SPLASH TALK: THE FOUNDATIONAL CRISIS OF MATHEMATICS

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1. Introduction: The idea of proof

This class will cover some of the mathematics, history, and philosophy of the so-called *foundational crisis in mathematics*. Broadly speaking, mathematics in the late nineteenth and early twentieth centuries was marked by an increased awareness of "foundational issues," prompted by a number of problems in the practice of mathematics that had accumulated over the years. We will discuss a few examples of some of these problems, and then discuss the three major schools of thought that emerged to deal with them and provide a coherent philosophical and methodological underpinning for mathematics.

So far, this is just a lot of big words, so let's consider a single, specific idea – the idea of a proof. The first mathematicians to be interested in the concept of proof were the ancient Greeks, and probably the most famous to us are the mostly geometric proofs contained in Euclid's *Elements*, which is a sort of encyclopedia or compendium of Greek geometric knowledge at the time. You have probably seen Euclid-style geometry, because to some extent it is still taught in schools: lots of angles and lines, questions about parallels and perpendiculars, etc. The main elements of Greek proof are the inclusion of some axioms, which are basic statements that you assume from the beginning, a diagram, labeling the figures involved, and a deduction from axioms or previous propositions to a new proposition.

Here's an example of an actual proof in Euclid. Claim: triangles which share a base and are "in the same parallels" (i.e., the line through their other points is parallel to the base) have equal area. To prove the claim, we draw two such triangles, ABC and DBC. We extend the line AD to E and F, letting BE be drawn parallel to AC and letting CF be drawn parallel to BD. Look at the parallelograms EBCA and DBCF. They share a base BC and are in the same parallels, so by a previous proposition in the Elements, they have equal area. The triangle ABC is half of EBCA, and the triangle DBC is half of DBCF. Therefore ABC and DBC also have the same area.

This may not be a particularly exciting proof, but it is typical: one relies on earlier results to make incremental progress, and a diagram is included. It is important to emphasize that all of the proofs in the *Elements* are like this – even proofs of results in what we would now call arithmetic! There is always a diagram, and what we would call a "number" is actually always a length, or an area, or a volume.

While arithmetic eventually became a study in its own right, separate from geometry, the Greek idea that a geometric proof requires a diagram – a sort of visual check of correctness – persisted well into the nineteenth century.[1] It was not sufficient to start with precisely-defined structures and axioms and proceed in a way that we would call "logically," nor was it always necessary to do this if the diagram was sufficiently convincing.

This framework only changed in the very late nineteenth century, and is perhaps best exemplified by a dictum of Dedekind, as popularized by Hilbert: it should be possible to replace the words "point," "line," and "plane" with the words "chair," "table," and "beer mug" without any difficulty whatsoever. That is, diagrams and geometric intuition should be eliminated from all proofs in geometry.[2] This is the way that mathematics, as modern mathematicians know it, is usually said to be done, and it is important to recognize this change in the status of proof. It coheres with what we will call the "formalist" philosophy, to be explained in greater detail later.

I should also mention that most mathematics is not done with a mind towards foundational issues. Mathematicians in their everyday lives do mathematics, not philosophy. But during the foundational crisis, a large number of prominent mathematicians weighed in on philosophical issues that arose as the idea of proof developed, the range of mathematics expanded, and certain "problems" in mathematical thought arose.

2. Problems

2.1. **What is a function?** The first "problem" we will highlight is that of the nature of a function on the real line. So let's start with an exercise: everyone come up to the blackboard and write down, or draw, or notate in some way, any function on the real line. I will too. Your function doesn't have to be interesting, but it can be if you want it to, and we'll discuss the results.

The modern definition of a function on the real line is as an abstract map of sets: we have a set of real numbers, and to each we assign another real number. This is a function defined as a rule, which can be arbitrarily complicated, perhaps so complicated that we can never even write down an explicit formula.

Let's look at continuity a bit more closely. In the modern formulation, if we have a function defined by a rule $y = f(x)$, a function is continuous at a point x_0 if for every $\epsilon > 0$, there exists a $\delta > 0$ such that for every x such that $|x - x_0| < \delta$, we have $|f(x) - f(x_0)| < \epsilon$. If you haven't seen this before, don't worry about it too much. It is the modern formulation (in the language of function-as-rule) of the intuitive idea that continuity should mean that one can draw the graph of the function without picking up one's pencil or chalk.

So why do we even want to think about discontinuous functions? Well, there are some rules that most people would think of as functions that are, in fact, discontinuous. For example, consider the Heaviside step function θ, defined by

$$\theta(x) = \begin{cases} 0 & \text{if } x < 0, \\ \frac{1}{2} & \text{if } x = 0, \\ 1 & \text{if } x > 0. \end{cases}$$

[1]For example, this kind of visual or intuitive thinking is claimed in Kant's *Critique of Pure Reason* to be necessary for the study of geometry.

[2]This is not to say that geometric intuition should be banished from mathematics – far from it! The proof itself, however, should be verifiable independently of intuition, at least in theory.

This is a perfectly simple thing to graph, and it has evident utility in physics, for example: a switch is turned on, and suddenly there is current; the boundary between one object and another can usually be considered discontinuous in this way; etc. Therefore from the perspective of the nineteenth-century mathematician, whose functions come from somewhere – number theory, or solutions to differential equations, or physics, or chemistry – it seems perfectly natural to consider this sort of rule to define a reasonable function.

If this is as complicated as things ever got, then there would probably have been little controversy. Unfortunately (or fortunately, if you have a taste for this sort of thing), things *do* get quite a bit more complicated. Consider the following rule:

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational,} \\ 0 & \text{otherwise (i.e., if } x \text{ is irrational).} \end{cases}$$

If we try to draw a graph of *this* rule, we run into difficulties: in any interval, no matter how small, this function takes on both the value 0 and the value 1, but no other values (in fact, it takes on both of these values *infinitely many times* in any interval). So our graph looks like a big jumble of points on the line $y = 0$, and a big jumble of points on the line $y = 1$. But we shouldn't just draw the lines $y = 0$ and $y = 1$ either, because both jumbles have lots of holes. So here we have an example of a rule (called the *indicator function of the rational numbers*, or sometimes the *Dirichlet function*) that *can't* be drawn nicely as a graph, and is *nowhere discontinuous* (this last claim needs to be proved formally, but it is fairly clear from our attempt to graph it: we need to lift up our pen at every point, in some sense).

