NOTES ON SEVERAL COMPLEX VARIABLES: THEOREMS OF REMMERT-STEIN, CHOW

EVAN WARNER

1. Statements and introduction

The goal is to outline how one proves the following two theorems, the second of which is a quick consequence of the first:

**Theorem 1.1** (Remmert-Stein). Let $D$ be the unit polydisc in $\mathbb{C}^m$ and $V$ a subvariety. Let $W$ be an irreducible subvariety of $D - V$, where $\dim W > \dim V$. Then $\overline{W} \subset D$ is also an irreducible subvariety, and $\dim \overline{W} = \dim W$.

**Theorem 1.2** (Chow). Every analytic subvariety of $\mathbb{P}^n$ is actually an algebraic subvariety.

We will assume the following theorem, due to lack of time:

**Theorem 1.3.** Let $(X, \mathcal{O})$ be an analytic space of pure dimension $d$, $F$ a Fréchet algebra of functions holomorphic on $X$ such that the injection $i : F \to \mathcal{O}$ is continuous. If $X$ is $F$-complete, then there exists a $g \in F^d$ such that $X$ is $g$-complete.

By definition, if a space $X$ is complete with respect to a collection of holomorphic functions $F$, it means that

$$\dim_x \mathcal{L}_x(F) = 0 \quad \forall x \in X.$$

That is, the simultaneous level sets of all functions in $F$ at all points are discrete. Theorem 1.3 therefore states that, given some assumptions on a Fréchet algebra of holomorphic functions, if the simultaneous level sets are discrete then we can find a single function mapping into $\mathbb{C}^d$ with discrete level sets. The proof of Theorem 1.3 uses category arguments and an induction on dimension.

Let’s get a little more intuition for Remmert-Stein before proving it. The basic idea is that it is conceivable that taking the closure of $W$ could cause pathologies on $V$, and the theorem states that as long as $V$ is small (in dimension), this doesn’t happen. Here’s an example showing that if $\dim V = \dim W$, we can get serious problems. The example isn’t quite a counterexample to the Remmert-Stein theorem as stated in this case, because it does not occur in a polydisc, but it illustrates the basic problem (and can be made totally kosher at the expense of some opacity). We will exploit the weirdness of an essentially singularity.

Take $\mathbb{C}^2$, let $V = \{z = 0\}$ and let $W = \{w = e^{1/z}, z \neq 0\}$, which is a subvariety of $\mathbb{C}^2 \setminus V$. I claim that $\overline{W}$ cannot possibly be a subvariety: if it were, then its intersection with a one-dimensional variety it does not contain will be zero-dimensional, hence discrete. But take $X = \{w = 1\}$, for instance. By Picard’s theorem, $(0, 1)$ must be a limit point of $X \cap \overline{W}$, hence this intersection is not discrete, hence not zero-dimensional. So $\overline{W}$ is not a variety.
2. Proof sketches

First, some definitions. A negligible set of an analytic space is a set that is nowhere dense and such that all bounded holomorphic functions defined away from the set can be extended to it. An analytic cover is a triple \((X, \pi, U)\) such that

(i) \(X\) is a locally compact Hausdorff space,
(ii) \(U\) is a domain in \(\mathbb{C}^n\),
(iii) \(\pi\) is a proper mapping \(X\) onto \(U\) with discrete fibers,
(iv) there is a negligible set \(A \subset U\) and a natural number \(\lambda\) such that \(\pi\) is a \(\lambda\)-sheeted covering map \(X \setminus \pi^{-1}(A) \to U \setminus A\), and
(v) \(X \setminus \pi^{-1}(A)\) is dense in \(X\).

We let \(X_0 = X - \pi^{-1}(A)\), so in particular \(\pi\) is a local homeomorphism on \(X_0\).

Given an analytic cover, we can define an analytic structure on \(X\) by pulling back along \(\pi\); that is, given an open set \(W \subseteq X\) and a function \(f\) on \(W\) we call \(f\) holomorphic if for every \(W' \subseteq X_0\) on which \(\pi\) is a homeomorphism we have that \(f|_{W'} \circ \pi^{-1}\) is holomorphic on \(\pi(W') \subseteq U\). The utility of analytic covers is that they provide a different way of thinking about analytic varieties (rather than as an “admissible representation”). It turns out that an irreducible variety is locally an analytic cover, and conversely that an analytic cover is itself an analytic variety (the latter is a theorem of Grauert and Remmert).

