot c_{\alpha,i-1}
\end{align*}
together with the initial value \( c_{0,0} = 1 \) and \( c_{\alpha,0} = 0 \) for \( \alpha > 0 \).

It remains to show that \( c_{\alpha,i} = A(\alpha, i - 1) \). By assumption, their initial values match. If we know \( A(\alpha - 1, k - 1) \) and \( A(\alpha - 1, k) \), then we can calculate \( A(\alpha, k) \) as follows, arguing combinatorially: all permutations with of length \( \alpha \) with precisely \( k \) ascents are achieved by inserting \( \alpha \) somewhere in a permutation of length \( \alpha - 1 \) with either \( k - 1 \) or \( k \) ascents. In the former case, placing \( \alpha \) anywhere except in between an ascent that already exists will increase the number of ascents by one. In the latter case, placing \( \alpha \) in any ascent, or at the beginning, will maintain the number of ascents. Therefore
\[
A(\alpha, k) = (\alpha - k)A(\alpha - 1, k - 1) + (k + 1)A(\alpha - 1, k).
\]

This recursion formula, when compared to (3), shows that \( c_{\alpha,i} = A(\alpha, i - 1) \) and completes the proof. \( \square \)

The combinatorial definition of the Eulerian numbers immediately implies the relations
\[
A(\alpha, k) = A(\alpha, \alpha - k - 1)
\]
and
\[
\sum_{k=0}^{\alpha-1} A(\alpha, k) = \alpha!
\]
for \( \alpha > 0 \). In particular, the latter relation implies that the coefficients of \( T_\alpha \) will grow quite rapidly in general. In addition, one can prove that
\[
A(\alpha, k) = \sum_{j=0}^k (-1)^j \binom{\alpha + 1}{j} (k + 1 - j)^\alpha,
\]
yielding the truly closed-form relation
\[
\sum_{i=0}^\infty R^i \cdot i^\alpha = \frac{\sum_{k=0}^\alpha (\alpha + 1)! \left( \sum_{j=0}^{k-1} \frac{(-1)^j}{j!(\alpha+1-j)!} (k-j)^\alpha \right) R^k}{(1 - R)^{\alpha+1}}.
\]
A calculation using the recurrence relation or the closed-form solution yields the first few values of $T_\alpha$:

$T_0(R) = 1,$
$T_1(R) = R,$
$T_2(R) = R^2 + R,$
$T_3(R) = R^3 + 4R + R,$
$T_4(R) = R^4 + 11R^3 + 11R^2 + R,$
$T_5(R) = R^5 + 26R^4 + 66R^3 + 26R^2 + R,$
$T_6(R) = R^6 + 57R^5 + 302R^4 + 302R^3 + 57R^2 + R,$
$T_7(R) = R^7 + 120R^6 + 1191R^5 + 2416R^4 + 1191R^3 + 120R^2 + R.$

We can also use the recurrence relation to calculate, for example, $A(\alpha, 1)$ and $A(\alpha, 2)$ for all $\alpha$. We know that $A(\alpha, 0) = 1$ for all $\alpha > 0$ and $A(\alpha, \alpha) = 1$, so the recurrence relation yields

$A(\alpha + 1, 1) = 2 \cdot A(\alpha, 1) + \alpha$

for all $\alpha > 1$. The general method of solving such recurrence relations, which is presented in some detail below in another context, gives us

$A(\alpha, 1) = 2^\alpha - \alpha - 1.$

Going one step further, the recurrence relation yields

$A(\alpha + 1, 2) = 3 \cdot A(\alpha, 2) + (\alpha - 1) \cdot A(\alpha, 1) = 3 \cdot A(\alpha, 2) + (\alpha - 1) \cdot (2^\alpha - \alpha - 1)$

for $\alpha > 2$, and the same general method (with a fair amount of irritating algebra) gives us

$A(\alpha, 2) = 3^\alpha - (\alpha + 1)2^\alpha + \frac{1}{2}a(\alpha + 1).$

We could, of course, continue in this way.