So does this rule define a "legitimate" function or not? In its favor: we have given a perfectly explicit rule defining it, and the rule isn't even particularly arbitrary – the distinction between rational and irrational numbers is of fundamental importance. Counting against it: we can't graph it very well, it is everywhere discontinuous, and it doesn't seem to have any physical significance (we certainly won't get this function as a solution to a differential equation, for instance).

Debates on the admissibility of this rule and similar ones[3] as legitimate functions were a preoccupation of mostly nineteenth-century mathematics. Even Weierstrass, who is generally credited with putting the study of functions on a rigorous footing, was not happy with these "pathological" examples. Nevertheless, the "function-as-rule" approach emerged victorious, and modern mathematicians consider all of our

[3]As another important set of examples, assuming people know what differentiability means, there exists functions that are everywhere continuous but nowhere differentiable. Graphs of such functions tend to look like fractals, which are shapes that look similarly complicated no matter how far you zoom in. For completeness' sake, here is the first known example of such a function, discovered by Weierstrass:

$$f(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi x),$$

where $0 \leq a \leq 1$, b is a positive odd integer, and $ab > 1 + 3\pi/2$. This is a primitive example of a lacunary Fourier series; i.e., a Fourier series that omits more and more terms the further out one goes. As a side note to this footnote, nowhere differentiable functions are actually "typical" of continuous functions, in that if you define any reasonable way of picking randomly from the set of all continuous functions, you are almost surely going to pick a nowhere differentiable one. So in the modern idea of a function on the real line, these "pathological" functions are not so pathological after all!

examples as legitimate functions. As we will see, however, this apparent consensus is not accepted by the intuitionists, who have their own idea of what a function on the real line entails.

2.2. **Bigger and bigger sets.**

2.2.1. *Sets and axioms.* The obvious question to ask here is "what is a set"? Whatever vague answer you are likely to give is probably pretty close to right mark – most (though not all) modern set theories take the concept of "set" as undefined, and then list the properties it should have, which are the axioms of set theory.

So for our purposes, a set is a collection of distinct objects. What sort of axioms should sets obey? Well, there are a few stupid ones: sets are equal if they have the same elements, and we want to be able to take unions of sets. There are some that are slightly less obvious: given two sets A and B, we want to be able to consider the set of sets $\mathcal{A} = \{A, B\}$ that has A and B as members. Along these lines, we want to be able to consider the set of all subsets of a given set (this is called the *power set*). And there are some more technical axioms that are usually taken that I won't mention at all.

There is one other axiom that I want to mention, though, which might seem like the most obvious axiom of all. There are two basic ways of building sets, and we want to be able to handle them both. One is just by listing elements: I specify a set $A = \{1, 4, 5, 7\}$ by writing down 1, 4, 5, and 7. Some sets are clearly too big to be written down like this, however: consider the set of all pieces of furniture in the world that are blue. Here we are specifying a set by writing down a property (here, the property is "being a blue piece of furniture"), and we want an axiom to ensure that this set exists, even though we have not written down every element of it. In other words, given any property, we want to be able to consider the set of all elements with this property. Call this the *axiom of comprehension*. It will be important momentarily.

2.2.2. *The size of sets.* Now let's talk about the size[4] of sets. We say that two sets are the same size if there exists a *bijection* between them; i.e., a function from one to the other that is both one-to-one and onto. More concretely, if A and B are sets, we say that they have the same size if there is a function $f : A \to B$ that "pairs off" elements of A with elements of B so that every element of B, and hence A, has exactly one partner (i.e., no elements are left out, and no elements are paired with more than one element of A). For finite sets, this gives us the answer we expect: one set is the same size as another if they have the same number of elements.

What about infinite sets? Using the above concepts, we can actually *define* an infinite set to be a set that can be put in a bijection with a *proper subset* of itself, where a proper subset means a subset not equal to the whole set. Let's check that this agrees with our intuition. If we have a finite set, say for example the set $A = \{1, 2, 3\}$, then any bijection from A to a proper subset of itself, say for example $B = \{1, 2\}$, will have to send 3 elements to 2 elements, and therefore cannot be one-to-one.[5] On the other hand, take a set like the natural numbers $\mathbb{N} = \{1, 2, 3, \ldots\}$

[4] The technical word for the size of a set – i.e., the word used in the mathematical literature – is *cardinality*.

[5] In general, we will always have to send n elements to $n - 1$ elements, and in this situation we will have to send two elements to the same place. This simple-sounding argument has a fancy name: the *Dirichlet pigeonhole principle*.

that our intuition says is infinite. Then the set $C = \{2, 3, 4, \ldots\}$ is a proper subset, because it omits the element 1. Take the function $f : \mathbb{N} \to C$ that maps 1 to 2, 2 to 3, 3 to 4, and in general n to $n+1$ for each integer n. Then I claim f is a bijection: every element of C is in the image of f, and no element of C gets mapped to more than once. We have "paired off" the elements of $\mathbb{N}$ and C. Therefore according to our definition, $\mathbb{N}$ is an infinite set (and according to our earlier definition, $\mathbb{N}$ and C have the same size, which may be somewhat counterintuitive).

2.2.3. *Diagonalization.* In general, infinite sets can have many different sizes, which may at first be surprising. Remember, according to our definition, this means simply that there exist infinite sets with elements that cannot be "paired off" in the way described above. The proof of this fact is a very clever argument first used by Cantor – so clever that I would be amiss in omitting it from this discussion. The method of proof is called the *diagonalization argument*, and we will use it to prove that there is no bijection between the natural numbers $\mathbb{N} = \{1, 2, 3, \ldots\}$ and the real numbers $\mathbb{R}$. Because the natural numbers are a subset of the real numbers, this will prove that the set of real numbers has a larger size than the set of natural numbers.

