NOTES ON SEVERAL COMPLEX VARIABLES:
AMALGAMATION OF SYZYGIES

EVAN WARNER

1. Statement and consequences

The goal is the following theorem, which should take about two lectures to prove:

Theorem 1.1 (Theorem of amalgamation of syzygies). Let \( K \) be a compact subset of a simply connected open polydomain \( D \subset \mathbb{C}^n \). Then there exists an open set \( U \) with \( K \subset U \subset \overline{U} \subset D \) such that any coherent analytic sheaf over \( D \) has a terminating chain of syzygies over \( U \).

In order to understand this statement, recall the following definitions:

- A polydomain is a product of domains (connected open subsets) of \( \mathbb{C} \).
- An analytic sheaf is a sheaf of \( \mathcal{O} \)-modules, where \( \mathcal{O} \) is the sheaf of germs of holomorphic functions in \( \mathbb{C}^n \).
- A coherent sheaf is a sheaf that locally admits arbitrarily long chains of syzygies. Remember that terms of syzygies, by definition, are finitely generated.
- A terminating chain of syzygies is chain of syzygies ending with zero on the left.

We will assume the following:

Theorem 1.2 (Hilbert syzygy theorem for sheaves). Any coherent analytic sheaf \( S \) locally admits a terminating chain of syzygies of length \( n \) of the form

\[
0 \rightarrow \mathcal{O}^{|p_n|_U} \xrightarrow{\mu_n} \ldots \xrightarrow{\mu_2} \mathcal{O}^{|p_1|_U} \xrightarrow{\mu_1} \mathcal{O}^{|p_0|_U} \xrightarrow{\mu} S|_U \rightarrow 0.
\]

One can therefore think of the theorem of amalgamation of syzygies as a statement about patching together lots of instances of the Hilbert syzygy theorem in a simply connected polydomain: we lose control over how long the chain of syzygies is, but it still must eventually terminate.

The major consequences are the following extremely important theorems:

Theorem 1.3 (Theorems A and B of Cartan). Let \( S \) be a coherent analytic sheaf defined on \( D \subset \mathbb{C}^n \), where \( D \) is an open, simply connected polydomain. Let \( K \) be also an open simply connected polydomain, with \( K \subset D \) compact. Then

(A) \( S|_K \) is generated by finitely many global sections (elements of \( \Gamma(K, S) \)).

(B) The sheaf cohomology \( H^q(K, S) \) vanishes for \( q \geq 1 \).

Proof. By amalgamation of syzygies, we get a terminating chain of syzygies in a neighborhood of \( K \), and therefore also in \( K \) itself, ending in the following:

\[
\ldots \mathcal{O}^{|p_j|_K} \xrightarrow{\mu} S|_K \rightarrow 0.
\]

Source: mostly Gunning’s text Analytic Functions of Several Complex Variables
The sheaf $\mathcal{O}^p_K$ is generated by the basis global sections $E_j$, so $\mathcal{S}|_K$ is generated by the global sections $\mu(E_j) \in \Gamma(K, \mathcal{S})$, which proves (A). The second part (B) follows from a theorem discussed a couple of classes ago that states directly that an analytic sheaf on a simply connected polydomain which has a terminating chain of syzygies has vanishing higher cohomology (this is not an obvious fact; it requires the development of both sheaf cohomology and Dolbeault cohomology). □

The plan of attack is as follows: we will first prove Cartan’s lemma, which allows us to “patch together” matrix-valued holomorphic functions. This is purely a theorem in analysis. Then we will prove a “modification of syzygies” theorem, which allows us to systematically modify terminating chains of syzygies over isomorphic spaces so they are isomorphic. This theorem is purely sheaf-theoretic. We will then put the two together to get the main result.

2. Cartan’s lemma

Even to state Cartan’s lemma, it is helpful to have a picture in mind:

We are given real numbers $a_1 < a_2 < a_3 < a_4$ and $b_1 < b_2$, from which we form the above three open rectangles in $\mathbb{C}$. Explicitly, thinking of $\mathbb{C}$ as $\mathbb{R}^2$,

$$K_1 = (a_2, a_3) \times (b_1, b_2),$$
$$K_1' = (a_1, a_3) \times (b_1, b_2),$$
$$K_1'' = (a_2, a_4) \times (b_1, b_2).$$

In particular, $K_1 = K_1' \cap K_1''$. Let $K_2, \ldots, K_n$ be simply connected domains in $\mathbb{C}$, and let

$$K = K_1 \times K_2 \times \ldots \times K_n,$$
$$K' = K_1' \times K_2' \times \ldots \times K_n',$$
$$K'' = K_1'' \times K_2'' \times \ldots \times K_n',$$

all three of which are obviously open in $\mathbb{C}^n$. In particular the picture above shows the projection of these three sets down to the first two coordinates.