For our purposes, we need only the following theorem, whose proof we will not include:

**Theorem 2.1.** Let \(D\) be a domain in \(\mathbb{C}^n\), \(X \subset D\) a subset, and \(g : D \to \mathbb{C}^k\) a holomorphic mapping such that \((X, g, g(D))\) is an analytic cover and \(X_0\) is a complex submanifold of \(\mathbb{C}^n\). Then \(X\) itself is a subvariety of \(D\). \(\square\)

The idea of the proof of Remmert-Stein is to set up all of these conditions with \(X = \overline{W}\). It suffices to consider the situation locally: we want to show that for every \(x \in \overline{W}\), there is a neighborhood \(\Delta\) of \(x\) such that \(\overline{W} \cap \Delta\) is a subvariety of \(\Delta\). If \(x \notin V\), there is nothing to prove, so we can assume \(x \in \overline{W} \cap V\). Let \(W'\) be any variety of the same dimension as \(W\) that contains \(V\) in some neighborhood \(U\) of \(x\) (such a variety is easily found). Abstractly, take the disjoint union restricted to \(U\),

\[ W^0 = (W' \cup W)|_U, \]

and the ordinary union restricted to \(U\),

\[ \mathcal{W} = (W' \cup W)|_U. \]

If we let \(\mathcal{F} = \mathcal{O}_{\mathcal{U}}/\mathcal{I}_{\mathcal{W'}}\), then we can consider \(\mathcal{F}\) as a Fréchet algebra on \(W_0\) with a topology finer than that of \(\mathcal{O}_{\mathcal{W}^0}\). Therefore \(W^0\) is \(\mathcal{F}\)-complete, so by Theorem 1.3 there is an \(F \in \mathcal{O}_{\mathcal{U}}^0\) such that \(F\) has zero-dimensional level sets on \(W^0\). We can assume without loss of generality that \(F(x) = 0\).

We can therefore easily restrict to a small enough disc so that \(F \neq 0\) on the boundary (because \(F\) has zero-dimensional level sets) and \(F\) is proper. The remaining steps are to strip away that “bad parts” of \(F\) to get an analytic cover. At each step, we need to show that the part we are throwing away is a negligible set. The three steps are:

1. Cut out \(F^{-1}(F(V))\) from the domain. This is negligible because \(\dim V < \dim W\). Call the new domain \(W_1\).
(2) Cut out $F^{-1}(F(S(W_1)))$ from the domain, where $S(W_1)$ are the singular points of $W_1$. This is negligible because $F(S(W_1))$ is a subvariety of dimension less that $\dim W$. Call the new domain $W_2$.

(3) Cut out the points of the domain where the rank of $F$ is less than full. Call the new domain $W_3$.

In all, we’ve thrown away a negligible set and we’re left with a proper, nonsingular map from the rest to some dense set of a polydisc. This map is therefore a covering map, so we can apply Theorem 2.1 to conclude that $\overline{W}$ is a variety. □

As a corollary, if $X$ is an analytic space, $Y$ a subvariety of $X$, and $W$ a subvariety of $X - Y$ with finitely many branches each with dimension greater than $\dim Y$, then $\overline{W}$ is a subvariety of $X$. This follows because we can consider each branch separately.

Proving Chow’s theorem is now straightforward. Let $\pi: \mathbb{C}^{n+1} - \{0\} \to \mathbb{P}^n$ be the usual projection to homogeneous coordinates, and consider an analytic subvariety $V$ of $\mathbb{P}^n$. Since $\pi$ has rank $n$ everywhere, $\pi^{-1}(V)$ has dimension $\geq 1$ everywhere (in fact, $\pi^{-1}$ is clearly a cone missing the origin). Since $V$ itself is compact, being a closed subset of a compact space $\mathbb{P}^n$, it has finitely many branches, so $\pi^{-1}(V)$ has finitely many branches as well. By (the corollary to) Remmert-Stein, $V' = \pi^{-1}(V) \cup \{0\}$ is an analytic variety of $\mathbb{C}^n$. Geometrically, $V'$ is a cone (including the origin).

Consider the germ $V'_0$ of $V'$ at $0$, which is the vanishing set of some $g^1, \ldots, g^k$ with $g^i \in \mathcal{O}_0$. Expand each $g^i$ into homogeneous polynomials:

$$g^i = \sum_{n=1}^{\infty} g^i_n z^n.$$ 

Then

$$g^i(tz) = \sum_{n=1}^{\infty} g^i_n(z) t^n$$

for all $z \in \mathbb{C}^{n+1}$, $t \in \mathbb{C}$. If $z \in V'$, then since $V'$ is a cone $tz \in V'$ for all $t \in \mathbb{C}$, so $g^i(tz) = 0$ identically. By the above relation, all coefficients $g^i_n(z)$ vanish. So in fact $V'_0$ is the vanishing set of all the homogeneous parts $g^i_k$, which are polynomials. By Noetherianity, finitely many such suffice, and we’re done. □

Some consequences of Chow’s theorem:

1. Every meromorphic function on $\mathbb{P}^n$ is rational (generalizing the fact for $n = 1$).

2. Every holomorphic map between nonsingular projective varieties is a morphism of algebraic varieties. Here one simply applies Chow’s theorem to the graph of the map.

3. Every compact Riemann surface is an algebraic curve. This is difficult: we need to show that there is an analytic embedding into $\mathbb{P}^n$, which is achieved either via the Riemann-Roch theorem and Serre duality or via $L^2$ methods.