To do this, let's go back to the definition of size – we need to prove that there is no bijection f from $\mathbb{N}$ to $\mathbb{R}$. How can we possibly prove a statement like this? There are many, many possible maps $\mathbb{N} \to \mathbb{R}$, and we have to show that *none* of them work.

We will employ a *proof by contradiction*: assume that such a function f exists, a derive a contradiction, showing that the original assumption must have been false after all and no such f exists. So let's assume that we have a bijection $f : \mathbb{N} \to \mathbb{R}$. If we think about what a function from $\mathbb{N}$ is, we realize quickly that this is just a list, indexed by the natural numbers in the way that lists usually are (starting at 1 and proceeding ad infinitum). So this given bijection is really a complete list of real numbers, such that no real number is repeated.

We will now use the only fact about real numbers we need: every real number has a decimal expansion.[6] So we can write out our list with these decimal expansions, and get a sort of infinite grid of digits. An example might be the following (obviously incomplete):

$$\begin{array}{rl}
1) & \texttt{1.}\underline{\texttt{7}}\texttt{04105739}\ldots \\
2) & \texttt{0.2}\underline{\texttt{4}}\texttt{1959904}\ldots \\
3) & \texttt{5.19}\underline{\texttt{2}}\texttt{850000}\ldots \\
4) & \texttt{1.000}\underline{\texttt{0}}\texttt{10010}\ldots \\
5) & \texttt{3.2958}\underline{\texttt{0}}\texttt{1992}\ldots \\
6) & \texttt{0.00194}\underline{\texttt{8}}\texttt{292}\ldots \\
\vdots & \qquad\qquad\quad \ddots
\end{array}$$

Now do the following weird-sounding procedure: take the first digit after the decimal place in the first element of the list, and add one. So in the above example we have a 7, so we take **8** as our first number. Then take the second digit after the decimal

[6] Most people think of the decimal expansion as defining the real number system, and this is not a totally unreasonable position to take for some purposes – for centuries, beginning with Simon Stevin, even mathematicians used this "definition."

place in the second element of the list, and add one. In this case, $4 + 1 = \mathbf{5}$. Then take the third digit after the decimal place in the third element of the list and and add one, here getting $2+1 = \mathbf{3}$. The digits chosen in this procedure are underlined in the above list. Proceed ad infinitum in the same way, taking the digits along the diagonal (hence the name diagonalization) and adding one (if you come across the digit 9, change it to 0). Then put all these digits together to form the decimal expansion of a real number, which in our case is the following:

$$r = 0.853119\ldots$$

This is a perfectly reasonable real number that we have constructed. Therefore by assumption, since our list is complete, it lies somewhere on our list; say it is the nth entry. But now we run into a problem: r *differs* from the nth entry on the list at the nth digit after the decimal place, by definition. That is, if the nth digit of r is a 1, then the nth digit of the nth entry on the list is a 0, and so on. So r is not the nth entry on the list after all![7] This contradiction shows that no bijection $\mathbb{N} \to \mathbb{R}$ exists.[8]

This proof can be adjusted appropriately to prove that no set is in bijection with its power set (which, recall, is the set of all of its subsets). So Cantor's diagonalization argument, together with the aforementioned quite reasonable axiom that power sets always exist, shows that no matter how big a set we have constructed, there is always an even bigger one.[9]

2.2.4. *The universe of sets.* The point of all this is that innocuous-sounding axioms for set theory can lead to the existence of some very, very large objects, objects that we have no reasonable chance of ever seeing in "ordinary" mathematics. To a turn-of-the-century mathematician worried about the foundations of mathematics, this infinitely proliferating universe of sets that we have unleashed is at least mildly disconcerting. All we wanted was an account of how sets work in "actual mathematics," like geometry or number theory or analysis, but we seem to have received more than we bargained for.

If these were all the consequences of our reckless axiomatization, then so be it; although we will meet a philosophy that rejects such objects later there seems to be

[7]If this is confusing, go through each possible n: r cannot be the first entry on the list because it differs from that entry at the 1st decimal place, r cannot be the second entry on the list because it differs from that entry at the 2nd decimal place, and so on for all entries on the list. Therefore it is not on the list.

[8]I should confess here that there is one subtlety that needs to be addressed in this proof; namely, the small nonuniqueness in decimal expansions exemplified by the equation

$$1.0000\ldots = 0.9999\ldots$$

It turns out this is the only kind of ambiguity that can occur, and it is not difficult to show that our proof goes through regardless.

[9]There is at least one other method of proving this theorem, which is arguably conceptually simpler, that essentially encodes the Liar's Paradox (how can we assign a truth value to the statement "this statement is false"?) to prove the nonexistence result. Extremely briefly: suppose for contradiction that there exists a set X whose power set is enumerated by X; i.e., we have that $2^X = \{A_x : x \in X\}$. Consider the set $Y = \{x \in X : x \notin A_x\}$, consisting of all elements of X that are not contained in the subset of X that they enumerate. Since Y is a subset of X, we have $A_y = Y$ for some $y \in X$. But if $y \in Y$, we conclude that $y \notin Y$, and if $y \notin Y$ then we conclude that $y \in Y$, contradiction. This argument is suspiciously similar to Russell's paradox, which we will encounter soon.

nothing so extraordinarily objectionable here. But there is a much larger problem lurking in the background of any naïve set theory: an actual paradox.

2.2.5. *Russell's paradox.* Russell's paradox is easy to state but difficult at first to internalize. We first note that there is nothing in our set theory that prevents sets from containing other sets; in fact, the existence of the power set relies on the admissibility of this construction. We can go one step further and note that there is nothing preventing us from considering the possibility of sets that contain *themselves* as elements.[10] So let's jump in with both feet and define the set S to be the *set of all sets that do not contain themselves.* So the set $\{1\}$ is such a set and hence is an element of S, and the set of all living presidents of the United States is an element of S, and the power set of the natural numbers is an element of S (its elements are all subsets of $\mathbb{N}$, none of which are equal to the power set of $\mathbb{N}$ itself). In fact, pretty much any set you would think of is in S, so S is a pretty large set indeed. No matter; it's a weird definition, but so far so good.

