MAT205A, FALL 2019 HOMEWORK

ASSIGNMENT 2, DUE OCTOBER 10

Problem 1. (Folland 1.26) Let μ be a Radon measure on $(\mathbb{R}, \mathcal{M})$. Show that if $E \in \mathcal{M}$ and $\mu(E) < \infty$ then for every $\varepsilon > 0$ there is a set A that is a finite union of open intervals such that $\mu(E \Delta A) < \varepsilon$, where $E \Delta A = (E \setminus A) \cup (A \setminus E)$.

Problem 2. (Folland 1.29) Let N be the following set discussed in the beginning of Lecture 1. For $x, y \in [0, 1)$ we say that $x \sim y$ if $x - y \in \mathbb{Q}$. Then $N \subset [0, 1)$ contains exactly one point in each equivalence class.
(a) Let now $F \subset N$ be a Lebesgue measurable set. Show that $m(F) = 0$.
(b) Prove that if E is a Lebesgue measurable set and $m(E) > 0$ then E contains a nonmeasurable set.
(Hint to (b): Assume that $E \subset [0, 1)$, define $N_r = (N \cap [0, r) + 1 - r) \cup (N \cap [r, 1) - r)$ and consider $E = \cup_{r \in \mathbb{Q} \cap [0,1)} E \cap N_r$).

Problem 3. Give an example of a Borel measure on $\mathbb{R} \times \mathbb{R}$ that is not a product measure of two Borel measures on real lines.

Problem 4. (Folland 2.8) Prove that any monotone function $f : \mathbb{R} \to \mathbb{R}$ is Borel measurable.

Problem 5. (Folland 2.6) Show that the supremum of an uncountable family of measurable functions can fail to be measurable.

Problem 6. (Qual, Folland 2.3) Let $\{f_n\}$ be a sequence of measurable functions on $(X, \mathcal{M})$. Show that the set $E = \{x \in X : \lim_{n \to \infty} f_n(x) \text{ exists}\}$ is measurable.

Problem 7. Suppose that $E_n \in \mathcal{M}$, χ_n is the characteristic function of E_n and χ_n converges to f in measure. Show that there exists $E \in \mathcal{M}$ such that $f = \chi_E$ almost everywhere.