**Theorem 2.1** (Cartan’s lemma). If $F(z)$ is a holomorphic, nonsingular, matrix-valued function on $K$, then there are holomorphic, nonsingular, matrix-valued functions $F'$ and $F''$ on $K'$ and $K''$, respectively, such that $F(z) = F'(z)F''(z)$.

By a holomorphic matrix-valued function, we mean that the function is holomorphic in each entry separately. Nonsingularity is required to hold everywhere.
Cartan’s lemma will (eventually) allow us to patch together terminating chains of syzygies so we can bootstrap our way from local to global results (i.e., from the Hilbert syzygy theorem to the theorem on amalgamation of syzygies).

Unfortunately, Cartan’s lemma requires a lot of preliminary work, so we’ll prove a few lemmas first and work up to it. Define, for \( z = (z_1, \ldots, z_n) \in \mathbb{C}^m \), a norm

\[
|z| = \left( \sum_{\nu} |z_{\nu}|^2 \right)^{1/2}.
\]

If \( A = (a_{\mu\nu}) \) is an \( n \times n \) matrix of complex numbers, define the norm of \( A \) to be the operator norm

\[
|A| = \sup_{|z| = 1} |Az|.
\]

Lemma 2.2. Let \( A = (a_{\mu\nu}), B = (b_{\mu\nu}) \) be \( m \times m \) matrices over \( \mathbb{C} \). Then

\begin{enumerate}[(a)]
  \item \( |a_{\mu\nu}| \leq |A| \) for each \( \mu, \nu \),
  \item \( |AB| \leq |A||B| \), and
  \item \( |A + B| \leq |A| + |B| \).
\end{enumerate}

**Proof.** Claims (b) and (c) are true in general for the operator norm, and are entirely standard exercises. For (a), let \( z = (0, 0, \ldots, 0, 1, 0, \ldots, 0) \), with the 1 in the \( \nu \)th entry. Then

\[
Az = (a_{1\nu}, a_{2\nu}, \ldots, a_{m\nu}),
\]

so

\[
|a_{\mu\nu}| \leq |Az| \leq |A|,
\]

where the first inequality holds by the definition of the norm of elements of \( \mathbb{C}^m \), and the second inequality holds by the definition of the operator norm. \( \square \)

Define the matrix exponential and logarithm by power series in the usual way:

\[
\exp(A) = \sum_{\nu=0}^{\infty} \frac{1}{\nu!} A^\nu, \quad \log(I - A) = -\sum_{\nu=1}^{\infty} \frac{1}{\nu} A^\nu,
\]

where \( I \) is the identity matrix. By part (b) of the above lemma, \( |A^\nu| \leq |A|^\nu \), so we have the inequalities

\[
|\exp(A)| \leq \sum_{\nu=0}^{k} \frac{|A|^\nu}{\nu!}, \quad |\log(I - A)| \leq \sum_{\nu=1}^{k} \frac{1}{\nu} |A|^\nu.
\]

Therefore \( \exp(*) \) converges uniformly on compact subsets of \( \mathbb{C}^{m \times m} \) while \( \log(I - *) \) converges uniformly on compact subsets of the domain \( \{ A \in \mathbb{C}^{m \times m} : |A| < 1 \} \). In particular, they are holomorphic on their respective domains.

It is easy to check the following:

\begin{enumerate}[(1)]
  \item \( \exp(\log(I - A)) = I - A \) where applicable, and
  \item \( \exp(A)\exp(-A) = I \).
\end{enumerate}

From (2), we see that \( \exp(A) \) is always nonsingular. If \( |I - A| < 1 \), then by putting \( B = \log A \) we have \( A = \exp B \) by (1), so \( A \) is nonsingular.

These notions obviously extend to matrix-valued functions, and the operations preserve holomorphicity.

The next lemma allows us to split nonsingular matrix-valued functions into a finite product, all of whose terms are close to the identity.
Lemma 2.3. Let $K$ be a compact subset of a simply connected open polydomain $D \subseteq \mathbb{C}^n$ and $F(z)$ a holomorphic nonsingular matrix-valued function on $D$. Then there exists an integer $r$ and holomorphic nonsingular matrix-valued functions $F_1, \ldots, F_r$ on $D$ such that

$$F(z) = \prod_{j=1}^{r} F_j(z) \quad \text{and} \quad |I - F_j(z)| < 1 \text{ for all } z \in K, 1 \leq j \leq r.$$  

Proof. By simple-connectedness and the Riemann mapping theorem, we can assume that $D$ is a unit polydisc centered at the origin (in fact, we will only need to make $D$ star-convex, but we might as well go all the way). By adding more points to $K$, we can without loss of generality assume that $K$ has nonempty interior.