Here's the problem. Is S an element of itself? I don't know, so let's check. If S is an element of itself, then, since S is defined as the set of all sets that do not contain themselves, S does not lie in S. But then S is an element of itself after all, so that can't be right. If S is *not* an element of itself, then, since S is defined as the set of all sets that do not contain themselves, S does lie in S. But then S is an element of itself after all, so that can't be right either. But these are the only two possibilities! Either a set is a member of itself, or it isn't. Therefore we arrive at a genuine contradiction in our set theory.[11] Russell's paradox is, in fact, deserving of the name "paradox."

We'll see later how to resolve this, but I do want to stress now that we will have to give something up – there is no way around Russell's paradox other than by changing or restricting our assumptions. We cannot get around it by cleverness, as it is a valid deduction from our naïve conceptions of set theory.

2.3. **The ambiguity of language.** Here's a paradox that's a bit more direct than Russell's. Consider the infinitude of natural numbers $\mathbb{N} = \{1, 2, 3, \ldots\}$. Since the English language has only a finite number of words and symbols, there are only finitely many possible phrases with twelve or fewer words. Some of these phrases may define natural numbers: for example, the phrase "the smallest natural number" certainly specifies the number 1, and the phrase "ten plus ten plus ten plus ten" defines the number 40. However, since there are only finitely many such phrases (with twelve or fewer words, remember), and there are infinitely many natural numbers, there must be some natural numbers that *cannot* be defined in an English phrase of twelve or fewer words. Let N be the smallest such natural number.[12]

What's wrong with N? Well, the phrase "the smallest natural number not definable in twelve or fewer words" has only eleven words! So we have, in fact,

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[10] It's not important at the moment that such sets actually exist, just that we are able to talk about this property as a possibility.

[11] If this paragraph doesn't make sense on a first reading, don't worry. This is genuinely confusing.

[12] There is a subtle issue here: how do we know that there is a smallest such number? There certainly does not exist a largest such number, for instance. The reason is something called the *well-ordering* of the natural numbers, which asserts that every set of natural numbers has a smallest element. This property is not difficult to prove by induction.

defined N in twelve or fewer words. So this is a contradiction. This line of reasoning is known as Berry's paradox, after a librarian at Oxford who apparently suggested something like it to Bertrand Russell (the namesake of Russell's paradox, above).

Philosophically, how can we resolve this statement? In contrast to Russell's paradox, there is really only one reasonable way to proceed. For now we'll leave it as a mystery.

2.4. **Choice.** The *axiom of choice* is, informally speaking, the following reasonable-sounding statement: given any set of nonempty sets, we can make a selection of exactly one object from each set. For example, in a proof we might be faced with a very large set of nonempty subsets of the real numbers, and for some construction we might want to select one real number from each of them. The axiom of choice asserts that we can do this.

To many people, this is an utterly reasonable statement, even if we have to take it as an axiom.[13] Furthermore, it turns out to be really helpful, especially in analysis. It was used even by nineteenth-century mathematicians, usually implicitly. Essentially all of functional analysis depends on it. Various useful facts in algebra, including the existence of algebraic closures of fields and the existence of bases in arbitrary vector spaces, depend on it. If none of these statements are meaningful, that's all right: the point is that in addition to being reasonable-sounding, it's also useful.

So what's the catch? Don't worry; this time we won't come up with a paradox. In fact, it has been proven that if the axioms of set theory are consistent (i.e., paradox-free) without the axiom of choice, then they are consistent with the axiom of choice as well, so using it won't lead to any problems we haven't already encountered. The problem here is not that we get *contradictory* results, it is that we get *counterintuitive* results. The most famous of these is the Banach-Tarski theorem, which asserts that there is a method of decomposing a solid ball into finitely many non-overlapping subsets, which can be reassembled by translations and rotations into *two* solid balls, each of the same volume as the original ball. This, clearly, violates our intuition about how volume should work, although it should be pointed out that the non-overlapping subsets in question are very complicated and in fact do not possess a well-defined volume themselves.[14]

As with the other "problems" above, we will return to the axiom of choice once we have met some of the competing philosophies of mathematics in the early twentieth century.

[13]We *do* need to take it as an axiom, because it turns out to be logically independent of the other standard axioms of set theory.

[14]Another commonly cited counterintuitive consequence of the axiom of choice which is slightly more complicated to explain is the *well-ordering principle*, which states that every set can be put in a well-order. Briefly, a well-order on a set S is a total order (i.e., a relation $\leq$ that obeys all the usual axioms for the relation $\leq$ on the real numbers: reflexivity, antisymmetry, transitivity, and *totality*, the property that $a \leq b$ or $b \leq a$) such that every nonempty subset has a least element. For example, the natural numbers are well-ordered. This is considered very counterintuitive because some sets are very, very large indeed, and a well-ordering is a very restrictive relation. For example, what would a well-ordering on the real numbers look like? The answer is that we can't actually write down a rule for one; coming up with such a rule would involve infinitely many arbitrary choices. All we "know" (assuming the axiom of choice) is that one must exist.

3. Philosophies

3.1. **Logicism.**

3.1.1. *Logic.* What is logic? As a subject, the study of logic goes back to Aristotle, but in a more "modern" formulation we can answer this by saying something like the following: we have some symbols like P and Q, which represent arbitrary propositions like "it is raining today" or "one plus one equals two," and we want to write down formal rules for manipulating them to get true statements. So for example, if we are given that P is true and P implies Q (written $P \implies Q$) is true, then our logic should be able to, purely formally, tell us that Q is true. If we are given that "P and Q" (written $P \,\&\, Q$) is true, we should be able to derive that Q is true. Essentially, logic should formalize what words like "and" and "or" and "implies" mean, and the axioms of the formalization should be self-evident or even tautological.