Although there was no real need to consider multi-parameter sums in the above theorem, there are alternative “ad-hoc” methods which, using the techniques of linear recurrence relations, evaluate particular sums without first splitting into monomials. An example follows, which arose in the context of a question in computational number theory:

**Theorem.** Let $n$ be a positive integer and $L_1, \ldots, L_n$ be complex numbers such that $|L_\ell| > 1$ for $1 \leq \ell \leq n$. Then

$$
\sum_{i_1, i_2, \ldots, i_n = 0}^{\infty} \frac{(-1)^{\sum_{\ell=1}^{n} i_\ell}}{\prod_{\ell=1}^{n} (L_\ell)^{i_\ell}} \left( \prod_{\ell=1}^{n} i_\ell + 1 \right) = \left[ \prod_{\ell=1}^{n} \frac{L_\ell}{(1 + L_\ell)^2} \right] \cdot \left[ \prod_{\ell=1}^{n} (1 + L_\ell) + (-1)^n \right].
$$

**Proof.** Absolute convergence is obvious, so rearrangements will be made at will. For $1 \leq k \leq n$, let

$$
S_k = \sum_{i_1, i_2, \ldots, i_k = 0}^{\infty} \frac{(-1)^{\sum_{\ell=1}^{k} i_\ell}}{\prod_{\ell=1}^{k} L_\ell^{i_\ell}} \left( \prod_{\ell=1}^{k} i_\ell + 1 \right).
$$
We will essentially work by induction on \( k \), so we first calculate
\[
S_1 = \sum_{i=0}^{\infty} \left( -\frac{1}{L_1} \right)^i (i + 1) = \frac{L_1^2}{(1 + L_1)^2}.
\]

By bringing the \( k \)th summation to the inside, we get
\[
S_k = \sum_{i_1, i_2, \ldots, i_{k-1}=0}^{\infty} \left[ \sum_{i_k=0}^{\infty} \frac{(-1)^{i_k} (-1)^{-k}}{k \prod_{\ell=1}^{k-1} L_\ell^{i_\ell}} \left( \prod_{\ell=1}^{k} i_\ell + 1 \right) \right]
\]\
\[
= \sum_{i_1, i_2, \ldots, i_{k-1}=0}^{\infty} \frac{(-1)^{i_k} (-1)^{-k}}{k \prod_{\ell=1}^{k-1} L_\ell^{i_\ell}} \left[ \sum_{i_k=0}^{\infty} \frac{(-1)^k}{L_k} \left( \prod_{\ell=1}^{k} i_\ell \cdot i_k + 1 \right) \right].
\]

Let
\[
D_k = \prod_{\ell=1}^{k-1} i_\ell, \quad R_k = -\frac{1}{L_k}.
\]

Then the sum in brackets becomes
\[
\sum_{i=0}^{\infty} (D_k \cdot i + 1) R_k^i = \frac{D_k \cdot R_k}{(1 - R_k)^2} + \frac{1}{1 - R_k} = -\frac{L_k}{(1 + L_k)^2} (D_k - L_k - 1),
\]

and the whole sum \( S_k \) is equal to
\[
-\frac{L_k}{(1 + L_k)^2} \sum_{i_1, i_2, \ldots, i_{k-1}=0}^{\infty} \frac{(-1)^{i_k} (-1)^{-k}}{k \prod_{\ell=1}^{k-1} L_\ell^{i_\ell}} [D_k - L_k - 1].
\]

In order to derive a relation for \( S_k \) in terms of \( S_{k-1} \), we split up the sum into two parts. In the second line, we use \([4]\).
\[
S_k = -\frac{L_k}{(1 + L_k)^2} \left[ \sum_{i_1, i_2, \ldots, i_{k-1}=0}^{\infty} \frac{(-1)^{i_k} (-1)^{-k}}{k \prod_{\ell=1}^{k-1} L_\ell^{i_\ell}} (D_k + 1) - \sum_{i_1, i_2, \ldots, i_{k-1}=0}^{\infty} \frac{(-1)^{i_k} (-1)^{-k}}{k \prod_{\ell=1}^{k-1} L_\ell^{i_\ell}} (L_k + 2) \right]
\]
\[
= -\frac{L_k}{(1 + L_k)^2} \left[ S_{k-1} - (L_k + 2) \prod_{\ell=1}^{k-1} \left( \sum_{i_\ell=0}^{\infty} \left( -\frac{1}{L_k} \right)^{i_\ell} \right) \right]
\]
\[
= -\frac{L_k}{(1 + L_k)^2} \left[ S_{k-1} - (L_k + 2) \prod_{\ell=1}^{k-1} \left[ \frac{1}{1 + L_\ell} \right] \right].
\]