We want to use the general fact that a connected topological group is generated by any open neighborhood of the identity. Let the group $G$ in question be the set of holomorphic nonsingular matrix-valued function on $D$ with multiplication given by pointwise matrix multiplication. Define

$$||F|| = \sup_{z \in K} |F(z)|$$

for every $F \in G$. Since $K$ has nonempty interior, by the identity theorem for holomorphic functions this is a legitimate norm (not just a seminorm). This gives rise to a topological group structure on $G$. We want to prove that $G$ is connected, which we can do by letting

$$F_t(z) = F(tz)$$

for $0 \leq t \leq 1$. This defines a continuous path from $F(z)$ to the constant function $F(0)$. The set of all constant functions in $G$ is certainly connected, so $G$ itself is connected. Therefore by the aforementioned general fact the neighborhood $\{F \in G : ||I - F|| < 1\}$ of the identity generates the whole group, which upon unraveling definitions is exactly what we wanted. \[ \square \]

Next we prove an approximation theorem.

Theorem 2.4 (Approximation by entire functions). Let $K$ be a compact, simply connected polydomain in $\mathbb{C}^n$, $F$ a holomorphic nonsingular matrix-valued function on an open neighborhood of $K$. Then for every $\epsilon > 0$, there exists a holomorphic nonsingular matrix-valued function $G$, defined on all of $\mathbb{C}^n$, such that

$$|F(z) - G(z)| < \epsilon$$

for all $z \in K$.

Proof. It is worth noting that most of the content of this theorem is in the word “nonsingular”: if we drop this condition then by Runge’s theorem we can actually take $G(z)$ to be a matrix of polynomial functions. To get nonsingularity, however, we will need to write things as exponentials.

Fix $0 < \epsilon < 1$ and an $F$. Using Lemma 2.3, find $F_1, F_2, \ldots, F_r$ such that $F = \prod_j F_j$ and $|I - F_j(z)| < 1$ for each $j$ and $z$. As discussed above, if we let $H_j = \log F_j$, then $F_j = \exp H_j$. The $H_j$ are holomorphic but possibly singular. We can use Runge’s theorem on a neighborhood of $K$ to find matrices of polynomials $P_j$ such that $|H_j(z) - P_j(z)|$ can be made uniformly small. Since $\exp$ is continuous, applying it to $H_j$ and $P_j$ we can write

$$|F_j(z) - G_j(z)| < \frac{\epsilon}{r^{M^r-1}},$$
where \( M \) is such that \(|F_j(z)| < M - 1\) for all \( z \in K \), which is possibly by compactness. Note that each \( G_j = \exp P_j \), being an exponential, is nonsingular, and we have a bound like \(|G_j(z)| < M\). Let \( G = \prod_j G_j \), which is also holomorphic and nonsingular. Expanding the product, we have

\[
|F(z) - G(z)| = |F_1(z)F_2(z) \ldots F_r(z) - G_1(z)G_2(z) \ldots G_r(z)|
\]
\[
\leq |F_1(z)F_2(z) \ldots F_r(z) - F_1(z)F_2(z) \ldots F_{r-1}(z)G_r(z)|
\]
\[
+ |F_1(z)F_2(z) \ldots F_{r-1}(z)G_r(z) - F_1(z) \ldots F_{r-2}(z)G_{r-1}(z)G_r(z)|
\]
\[
+ \ldots
\]
\[
+ |F_1(z)G_2(z) \ldots G_r(z) - G_1(z) \ldots G_r(z)|
\]
\[
\leq M^{r-1}|F_r(z) - G_r(z) + \ldots + M^{r-1}|F_1(z) - G_1(z)|
\]
\[
\leq \epsilon. \tag*{□}
\]

Another necessarily but seemingly random lemma:

**Lemma 2.5.** Let \( \sum_{\nu=1}^{\infty} a_{\nu} \) be a convergent series of positive reals and \( F_\nu \) a sequence of holomorphic matrix-valued functions on some \( D \subseteq \mathbb{C}^n \), termwise bounded in \( D \) by \( a_\nu \). Then

\[
F(z) = \prod_{\nu=1}^{\infty} (I + F_\nu(z))
\]

converges to a holomorphic matrix-valued function on \( D \). If all terms are nonsingular, so is the product.

**Proof.** The first part is standard; for example, it is in E. Stein’s complex analysis text (except for the matrix part, but the proof is the same: we just show that the partial products are Cauchy).

