MAT205A, FALL 2019 HOMEWORK

ASSIGNMENT 4, DUE OCTOBER 24

Problem 1. (Folland 2.48) Let $X = Y = \mathbb{N}$, $\mathcal{M} = \mathcal{N} = \mathcal{P}(\mathbb{N})$, and $\mu = \nu$ be counting measures. Define $f(m, n) = 1$ if $m = n$, $f(m, n) = -1$ if $m = n + 1$, and $f(m, n) = 0$ otherwise. Show that $\int |f| \, d(\mu \times \nu) = +\infty$, and the integrals $\int \int f d\mu d\nu$ and $\int \int f d\nu d\mu$ exist and are not equal.

Problem 2. (Folland 2.56) Let f be a Lebesgue integrable function on $(0, a)$. Define the function $g(x) = \int_{[x,a]} t^{-1} f(t) dm(t)$ for $0 < x < a$. Show that g is Lebesgue integrable on $(0, a)$ and $\int_{(0,a)} g(x) dm(x) = \int_{(0,a)} f(x) dm(x)$.

Problem 3. Suppose that $f : X \to \mathbb{R}$ is a non-negative integrable function on a measure space $(X, \mathcal{M}, \mu)$ with a σ-finite measure μ. Show that

$$\int f d\mu = \int_0^\infty \mu\{f \geq t\} dt.$$

Problem 4. (Folland 3.6) Let $(X, \mathcal{M}, \mu)$ be a measure space with non-negative measure μ and f be a measurable function such that $\int_X f d\mu$ is defined. For each $E \in \mathcal{M}$ define $\nu(E) = \int_E f d\mu$ (show that the integral makes sense). Prove that ν is a signed measure on $\mathcal{M}$ and describe the Hahn decomposition and total variation of ν it terms of f and μ.

Problem 5. (Folland 3.7) Let ν be a signed measure on $(X, \mathcal{M})$ and $\nu = \nu^+ - \nu^-$ be its Jordan decomposition. Prove that for any $E \in \mathcal{M}$
(a) $\nu^+(E) = \sup\{\nu(F) : F \subset E, F \in \mathcal{M}\}$ and $\nu^-(E) = -\inf\{\nu(F) : F \subset E, F \in \mathcal{M}\}$.
(b) $|\nu|(E) = \sup\{\sum_1^n |\nu(E_j)| : E_1, ..., E_n \in \mathcal{M},$ disjoint, and $\cup E_j = E.\}$

Problem 6. (Folland 3.11) Let μ be a positive measure. A collection of functions $\{f_\alpha\}_{\alpha \in A} \subset L^1(\mu)$ is called uniformly integrable if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\left|\int_E f_\alpha d\mu\right| < \varepsilon$ for any $\alpha \in A$ and every $E \in \mathcal{M}$ such that $\mu(E) < \delta$.
(a) Show that any finite set of functions in $L^1(\mu)$ is uniformly measurable.
(b) Suppose that $\{f_n\}_n$ is a sequence that converges to f in $L^1(\mu)$. Show that $\{f_n\}_n$ is uniformly integrable.

Problem 7. (Folland 3.21 part) Let ν be a signed measure on $(X, \mathcal{M})$. Show that, for any $E \in \mathcal{M}$, $|\nu|(E) = \sup\left\{\left|\int_E f d\nu\right| : |f| \leq 1\right\}$.