This is a bit abstract, so let's get more specific. Let P be the proposition "it is raining today" and let Q be the proposition "I have my umbrella." Suppose that we are given that P and $P \implies Q$. Then it is an axiom of our logical system – called *modus ponens*, although the name is not terribly important – that we can conclude that Q is true. Translated into English: if it is raining today, and if we know that if it is raining today then I have my umbrella, then we can conclude that I have my umbrella. Obvious stuff! But it is precisely the strength of logic that our axioms are so obvious: the idea is that we take only extremely simple, self-evident axioms, and perhaps at the end of the day get something highly nontrivial.

The aim of logicism is to reduce *all of mathematics* to such logical trivialities. Mathematics, then, becomes a part of logic, and can be expressed entirely in terms of pure relations among concepts, just like our formal manipulation of Ps and Qs above.

For what follows, we'll need a few more logical concepts. We introduce variables, like x, y, and z, and let propositions depend on them. For example, $P(x)$ could stand for "x is red." The use of variables in propositions also allows us to consider sets; $P(x)$ can equally well represent the set of all red things. Then we allow the use of the phrases "for all" and "there exists" to express sentences like "for all x, x is red" (meaning, of course, "all things are red"), and "there exists an x such that x is red" (meaning "there exists a red thing"). We also have a concept of equality. We fix the following notation as short-hand for the phrases:[15]

$\&$	and
$\vee$	or
$\neg$	not
$\implies$	implies
$\forall$	for all
$\exists$	there exists
$=$	equals

[15]There is actually no need to include all of the following notations. For example, $A \implies B$ turns out to be the same as $(\neg A) \vee B$ under the usual axioms, so we don't need an $\implies$ symbol. But it's convenient to include anyway.

Like the axioms that relate to &, $\implies$, and so on, there are simple axioms that govern the use of $\forall$, $\exists$, and $=$, but we won't need to know specifically what they are.

3.1.2. *The concept of twoness.* With this background, let's look at how a logicist would define the number two. Remember, in the logicist philosophy, all of mathematics should be reducible to logic, so we should certainly at least be able to define things like the natural numbers. In this conception, "having two elements" is a property that sets (i.e., propositions with one variable) can have, so we *define* the number two as a property of propositions with one variable, as follows:

$$2(P) = (\exists x)(\exists y)[P(x)\&P(y)\&x \neq y\&\forall z(P(z) \implies z = x \vee z = y)].$$

It's not worth getting too worked up over the details of this formula if you don't have the inclination, but in English what we are saying is something like "we say P has two elements if there exist distinct elements x and y such that $P(x)$ and $P(y)$, but for every other element z we have that $P(z)$ is false." This is undoubtedly complicated and a bit silly-seeming, but the point is that everything does work out: all of the relational properties of the number "two" are captured by the above definition.[16]

Just as we can define the number "two," we define the other natural numbers. Though I'm not going to write it out, we can also define the operation of addition, from which one can express various other important concepts: multiplication of natural numbers, whether a number is prime, and so on. And when one has constructed the natural numbers, one can construct the rational numbers, and then (via a clever construction of Dedekind, which I will not go into) the real numbers. Everything is constructed, ultimately, from logic.

3.1.3. *Logicism and its problems.* Of course, things didn't turn out so simply. The first logicist program, carried out by Frege in the late nineteenth century, suffered the fatal blow of Russell's paradox and had to be severely modified (more on this later). The next major attempt, carried out by Russell and Whitehead in their famous *Principia Mathematica*, avoided these problems, but at the cost of elegance.

At a more fundamental level, these programs quickly seemed to overstep the bounds of logic itself. Recall that the goal of the logicist program was to reduce all of mathematics to "pure relations between concepts;" i.e., logic. As more and more axioms were introduced, it became harder and harder to argue that the entire formal system was really a part of logic at all.

For instance, to construct the set of natural numbers *as a totality*, not just one by one, it turns out to be necessary to include the axiom that there exists an infinite set. It is very difficult to argue that this so-called "axiom of infinity" is a logical axiom! In fact, Russell and Whitehead did not even take this approach, choosing instead to write all the theorems that depended on the axiom of infinity as conditional statements (e.g. Theorem: If the axiom of infinity is true, then so-and-so is true). At the very least, this approach is awkward.

Furthermore, although we've sketched how to construct most of mathematics from logic, there's a lot more to mathematics than that: we also need to reduce

[16]Perhaps we should not have expected a cleaner definition anyway, given that our logic was purposely designed to be as simple-minded as possible. The idea is that it should be *possible* to reduce mathematics all the way to logic, not that it should be *easy*.

mathematical proof to logic! And there are some fundamental difficulties with this. Even the notion of iterating an operation an arbitrary finite number of times, which is absolutely necessary for mathematical induction and proof in general, seems to have only a tenuous relationship to "pure relations among concepts." The axiom of choice, which we have already discussed, runs into the same issue. These problems, together with further philosophical issues with *Principia Mathematica*,[17] led to a general abandonment of logicism in the early 20th century.[18]

3.2. **Intuitionism.**

3.2.1. *Philosophical background.* Intuitionism was the name for a radically new way of viewing the foundations of mathematics, largely conceived and popularized by L.E.J. Brouwer, a Dutch topologist. His ideas were diverse and somwhat difficult to summarize, but here's a brief summary anyway. Brouwer's metaphysical philosophy views individual consciousness as the *only* source of knowledge. Mathematical objects, like numbers, do not exist outside of human thought – there is no "Platonic realm" of pure ideas that mathematicians access and discover, only conscious mental constructions that create those objects. The most basic of these objects are the natural numbers themselves.[19]

Unlike logicism, therefore, intuitionism takes natural numbers as the foundation of mathematics. Somewhat like logicism, intuitionism is *constructive*. In fact, in the intuitionist framework, to prove something is to offer a mental construction of it. Such constructions are not formalized, because they cannot be formalized – they are the product of intuition. Truth itself is a subjective concept, and can be verified only by intuition.