Nonsingularity is trickier. Let \( P_\nu \) be the \( \nu \)th partial product. Then by continuity and multiplicativity of the determinant,

\[
\det F(z) = \det \lim_{\nu \to \infty} P_\nu(z) = \lim_{\nu \to \infty} \det P_\nu(z) = \lim_{\nu \to \infty} \prod_{j=1}^{\nu} \det(I + F_j(z)).
\]

Each term \( \det(I + F_j(z)) \) is a polynomial in the entries of \( F_j(z) \) with constant term 1 and some finite number \( N \) of monomials. Pick some \( k \) such that \( a_j < 1/N \) for all \( j \geq k \). Then since \(|F_j(z)| < a_j\), we have

\[
|\det(I + F_j(z))| > 1 - Na_j > 0
\]

for all \( j \geq k \). It is a basic fact that an infinite product \( \prod(1 + a_i) \), where the \( a_i \) converge absolutely, is zero if and only if one of the terms is zero. None of the above terms is zero, and \( \sum Na_j \) converges absolutely, so

\[
\prod_{j=k}^{\infty} |\det(I + F_j(z))| \geq \prod_{j=k}^{\infty} (1 - Na_j) > 0.
\]

We can multiply back in the finitely many terms we took out without affecting the nonsingularity, since they are all nonsingular. It follows that \( \det F(z) \neq 0 \), as desired. \( \tag*{□} \)

Finally, one last lemma:
Lemma 2.6. There exists a number \( P > 0 \) such that for any \( n \times n \) matrices \( A, B \) with \( |A|, |B| < 1/2 \), and any \( C \) such that \( (I + A)(I + C)(I + B) = I + A + B \), then \( |C| \leq P|A||B| \). In plainer language, we can bound any \( C \) satisfying the above equation in terms of \( A \) and \( B \) for \( A, B \) small enough.

Proof. This is almost entirely unenlightening, and not very difficult. One uses compactness to bound things and then rearranges. \( \Box \)

Now we get back to the setting of Cartan’s lemma. Recall the picture, and let \( K_j(\delta) \) be \( \delta \)-neighborhoods of the \( K_j \) (where \( \delta > 0 \)). Likewise construct \( K_j'(\delta) \) and \( K_j''(\delta) \). Let

\[
K(\delta) = K_1(\delta) \times K_2(\delta) \times \ldots \times K_n(\delta),
\]

and likewise for \( K'(\delta) \) and \( K''(\delta) \). We get \( K(\delta) = K'(\delta) \cap K''(\delta) \), as expected. Let \( L \) be the length of the boundary of \( K_1(\delta) \). Finally, let

\[
U_\nu = K_1(2^{-\nu} \delta) \times K_2\left(\frac{1}{2}\delta\right) \times \ldots \times K_n\left(\frac{1}{2}\delta\right),
\]

and similarly for \( U'_\nu \) and \( U''_\nu \). All in all, we get nested sets

\[
K(\delta) \supset U_1 \supset U_2 \supset \ldots \supset K,
\]

and similarly for the primed and double-primed sets. This nesting is really the important part; the details of the construction are almost irrelevant. The method of proof of Cartan’s lemma will be to prove an additive version with some control on size, then apply it countably many times, once for each \( U_\nu \): each time we apply our machine, we will need to sacrifice some of the boundary.

Lemma 2.7 (Additive Cartan’s lemma with size control). Let \( G \) be a holomorphic matrix-valued function on an open neighborhood of \( \overline{U_\nu} \) such that \( |G(z)| \leq M \) uniformly on \( \overline{U_\nu} \). Then there are holomorphic matrix-valued functions \( G' \) and \( G'' \) on \( U_\nu \) and \( U''_\nu \), respectively, such that

\[
G(z) = G'(z) + G''(z) \quad \text{for all } z \in U_\nu,
\]

\[
|G'(z)| \leq \frac{2^\nu ML}{\pi \delta} \quad \text{for all } z \in \overline{U'_{\nu+1}}, \text{ and}
\]

\[
|G''(z)| \leq \frac{2^\nu ML}{\pi \delta} \quad \text{for all } z \in \overline{U''_{\nu+1}}.
\]

Proof. We will use the Cauchy integral formula to define \( G' \) and \( G'' \) and then prove that they satisfy the given bounds. Pick an \( a \) such that \( a_2 < a < a_3 \) (remember that, in the setup of Cartan’s lemma, we fixed some numbers defining our rectangles).

Define contours \( \gamma, \gamma' \), and \( \gamma'' \) as follows. The contour \( \gamma' \) will be comprised precisely of the part of the boundary of \( K_1(2^{-\nu} \delta) \) to the right of the line demarcated by \( a \), whereas \( \gamma'' \) will be the rest of said boundary. Then let \( \gamma = \gamma' + \gamma'' \) be the whole boundary. Draw a picture to make this clearer, noting that \( \gamma' \) is disjoint from \( K_1'(2^{-\nu} \delta) \) and \( \gamma'' \) is disjoint from \( K_1''(2^{-\nu} \delta) \).