This is a first-order inhomogeneous recurrence relation. We solve it as follows: let
\[
A_k = -\frac{L_{k+1}}{(1 + L_{k+1})^2}, \quad B_k = \frac{L_{k+1} (L_{k+1} + 2)}{(1 + L_{k+1})^2} \prod_{\ell=1}^{k} \frac{L_\ell}{1 + L_\ell},
\]

so
\[
S_{k+1} = A_k S_k + B_k.
\]

As \( |L_\ell| > 1 \) for each \( \ell \), we see that \( A_\ell \neq 0 \) for each \( \ell \), so upon dividing \([7]\) by \( \prod_{\ell=1}^{k} A_\ell \), we get
\[
\frac{S_{k+1}}{\prod_{\ell=1}^{k} A_\ell} - \frac{S_k}{\prod_{\ell=1}^{k-1} A_\ell} = \frac{B_k}{\prod_{\ell=1}^{k} A_\ell}.
\]
Note that if \( k = 1 \), then \( \prod_{\ell=1}^{k-1} A_{\ell} \) is the empty sum; i.e., equal to 1. Now sum up these equations from \( k = 1 \) to \( k = n - 1 \). This sum telescopes, and we are left with

\[
\frac{S_n}{\prod_{\ell=1}^{n-1} A_{\ell}} - S_1 = \sum_{k=1}^{n-1} \frac{B_k}{\prod_{\ell=1}^{k} A_{\ell}},
\]

or upon rearrangement,

\[
S_n = \left[ \prod_{\ell=1}^{n-1} A_{\ell} \right] \cdot \left[ S_1 + \sum_{k=1}^{n-1} \frac{B_k}{\prod_{\ell=1}^{k} A_{\ell}} \right].
\]

Upon plugging in (5) and (6),

\[
S_n = \left[ \prod_{\ell=1}^{n-1} A_{\ell} \right] \cdot \left[ \frac{L_1}{(1 + L_1)^2} - \frac{L_1^2}{(1 + L_1)^2} + \sum_{k=1}^{n-1} \left( \frac{L_{k+1}(L_{k+1} + 2)}{(1 + L_{k+1})^2} \prod_{\ell=1}^{k} \frac{L_{\ell}}{1 + L_{\ell}} \right) \right].
\]

Consider the expression in the second pair of straight brackets. By writing \( L_{k+1} + 2 = (L_{k+1} + 1) + 1 \), we see that it is equal to

\[
L_1 + \sum_{k=1}^{n-1} \left[ (-1)^k (L_{k+1} + 1) \prod_{\ell=1}^{k} (1 + L_{\ell}) + (-1)^k \prod_{\ell=1}^{k} (1 + L_{\ell}) \right] = L_1 + \sum_{k=1}^{n-1} \left[ (-1)^k \prod_{\ell=1}^{k+1} (1 + L_{\ell}) + (-1)^k \prod_{\ell=1}^{k} (1 + L_{\ell}) \right].
\]

This sum clearly telescopes due to the alternating signs \((-1)^k\), and we are left with

\[
L_1 + (-1)^{n-1} \prod_{\ell=1}^{n} (1 + L_{\ell}) + (-1)^1 (1 + L_1)
\]

\[
= (-1)^{n-1} \prod_{\ell=1}^{n} (1 + L_{\ell}) - 1.
\]

Plugging this back into our expression for \( S_n \) yields

\[
S_n = \left[ (-1)^{n-1} \prod_{\ell=1}^{n} \frac{L_{\ell}}{(1 + L_{\ell})^2} \right] \cdot \left[ (-1)^{n-1} \prod_{\ell=1}^{n} (1 + L_{\ell}) - 1 \right]
\]

\[
= \left[ \prod_{\ell=1}^{n} \frac{L_{\ell}}{(1 + L_{\ell})^2} \right] \cdot \left[ \prod_{\ell=1}^{n} (1 + L_{\ell}) + (-1)^n \right],
\]

which is the desired expression. \( \square \)