MAT205A, FALL 2019 HOMEWORK

ASSIGNMENT 3, DUE OCTOBER 17

Problem 1. (Folland 2.14) Let $(X, \mathcal{M}, \mu)$ be a measure space and f be a non-negative measurable function. For each $E \in \mathcal{M}$ define $\lambda(E) = \int_E f d\mu = \int_X f\chi_E d\mu$. Show that λ is a measure on $\mathcal{M}$ and that $\int_X g d\lambda = \int_X gf d\mu$ for any non-negative measurable function g. (Hint: consider the case when g is simple first.)

Problem 2. (Folland 2.16) Let $f \geq 0$ be a measurable function on $(X, \mathcal{M}, \mu)$ such that $\int_X f d\mu < \infty$. Prove that for any $\varepsilon > 0$ there exists $E \subset X$ such that $\mu(E) < \infty$ and $\int_E f d\mu > \int_X f d\mu - \varepsilon$.

Problem 3. (Qual) Suppose that f is a non-negative Lebesgue measurable function on $[0, 1]$ such that $f > 0$ almost everywhere. Show that for any $\varepsilon > 0$ there is $\delta > 0$ such that $\int_A f(x)dx \geq \delta$ for any Lebesgue measurable subset A of $[0, 1]$ with measure $m(A) \geq \varepsilon$.

Problem 4. (Folland 2.19) Suppose that $\{f_n\}$ is a sequence of measurable integrable functions, $f_n \in L^1(X, \mu)$ for each n, and f_n converges uniformly on X to f.
(a) Show that if $\mu(X) < \infty$ then $f \in L^1(X, \mu)$ and $\int f_n d\mu \to \int f d\mu$.
(b) Give an example when $\mu(X) = \infty$ and the conclusion of (a) fails. (Hint: find an example on $(\mathbb{R}, m)$.)

Problem 5. (Folland 2.25) Let $f(x) = x^{-1/2}$ if $0 < x < 1$ and $f(x) = 0$ otherwise. Let r_n be an enumeration of all rational numbers on $\mathbb{R}$ and define $g(x) = \sum_n 2^{-n} f(x - r_n)$. Show that
(i) $g \in L^1(m)$
(ii) if $g_1 = g$ a.e. then g_1 is unbounded on any interval
(ii) $g^2 < \infty$ a.e. but g^2 is not integrable on any interval.

Problem 6. (Folland 2.27) Let $f_n(x) = ae^{-nax} - be^{-nbx}$ for $n \geq 1$ and $0 < a < b$. Prove that
(a) $\sum_1^\infty \int_0^\infty |f_n(x)|dx = \infty$.
(b) $\sum_1^\infty \int f_n(x)dx = 0$.
(c) $f = \sum_1^\infty f_n \in L^1([0, +\infty), m)$ and $\int_0^\infty f = \log(b/a)$.

Problem 7. (Folland 2.32) Suppose that $\mu(X) < \infty$. For measurable functions f and g, we define

$$\rho(f, g) = \int_X \frac{|f - g|}{1 + |f - g|} d\mu.$$

Prove that ρ is a metric on the space of measurable functions (if we identify functions that are equal μ-a.e.), and that $f_n \to f$ with respect to this metric if and only if $f_n \to f$ in measure.