Define

\[
G'(z) = \frac{1}{2\pi i} \int_{t \in \gamma'} \frac{1}{t - z_1} G(t, z_2, \ldots, z_n) \, dt
\]

and

\[
G'(z) = \frac{1}{2\pi i} \int_{t \in \gamma''} \frac{1}{t - z_1} G(t, z_2, \ldots, z_n) \, dt,
\]
which are clearly analytic (by the aforementioned disjointness) in $U'_\nu$ and $U''_\nu$, respectively. By the Cauchy integral formula in one variable, $G(z) = G'(z) = G''(z)$.

The estimates follow easily: if $z \in U'_{\nu+1}$ and $t \in \gamma'$, then $|t - z| \geq 2^{-\nu-1}\delta$ by the definition of $U_\nu$, so just by pulling absolute values inside the integral we get

$$|G'(z)| \leq \frac{1}{2\pi} \frac{2^{\nu+1}}{\delta} M L.$$ 

The same procedure bounds $G''(z)$. 

Finally we will tackle Cartan's lemma directly.

**Proof of Cartan’s lemma.** We will proceed by first assuming that $F$ is very close to $I$, and then remove this constraint. The first part is the more difficult.

Pick $\delta$ so that $F$ is holomorphic and nonsingular in $K(\delta)$, and let $P$ be as given by Lemma 2.6. Let

$$\rho < \min \left(1, \frac{2^{\nu-1}\pi \delta}{L}, \frac{\pi^2 \delta^2}{4L^2P} \right).$$

First suppose that

$$|I - F(z)| < \frac{1}{4}\rho$$

for all $z \in U_1$. We will inductively construct three sequences of functions $G_\nu$, $G'_\nu$, and $G''_\nu$. First let $G_1(z) = F(z) - I$. Then inductively we let $G'_\nu$ and $G''_\nu$ be as given by the additive Cartan’s lemma (Lemma 2.7). The estimates in that lemma, together with (1), yield

$$|G'_\nu(z)| \leq \frac{1}{2} \quad \text{for all } z \in U'_{\nu+1},$$

$$|G''_\nu(z)| \leq \frac{1}{2} \quad \text{for all } z \in U''_{\nu+1}.$$ 

This means automatically that $I + G_\nu(z)$ and $I + G''_\nu(z)$ are nonsingular in an open neighborhood of $U_{\nu+1}$, so there is a (unique) $G_{\nu+1}(z)$ such that

$$(I + G'_\nu(z))(I + G_{\nu+1}(z))(I + G''_\nu(z)) = (I + G_\nu(z))$$

for all $z \in U_{\nu+1}$.

By Lemma 2.6 and the definition of $\rho$, we have

$$|G_{\nu+1}(z)| \leq P \left( \frac{L\rho}{2^{\nu+1}\pi \delta} \right)^2 \leq \frac{\rho}{4^{\nu+1}}.$$ 

Now define

$$F_\nu(z) = I + G_\nu(z),$$

$$F'_\nu(z) = I + G'_\nu(z),$$

$$F''_\nu(z) = I + G''_\nu(z).$$

These are holomorphic and nonsingular in neighborhoods of $K$, $K'$, and $K''$, respectively, where nonsingularity follows by the above bounds. By the definition of $G_{\nu+1}$ used repeatedly, we get

$$(2) \quad F(z) = [F'_1(z)F'_2(z) \ldots F'_\nu(z)]F_{\nu+1}(z)[F''_1(z) \ldots F''_\nu(z)F''_\nu(z)].$$

By Lemma 2.5, the function

$$F'(z) = \lim_{\nu \to \infty} [F'_1(z)F'_2(z) \ldots F'_\nu(z)]$$
is holomorphic and nonsingular, and similarly if we define

\[ F''(z) = \lim_{\nu \to \infty} [F''_1(z) \ldots F''_2(z) F''_\mu(z)]. \]

By the bound on \( G_\nu \) - namely, that \( |G_\nu(z)| \leq \rho/4^\nu \) - it is clear that

\[ \lim_{\nu \to \infty} F_\nu(z) = I \]

for all \( z \in K \). Therefore taking \( \nu \to \infty \) in (2), we get \( F(z) = F'(z) F''(z) \) for all \( z \in K \), as desired.

