MAT205A, FALL 2019 HOMEWORK

ASSINGMENT 1, DUE OCTOBER 3

Problem 1. (Qual) Define the function ν on all subsets of the real line as follows. If $A \subset \mathbb{R}$ then $\nu(A) = \infty$ if 0 is in the closure of A and $\nu(A) = 0$ otherwise. Prove that ν is finitely additive but it is not countably additive.

Problem 2. (Folland 1.14) If μ is a semifinite measure and $\mu(E) = \infty$ then for any $C > 0$ there exists $F \subset E$ such that $C < \mu(F) < \infty$.

Problem 3. (Folland 1.23) Let $\mathcal{A}$ be the collection of finite unions of sets of the form $(a, b] \cap \mathbb{Q}$, where $-\infty \leq a, b \leq +\infty$.
(a) Show that $\mathcal{A}$ is an algebra on $\mathbb{Q}$.
(b) The σ-algebra generated by $\mathcal{A}$ is $\mathcal{P}(\mathbb{Q})$.
(c) Define μ_0 on $\mathcal{A}$ by $\mu_0(\emptyset) = 0$ and $\mu_0(A) = \infty$ when $A \in \mathcal{A}$ and $A \neq \emptyset$. Then μ_0 is a premeasure and there is more than one measure on $\mathcal{P}(\mathbb{Q})$ whose restriction to $\mathcal{A}$ is μ_0.

Problem 4. (Folland 1.30) If E is a Lebesgue measurable subset of the real line and $m(E) > 0$, (here and below m is the Lebesgue measure on $\mathbb{R}$), then for any $\alpha \in (0, 1)$ there is an open interval I such that $m(E \cap I) > \alpha m(I)$.

Problem 5. (Folland 1.31) If $E \subset \mathbb{R}$ is maeasurable and $m(E) > 0$ then the set

$$E - E = \{x - y : x, y \in E\}$$

contains an interval centered at 0. (You may use the result of Problem 4.)

Problem 6. (Folland 1.33) There exists a Borel set $A \subset [0, 1]$ such that

$$0 < m(A \cap I) < m(I)$$

for any open (non-empty) subinterval of $[0, 1]$.

Problem 7. (Qual) Suppose $A \subset \mathbb{R}$ is a Borel set, and T is a dense subset of $\mathbb{R}$ such that $\tau_t(A) \setminus A$ has Lebesgue measure zero for each $t \in T$, where $\tau_t : \mathbb{R} \to \mathbb{R}$ is the translation $x \to x + t$. Prove that either A or $\mathbb{R} \setminus A$ has Lebesgue measure zero.