3.2.2. *Mathematical implications.* The effect of this philosophy on mathematics is profound and, at least when first encountered, somewhat unsettling. We first discard essentially all of set theory, because the intuitionists thought our intuition for how sets behave comes solely from the study of finite sets, and it would not be permissible to arbitrarily extend this intuition to infinite sets as well. If we use logic at all, it is in a subsidiary role: instead of mathematics founded on logic, our logic is founded on mathematics and mathematical intuition.

Most worryingly, we are forced to abandon all *non-constructive* mathematics. What does this mean? Recall that to the intuitionist, a proof is a mental construction. If your "proof" proceeds by assuming the opposite of what you are trying to prove and derives a contradiction,[20] you have not provided a construction at all! Thus the intuitionist would not accept your result as mathematically sound. As another way of stating this, there are some instances in which the intuitionist would reject the *principle of the excluded middle*, which states that a proposition is either

[17]Examples of these involve the dropping of the axiom of reducibility in the 2nd edition, which apparently made it impossible to define the real number system, and arguments over the admissibility of so-called "impredicative" definitions. I will not describe these technical points further.

[18]This is, of course, an absurd oversimplification. There is, for example, a "neo-logicist" school in mathematical philosophy even today, although many of the fundamental problems still remain.

[19]In the words of Leopold Kronecker, a pre-intuitionist mathematician, "God made natural numbers; all else is the work of man."

[20]We have encountered this type of proof before, in the form of Cantor's diagonalization argument, though in this case it is not really necessary.

true or false. A statement can be intuitionistically neither, and will continue to be so until a construction is put forth either for its confirmation or its refutation.[21]

To clarify this issue, let's consider two examples: one in which the principle of the excluded middle does hold in the intuitionistic framework, and one in which it does not. For our first example, take Euclid's ancient proof of the infinitude of the prime numbers. The proof proceeds something like this: assume that there are only finitely many prime numbers, $p_1, p_2, \ldots, p_n$. Consider the number

$$N = p_1 \cdot p_2 \cdot \ldots \cdot p_n + 1.$$

Now N is greater than all of the p_i, so it is not prime itself. Therefore some prime factor divides it, which must be one of the p_i; without loss of generality, assume p_1 divides N. Then since p_1 also obviously divides $p_1 \cdot p_2 \cdot \ldots \cdot p_n$, it divides the difference

$$N - p_1 \cdot p_2 \cdot \ldots \cdot p_n = 1.$$

But this is absurd; no prime number divides 1. This proof appears to be a proof by contradiction, and therefore intuitionistically inadmissible. In fact, it is perfectly valid, providing we rephrase things appropriately! We cannot meaningfully speak of an infinite set in intuitionistic mathematics,[22] but we can translate it to the statement "for every finite set of primes, there exists another prime not in the set." To prove this in an intuitionistic framework, we have to construct this prime, which we do in essentially the same way as before: given our finite set $\{p_1, p_2, \ldots, p_n\}$, construct $N = p_1 \cdot \ldots \cdot p_n + 1$. If N is prime, then we're done, and otherwise factor it into primes (a perfectly valid, algorithmic construction). For the same reason as before, none of the p_i can be factors of N, so in writing down its prime factorization we will find a prime number not in the given set. Euclid's proof, therefore, is intuitionistically valid.

For the second example – where the intuitionists would reject the principle of excluded middle – consider any as-yet-unproven statement in mathematics. For definitiveness, let's look at Goldbach's conjecture, which is the statement that every even integer greater than two can be written as the sum of two primes. This conjecture has not yet been proved and no counterexample has been found, despite extensive computer searches. Therefore, since we have no proof and no counterexample, Goldbach's conjecture is neither true nor false to the intuitionist. In other words, the principle of the excluded middle does not hold for Goldbach's conjecture. Note in particular that mathematics, as practiced by the intuitionists, is rooted in the actual mathematical practice of a particular time. The truth value of statements can change, because to be true is to have a proof, and proofs are constructed at a particular point in time; to be false is to have a counterexample, and counterexamples are found at a particular point in time.

3.3. **Formalism.**

[21]If this seems absurd, consider that to the intuitionist truth and falsehood have different meanings than in "standard" mathematics. For something to be true intuitionistically, it means that there is a proof (i.e., a construction); for something to be false intuitionistically, it means there is a refutation (also a construction).

[22]Actually, this is not quite accurate – different intuitionists have different positions with respect to infinite objects. Brouwer distinguished between the "potential infinite" (a procedure consisting of and unending series of steps, like counting) and the "actual infinite" (an actual infinite set, like $\mathbb{N}$), allowing the former and disallowing the latter.

3.3.1. *"Rescuing" classical mathematics.* Formalism is now our third major school in the philosophy philosophy of mathematics. In its most essentialist form, formalism holds that mathematics and logic are the result of formal manipulations of more or less arbitrary sets of axioms. Like in logicism, formalism proceeds by formal manipulations. In logicism, however, we start with something "at the bottom" – namely, the supposedly self-evident manipulations of formal logic. In formalism, we get to choose the axioms to suit our own purposes, and results do not have inherent meaning until we give them an interpretation. At its most banal level, mathematics is a "game" played with symbols.[23]

Recall that the natural numbers are constructed out of logic in the logicist program and taken as a basic intuition in the intuitionist program. In formalism, by contrast, to define the natural numbers we write down a set of axioms that seem to capture all of their properties, and then try to show that everything we can "classically" prove about the natural numbers can be proven starting with these axioms. In the case of the natural numbers, the axioms that are generally taken go by the name of Peano arithmetic. I will not go into any detail about these axioms; for our purposes it suffices to know that they exist and have stood the test of time.

Historically, the formalist program is most associated with the mathematician David Hilbert. It was, in a large part, a reaction to the intuitionists – the formalists wished to preserve as much of "classical mathematics" as possible, including set theory and Weierstrass-style analysis on the real numbers, while still giving an unobjectionably rigorous foundation to mathematics.