We now have to fill in the gap: what if \( |I - F(z)| \) is greater than or equal to \( \rho/4^\nu \)? Let \( C \) be such that \( |F(z)^{-1}| \leq C \) for all \( z \in U_1 \), which we can do by compactness. By Theorem 2.4, we can find an entire nonsingular matrix-valued function \( H(z) \) such that \( |F(z) - H(z)| < \rho/4C \) for all \( z \in U_1 \). Therefore

\[ |I - F(z)^{-1} H(z)| < \frac{\rho}{4} \]

in the same set. Apply the first part of this proof to \( F(z) H(z)^{-1} \) to get \( H'(z) \) on \( K' \) and \( H''(z) \) on \( K'' \) such that

\[ F(z)^{-1} H(z) = H''(z) H'(z) \]

for all \( z \in K \) (note the switched indices, which is clearly permissible). Then defining \( F'(z) = H(z) H'(z)^{-1} \) and \( F''(z) = H''(z)^{-1} \) give us what we want. \( \square \)

3. Modification of syzygies theorem

We now deal with the necessary sheaf theory. Define a modification of a chain of syzygies at the \( j \)th place for an analytic sheaf \( S \) given by

\[ \mathcal{O}^{p_m} \xrightarrow{\mu_{m+1}} \mathcal{O}^{p_{m-1}} \rightarrow \ldots \rightarrow \mathcal{O}^{p_1} \xrightarrow{\mu_1} \mathcal{O}^{p_0} \xrightarrow{\mu_0} S \rightarrow 0 \]

to be the direct sum of the original chain and the chain

\[ 0 \rightarrow 0 \rightarrow \ldots \rightarrow \mathcal{O}^q \xrightarrow{i} \mathcal{O}^q \rightarrow \ldots \rightarrow 0 \rightarrow 0, \]

where the two nonzero terms are in the \( j \)th and \((j - 1)\)st places. By a direct sum, we mean take the direct sum of each term and define the maps componentwise, so we get something like

\[ \mathcal{O}^{p_m} \rightarrow \mathcal{O}^{p_{j+1}} \xrightarrow{\mu_{j+1}} \mathcal{O}^{p_{j+q}} \xrightarrow{\mu_j} \mathcal{O}^{p_{j+q-1}} \xrightarrow{\mu_{j-1}} \mathcal{O}^{p_{j-1}} \rightarrow \ldots \rightarrow \mathcal{O}^{p_0} \xrightarrow{\mu_0} S \rightarrow 0, \]

where \( \mu_j \) acts like \( \mu_j \) on the first \( p_j \) components and is the identity on the last \( q \), and \( \mu_{j-1} \) acts like \( \mu_{j-1} \) on the first \( p_{j-1} \) components and is the zero map on the last \( q \). (I suppose technically \( \mu_{j+1} \) is now a different map as well, but what it does should be obvious).

If we have two sheaves \( S \) and \( T \), each with their own chains of syzygies of length \( m \), and a sheaf homomorphism \( \phi : S \rightarrow T \), we can form the following diagram:

\[ \mathcal{O}^{p_m} \xrightarrow{\mu_m} \ldots \xrightarrow{\mu_1} \mathcal{O}^{p_0} \xrightarrow{\mu_0} S \rightarrow 0 \]

\[ \phi \downarrow \]

\[ \mathcal{O}^{q_m} \xrightarrow{\nu_m} \ldots \xrightarrow{\nu_1} \mathcal{O}^{q_0} \xrightarrow{\nu_0} T \rightarrow 0. \]

A homomorphism of chains lying over \( \phi \) is a sequence of sheaf homomorphisms \( \phi_i : \mathcal{O}^{p_i} \rightarrow \mathcal{O}^{q_i} \) that make the above diagram commute. We call it an isomorphism if \( \phi \) and all of the \( \phi_i \) are sheaf isomorphisms.
The following theorem lets us modify chains over isomorphic sheaves to get an isomorphism of chains.

**Theorem 3.1 (Modification of syzygies).** Let \( D \) be a simply connected open polydomain, \( S \) and \( T \) analytic sheaves which admit terminating chains of syzygies over \( D \), and \( \phi : S \to T \) an isomorphism of sheaves. Then after a finite number of modifications there will exist an isomorphism between the chains lying over \( \phi \).

**Proof.** This is almost all algebra (though we do need to use the hypothesis that \( D \) is a polydomain!), and, of course, will proceed by induction. Assume we start with chains of syzygies

\[
\mathcal{O}^{p_m} \xrightarrow{\mu_m} \mathcal{O}^{p_{m-1}} \to \cdots \to \mathcal{O}^{p_1} \xrightarrow{\mu_1} \mathcal{O}^p \xrightarrow{\mu} S \to 0
\]

and

\[
\mathcal{O}^{q_r} \xrightarrow{\nu_r} \mathcal{O}^{q_{r-1}} \to \cdots \to \mathcal{O}^{q_1} \xrightarrow{\nu_1} \mathcal{O}^q \xrightarrow{\nu} T \to 0.
\]

In the base case, \( m = r = 0 \), so we have the diagram

\[
\begin{array}{c}
0 \\
\downarrow \\
0
\end{array} \xrightarrow{\quad} \begin{array}{c}
\mathcal{O}^p \\
\downarrow \\
\mathcal{O}^q
\end{array} \xrightarrow{\quad} S \xrightarrow{\quad} 0
\]

Every nonzero map in this diagram must actually be an isomorphism, so we can induce an isomorphism \( \phi_0 : \mathcal{O}^p \to \mathcal{O}^q \) making the diagram commute.

