MAT205A, FALL 2019 HOMEWORK

ASSIGNMENT 5, DUE NOVEMBER 7

Problem 1. (Folland 3.22) Let $f \in L^1(\mathbb{R}^n)$ and $\|f\|_1 > 0$. We define the maximal function Mf by

$$Mf(x) = \sup_{r>0} \frac{1}{m(B_r(x))} \int_{B_r(x)} |f(y)| dm(y).$$

Show that there exist C and R such that $Mf(x) \geq C|x|^{-n}$ for $|x| > R$.

Problem 2. (Folland 3.31) Let $F(x) = x^2 \sin(x^{-1})$ and $G(x) = x^2 \sin(x^{-2})$ for $x \neq 0$ and $F(0) = G(0) = 0$.
(a) Show that F and G are differentiable everywhere, including $x = 0$.
(b) Prove that $F \in BV([-1, 1])$ but $G \notin BV([-1, 1])$.

Problem 3. (Qual) Show that if $f : [0, 1] \to \mathbb{R}$ is absolutely continuous, $A \subset [0, 1]$ is Lebesgue measurable and $m(A) = 0$ then $f(A)$ is Lebesgue measurable and $m(f(A)) = 0$.

Problem 4. (Folland 3.33, Qual) Suppose that F is an increasing function on $[a, b]$, show that $F(b) - F(a) \geq \int_a^b F'(t)dt$.

Problem 5. Let $\{F_j\}$ be a sequence of nonnegative increasing functions on $[a, b]$, and let $F(x) = \sum F_j(x)$. Prove that $F'(x) = \sum_j F_j'(x)$ for a.e. $x \in [a, b]$. (Hint: Assume first that F_j are right continuous and consider the measures μ_{F_j}.)

Problem 6. The Cantor function is defined as follows. Let C be the Cantor set, $x \in C$ if there exist a sequence $\{c_j\}$ with $c_j \in \{0, 2\}$ such that $x = \sum_1^\infty c_j 3^{-j}$. We define $f(x) = \sum_j c_j 2^{-j-1}$, for $y \notin C$ we define $f(y) = \sup\{f(x) : x \in C, x < y\}$. Then f is an increasing function on $[0, 1]$. Show that $f' = 0$ a.e.

Problem 7. (Folland 3.41) Let $A \subset [0, 1]$ be a Borel set such that $0 < m(A \cap I) < m(I)$ for any non-empty open interval $I \subset [0, 1]$. Define $F(x) = m([0, x] \cap A)$ and $G(x) = m([0, x] \cap A) - m([0, x] \setminus A)$.
(a) Show that F is strictly increasing on $[0, 1]$, absolutely continuous and $F' = 0$ on a set of positive measure.
(b) Show that G is absolutely continuous on $[0, 1]$ but G is not monotone on any subinterval of $[0, 1]$.