Here's an example of the type of reasoning that the intuitionists would reject and the formalists wanted to save. Define the quantity ϵ_m to be equal to zero if $2m$ is the sum of two prime numbers, and 1 otherwise. Let f be a function on the interval $[0,1]$ of real numbers, defined so that it is linear on the three pieces $[0,1/3]$, $[1/3,2/3]$, and $[2/3,1]$, and the value of the function at those three points is given by

$$f(0) = -1, \quad f(1/3) = -\sum_{n=1}^{\infty} \frac{\epsilon_{2n}}{2^n}, \quad f(2/3) = \sum_{n=1}^{\infty} \frac{\epsilon_{2n-1}}{2^n}, \quad f(1) = 1.$$

Since we have defined f to be piecewise linear, it is certainly continuous, so because $f(1/3) \leq 0 \leq f(2/3)$ the intermediate value theorem of real analysis shows that f has a root in $[1/3, 2/3]$. Where is this root, exactly? Well, we don't know! In fact, we can't estimate it any more closely than we already have without knowing a case of Goldbach's conjecture: for example, if we knew that $f(1/3) \neq 0$, then $\epsilon_{2n} = 1$ for some n, which means that there is a counterexample to Goldbach's conjecture for some n divisible by 4. Similarly, if we knew that $f(2/3) \neq 0$, then $\epsilon_{2n-1} = 1$ for some n, which means that there is a counterexample to Goldbach's conjecture for some n divisible by 2 but not by 4. Similarly, if we have some counterexample, we know that these are not zeroes. Since we do not, as of 2013, know whether there is exist any counterexamples to Goldbach's conjecture, we cannot pin down a root of f to within an error smaller than $1/6$: we just know that at least one exists.

What have we shown? This function is certainly a bit contrived, for the purposes of exposition, but we get a real payoff for it: if we were intuitionists, then asserting the existence of a root for f would mean that we have constructed such a root,

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[23]I am deliberately overstating things here. In fact, Hilbert himself, the most famous formalist, believed that mathematics contained real meaning: some choices of rules for our symbolic "games" are more meaningful than others. But this reduction is useful to keep in mind.

which means in particular we can pin it down to arbitrary accuracy. But we cannot do that right now, so we have a problem. The only way out is to realize that the intermediate value theorem – a basic, uncontroversial theorem of ordinary single-variable calculus – is not valid intuitionistically![24] One can see, therefore, some glimmer of what has to be discarded in intuitionistic mathematics, and why the formalists wanted to be able to preserve as valid this kind of non-constructive existence proof.

3.3.2. *Hilbert's program.* In order to satisfy the demands of the intuitionists *and* keep using the traditional tools of mathematics (such as infinite sets and non-constructive proofs), Hilbert proposed a search for a complete and consistent foundation to mathematics. Such a foundation would consist of a finite set of axioms and a finite set of uncontroversial deductive rules. Then one would find a proof that *the system itself* was both consistent (could not be used to derive any contradictions or paradoxes) and complete (all true statements could be proved in the formalism). Furthermore, these proofs of consistency and completeness should consist only of "finitistic" reasoning which would be acceptable to intuitionists as well as mainstream mathematicians.

If this were accomplished, then intuitionism would be shown to be unnecessary, as the usual reasoning of classical mathematics would be shown to be intuitionistically valid. The well-foundedness of a complicated system (mathematics) would rest on the well-foundedness of an uncontroversial simple system (finitistic reasoning, as employed in the *metamathematical* consistency and completeness proofs), thus justifying use of the complicated system.

Half of this goal – the reduction of all mathematics to formal axiomatic systems – was essentially provided by the logicist program. The metamathematics, however, soon ran into insurmountable difficulties.

3.3.3. *Gödel and the end of Hilbert's program.* Hilbert's program was convincingly derailed in 1931, when the logicist Kurt Gödel published his famous First Incompleteness Theorem. In nontechnical language, the theorem used clever metamathematical tricks to prove that any finitely axiomatized theory that is strong enough to express the elementary arithmetic of the natural numbers can never be both consistent and complete. In particular, if (as most people believe) basic arithmetic is consistent, then there exist statements expressible in the given theory that are *true* without being *provable* in the theory.

Gödel's proof is worth studying, although we will not go into any details here. We will simply note that the upshot of the incompleteness theorem is that Hilbert's program is doomed to failure: we will never be able to use a weaker system to prove the consistency of a stronger, so long as both systems can express elementary arithmetic.

4. Clash and aftermath

Now that we have seen a quick overview of the competing philosophies, we will make a brief historical digression and then return to our original problems to see what these philosophies bring to bear.

[24]There is an "intuitionistic intermediate value theorem," but it is weaker and takes the form of an approximation statement.

4.1. **Historical digression: Hilbert and Brouwer.** In 1920, *Mathematische Annalen* was the leading journal in mathematics, and both David Hilbert (the formalist) and L.E.J. Brouwer (the intuitionist) held positions on its board. Hilbert, due to his greater age and cumulative influence in mathematics, often acted as a *de facto* editor-in-chief, and a majority of the board generally supported him. But by 1920, Hilbert was ill, and at the same time began to feel threatened by Brouwer's intuitionism, which he had come to believe was a threat to mathematics itself. Hilbert acted by circulating a letter to the other editors of the *Annalen*, seeking to have Brouwer removed from the board. Although Hilbert enjoyed general support, the conflict became acrimonious, especially when Brouwer discovered the plan and intervened energetically on his own behalf. Einstein, perhaps the most famous member of the board, stated that he wanted no part in Hilbert's plan and stayed out of the fray. The mathematician Carathéodory, who was a friend of Brouwer though not an intuitionist, tried and failed to mediate on his behalf. Ultimately, however, Hilbert and his supporters managed to dissolve the board and form it anew, minus Brouwer and Carathéodory.

This episode is certainly the most famous of the foundational crisis in mathematics, and if nothing else exemplified the growing dominance of formalist philosophy in mathematical practice.

4.2. **Back to our problems.** It is now time to return to our original problems, both to see what the three discussed philosophical approaches have to say about them and to see how most mathematicians today view them.