For the inductive step, assume without loss of generality that \( m \geq r \) and that we’ve proven the theorem for all chains of length \( \leq m - 1 \). We will construct an isomorphism at the 0th step. Let \( E_1, \ldots, E_p \) be the canonical generators of \( \mathcal{O}^p \). Since \( D \) is a polydomain, we know that the functor of “taking the section over \( D \)” is exact, so we get another similar commutative diagram with the sheaves themselves replaced by the sections over \( D \); as usual we will refer to the induced maps by the same name as previously. Using that \( \mu \) and \( \nu \) are surjections and tracing around, we can find \( F_1, \ldots, F_p \in \Gamma(D, \mathcal{O}^q) \) such that \( \nu(F_j) = \phi(\mu(E_i)) \). We can then define a homomorphism \( \sigma : \mathcal{O}^p \to \mathcal{O}^q \) such that \( \sigma(E_i) = F_i \), since then \( E_i \) are free generators. By the definition of this map, \( \nu \sigma = \phi \mu \). Do the same to the inverse isomorphism \( \phi^{-1} \), getting \( \tau : \mathcal{O}^q \to \mathcal{O}^p \) such that \( \mu \tau = \phi^{-1} \nu \). We get the following diagram, where both squares commute:

\[
\begin{array}{c}
\mathcal{O}^{p_1} \xrightarrow{\mu_1} \mathcal{O}^p \xrightarrow{\mu} S \xrightarrow{\phi} 0 \\
\mathcal{O}^{q_1} \xrightarrow{\nu_1} \mathcal{O}^q \xrightarrow{\nu} T \xrightarrow{\tau} 0
\end{array}
\]

We now do the obvious modifications at the first places to get the diagram

\[
\begin{array}{c}
\mathcal{O}^{p_1+q} \xrightarrow{\nu'_1} \mathcal{O}^{p+q} \xrightarrow{\mu'} S \xrightarrow{\phi} 0 \\
\mathcal{O}^{q_1+p} \xrightarrow{\nu'_1} \mathcal{O}^{p+q} \xrightarrow{\nu'} T \xrightarrow{\phi} 0
\end{array}
\]

with the maps given as follows: \( \phi_0 \) takes an element \((F, G) \in \mathcal{O}^p \oplus \mathcal{O}^p\) to the element \((G - \sigma \tau G + \sigma F, F, G)\), while \( \phi'_0 \) takes an element \((K, L) \in \mathcal{O}^p \oplus \mathcal{O}^q\) to the element \((L - \tau \sigma L + \tau K, K - \sigma L)\).
It is a trivial exercise to check that these definitions make the above diagram commute, even to the point where $\phi_0'\phi_0'$ and $\phi_0'\phi_0'$ are identity maps. Therefore they are inverse isomorphisms.

Let $S_0 = \ker \mu'$ and $T_0 = \ker \nu'$. We get the diagram

\[
\begin{array}{cccccc}
0 & \to & \cdots & \to & \mathcal{O} & \to & S_0 & \to & 0 \\
\downarrow & & & & \mu_0' & & \\
0 & \to & \cdots & \to & \mathcal{O} & \to & T_0 & \to & 0.
\end{array}
\]

Since this has length $m - 1$, we can use the inductive hypothesis to get the rest of the isomorphisms. Combining these with (3) gives the result. \qed

4. Putting everything together

We need one last lemma before proving the amalgamation of syzygies result, applying Cartan’s lemma to extend terminating chains of syzygies semi-locally. Let $K, K'$, and $K''$ be sets as before. Neighborhoods are always open by definition.

**Lemma 4.1.** If $S$ is an analytic sheaf over a neighborhood of $K' \cup K''$ which admits a terminating chain of syzygies over a neighborhood $K'$ and another over a neighborhood $K''$, then it has a terminating chain of syzygies over a neighborhood of $K' \cup K''$.

**Proof.** By possibly enlarging things slightly, it suffices to show that we can form a terminating chain of syzygies over $K' \cup K''$ itself.

Let the neighborhoods given by assumption in the lemma be $U' \supset K'$ and $U'' \supset K''$; that is, $S$ has terminating chains of syzygies over both $U'$ and $U''$. Let $U = U' \cap U''$. Thus $S|_U$ has two terminating chains lying over an identity homomorphism $S|_U \to S|_U$. By the modification theorem, we can modify to isomorphic chains. A modification over $U$ can clearly be extended to $U'$ and to $U''$, so we get

\[
\begin{array}{cccccc}
0 & \to & (\mathcal{O}|_{U'})^{p_m} & \xrightarrow{\mu_m} & \cdots & \xrightarrow{\mu_1} & (\mathcal{O}|_{U'})^{p} & \xrightarrow{\mu} & S|_{U'} & \to & 0 \\
\downarrow \lambda_m & & \downarrow \lambda_0 & & \downarrow i & & \downarrow & & \\
0 & \to & (\mathcal{O}|_{U''})^{p_m} & \xrightarrow{\nu_m} & \cdots & \xrightarrow{\nu_1} & (\mathcal{O}|_{U''})^{p} & \xrightarrow{\nu} & S|_{U''} & \to & 0
\end{array}
\]

Here and in the future, dotted lines indicate that the maps in question are only defined over $U$. The $\lambda_i$ are, in fact, isomorphisms when restricted to $U$. Each is therefore represented by a nonsingular, holomorphic, matrix-valued function $F_i(z)$ defined over $U$. By Cartan’s lemma, there exists nonsingular, holomorphic, matrix-valued functions $F'_i(z)$ and $F''_i(z)$, defined over $K'$ and $K''$, such that $F_i(z) = F''_i(z)^{-1}F'_i(z)^{-1}$ for all $z \in K$. These in turn define isomorphisms

\[
\begin{align*}
\lambda'_i : (\mathcal{O}|_{K'})^{p_i} & \to (\mathcal{O}|_{K'})^{p_i} \\
\lambda''_i : (\mathcal{O}|_{K''})^{p_i} & \to (\mathcal{O}|_{K''})^{p_i}
\end{align*}
\]
such that $\lambda_i = (\lambda_i')^{-1}(\lambda_i')^{-1}$. We get the following commutative diagram:

\[
\begin{array}{c}
\begin{array}{c}
0 \rightarrow (O|K)_p \xrightarrow{\mu_1} \cdots \xrightarrow{\mu_m} (O|K')_p \xrightarrow{\mu'} \rightarrow S|K' \xrightarrow{\lambda_0} \lambda_0' \rightarrow 0
\end{array}
\begin{array}{c}
0 \rightarrow (O|K)_p \xrightarrow{\mu_1} \cdots \xrightarrow{\mu_m} (O|K')_p \xrightarrow{\mu'} \rightarrow S|K' \xrightarrow{\lambda_0} \lambda_0' \rightarrow 0
\end{array}
\begin{array}{c}
0 \rightarrow (O|K')_p \xrightarrow{\mu_1} \cdots \xrightarrow{\mu_m} (O|K')_p \xrightarrow{\mu'} \rightarrow S|K' \xrightarrow{\lambda_0} \lambda_0' \rightarrow 0
\end{array}
\begin{array}{c}
0 \rightarrow (O|K')_p \xrightarrow{\mu_1} \cdots \xrightarrow{\mu_m} (O|K')_p \xrightarrow{\mu'} \rightarrow S|K' \xrightarrow{\lambda_0} \lambda_0' \rightarrow 0
\end{array}
\begin{array}{c}
0 \rightarrow (O|K')_p \xrightarrow{\mu_1} \cdots \xrightarrow{\mu_m} (O|K')_p \xrightarrow{\mu'} \rightarrow S|K' \xrightarrow{\lambda_0} \lambda_0' \rightarrow 0
\end{array}
\end{array}
\]

The maps $\mu_i'$ and $\nu_i'$ are uniquely determined so that everything commutes, and as before dotted lines indicate maps defined only over $K$. By construction, when restricted to $K$ the composition $\lambda_i'\lambda_i'\lambda_i'$ is the identity map. Therefore over $K$ the top and bottom resolutions are actually identical. Hence they can be patched together without any problems to define a single chain of syzygies over $K'\cup K''$. □

With this in hand, we turn immediately to the proof of the theorem of amalgamation of syzygies.

Proof of the theorem of amalgamation of syzygies. By the Riemann mapping theorem we may assume that $D$ is a polysquare, and then pick $U$ to be an open polysquare. By the Hilbert syzygy theorem, we can cover $U$ by neighborhoods $\{U_i\}$ on which there exists a terminating chain of syzygies. Take a subdivision of $U$ into small polysquares subordinate to the cover $\{U_i\}$. By restricting to these polysquares and applying Lemma 4.1 repeatedly, we eventually get a terminating chain of syzygies over all of $U$. □