Our first problem, recall, was about the nature of functions on the real line. Of course, before we can discuss functions on the real line, we have to discuss the real line itself. The logicist approach to the real numbers, as we have seen, is to construct them as a set, starting with some basic axioms (the formalists largely copied the logicists' approach here). The only natural definition of a function on the real line, then, is as an arbitrary map from the set of real numbers to the set of real numbers: that is, the "modern" definition. The logicists and formalists accept nowhere continuous functions as readily as they accept smooth functions.

The intuitionists' approach is very interesting, although it would take us too far afield to discuss in any reasonable level of detail. Recall that as intuitionists it makes no sense for us to discuss infinite sets; thus, any effort to discuss the real line as a set of points is a non-starter.[25] Instead, arbitrary sets of real numbers are replaced by concepts called *spreads* and *species*, which are much more firmly connected to the natural numbers (the source of all intuition in mathematics). A spread is sort of like sequence of natural numbers given by an effective rule. A function on the real line is then defined to be an assignment of values to a spread.

[25]In this respect, the intuitionist approach mimics Aristotle, who thought that the idea that a continuum was comprised of points was ridiculous. As far as I am aware, the first mathematician to posit the correspondence between a set of numbers and a continuum was Descartes, the founder of analytic geometry.

Interestingly, by using this approach we get a totally different concept of function than the logicists and formalists. For example, it is a theorem in intuitionistic mathematics that all functions are continuous![26] We have banished the nastiest functions from our arsenal. Unfortunately, we have also banished some not-so-nasty functions, including the aforementioned Heaviside step function θ. Intuitionistic analysis is fiendishly difficult, and is one of the major reasons that few mathematicians have ever become intuitionists.

Now let's move on to set theory. The intuitionist response to the problems of Cantor and Russell and Berry is simple: we give up set theory. So that's that. The logicist approach to Russell's paradox, as given by Russell himself in *Principia Mathematica*, is to give up our ability to allow sets to be members of themselves. In practice, what this means is that we must set up a *type theory*: to each set, we attach a label (its *type*) explaining whether it is a set of bare elements, or a set of sets, or a set of set of sets, and so on, and we only allow sets of one type to be elements of sets of the next larger type. This certainly gets rid of Russell's paradox, but it has two disadvantages: first, it adds another layer of complication to set theory (bringing it still further from basic predicate logic), and second, it is sometimes useful to consider sets of mixed type (for example, the set $\{1, \{2, 3\}\}$).

In contrast, the formalist approach – which also happens to be the current widely-accepted approach – is to modify the axiom of comprehension. Recall that the axiom of comprehension states, essentially, that given any property, we can create the set of all objects that possess that property. It is used in an essential way in Russell's paradox. We now modify it to the *axiom of restricted comprehension*, which states that given any property, we can create the set of all objects *that are elements of any given set.* In other words, we can turn properties into sets, provided we first restrict ourselves to elements that all lie in one big set. It is easy to see that Russell's paradox dissolves, for we can no longer form the "set" S of all sets that do not contain themselves – we have not restricted our universe to a set first. In this framework, we can use Russell-like arguments to show that S is not a set, and that there can be no "set of all sets." With restricted comprehension, some things are just too big to include in our set theory.

The resolution of Berry's paradox centers around the ambiguities relating to the word "definable." In short, definability is not, and cannot be, a mathematical concept. We dissolve the paradox only by formalizing our language. *Within a given formalized language*, it is possible to express something like definability only by going outside the language. The semantic ambiguities of English, which is not formalized and therefore has no such restriction on the use of the word "definable," are what lead to the paradox.

Finally, I should mention the status of the axiom of choice. Modern mathematicians now regularly accept it, although in most fields of study (analysis possibly excluded) it is considered in good taste to note when it is being used. The Banach-Tarski paradox and the well-ordering principle are considered counterintuitive but acceptable results, given the axiom of choice's utility in much of mathematics. In fact, although the usual form of the axiom of choice is generally considered to be

[26] In fact, uniformly continuous. In extremely vague terms, the reason is that we can never specify the input with perfect accuracy using only finitely many natural numbers, and therefore in order for a function to make sense the output had better not depend very heavily on a precise specification.

intuitionistically unacceptable, there are restricted forms that *are* accepted by some intuitionists.

4.3. **Where do we stand today?** Most modern mathematicians, it should be admitted, do not give much mind to foundational issues. This can either be interpreted as a tacit acceptance of formalism, which as we have seen has be default become the dominant approach (the failure of Hilbert's program notwithstanding), or as a dismissal of the relevance or importance of philosophy in general. It is still, however, worth tracing the strands of the other two philosophies to the present day.

After the rise of formalism and its assimilation of logicist methods, the school of logicism has largely faded away, save for a small school of neo-logicist philosophers who have attempted to ground as much mathematics as possible in quasi-logical axioms such as Hume's principle.[27]

Intuitionism, though consistently somewhat marginalized in mathematics, has enjoyed a continuous school of thought to the present day, via logicians like Arend Heyting, who introduced a formalization of intuitionistic reasoning,[28] and Errett Bishop, who developed constructive analysis far beyond Brouwer's work. Additionally, there are several related schools of thought, e.g., finitism, which rejects any construction that cannot be derived in finitely many steps starting from the natural numbers. The fact that intuitionism does not seek to give a foundation to mathematics as it is commonly practiced, but rather asserts how mathematics should be practiced, has consistently hindered its appeal.

Although the triad of logicism, intuitionism, and formalism serves reasonably well to explain the diversity of philosophical thought in the early part of the twentieth century, many other strands have emerged since, including structuralism, fictionalism, various forms of empiricism, and various forms of realism (Platonism). In the last few decades, there has been somewhat of a shift in focus among many philosophers, giving up the search for mathematical foundations and instead trying to understand "mathematics as it is actually done."

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[27] Hume's principle is the aforementioned supposition that two sets have the same size exactly when they can be put in a bijection.

[28] Brouwer, it must be said, disapproved of the formalization of intuitionistic thought. Interestingly, much later work in a field called *topos theory* has brought about a resurgence in the importance of these so-called intuitionistic logics.

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