

MAT205A, FALL 2019 HOMEWORK

ASSIGNMENT 6, DUE NOVEMBER 14

Problem 1. (Folland part of 6.2) Prove that $L^\infty(\mu)$ is a Banach space.

Problem 2. (Folland 6.4) Suppose that $1 \leq p < r \leq \infty$. Prove that $L^p + L^r$ is a Banach space with norm $\|f\| = \inf\{\|g\|_p + \|h\|_r; f = g + h\}$. Show that when $q \in (p, r)$ the inclusion $L^q \rightarrow L^p + L^r$ is bounded.

Problem 3. (Folland 6.7, Qual) Suppose that $F \in L^p(\mu) \cap L^\infty(\mu)$ for some $p \in [1, \infty)$. Show that $f \in L^q$ for any $q > p$ and $\|f\|_\infty = \lim_{q \rightarrow \infty} \|f\|_q$.

Problem 4. (Qual) Suppose (X, \mathcal{M}, μ) is a measure space.

(i) If $\mu(X) < \infty$, are there any inclusions among the spaces $L^1(X, \mu), L^2(X, \mu), L^\infty(X, \mu)$? (List any inclusions you can, and provide a proof for these.)

(ii) If $\mu(X) = \infty$, but μ is σ -finite, can the reverse of these inclusions hold? (Give an example or provide a proof to the contrary.)

Problem 5. (Folland 6.27) Define the operator T on $L^p((0, \infty), m)$, $1 < p < \infty$ by $Tf(x) = \int_0^\infty (x+y)^{-1} f(y) dy$. Prove that T is a bounded operator from $L^p((0, \infty), m)$ to $L^p((0, \infty), m)$ and $\|T\| \leq C_p$, where $C_p = \int_0^\infty x^{-1/p} (x+1)^{-1} dx$.

Problem 6. (Folland 6.32) Suppose that (X, \mathcal{M}, ν) and (Y, \mathcal{N}, ν) are measure spaces with σ -finite measures. Let $K \in L^2(\mu \times \nu)$, define $Tf(x) = \int K(x, y) f(y) d\nu(y)$ for $f \in L^2(\nu)$, show that the integral converges for μ -a.e. $x \in X$ and that $Tf \in L^2(\mu)$ with $\|Tf\|_2 \leq \|K\|_2 \|f\|_2$.

Problem 7. (Folland 6.33) Let $1 < p < \infty$, $1/p + 1/q = 1$. We consider the space $L^q((0, \infty), m)$ and define $Tf(x) = x^{-1/p} \int_0^x f(t) dt$. Prove that T is a bounded linear map from $L^q((0, \infty), m)$ to $C_u(0, \infty)$, where $C_u(0, \infty)$ is the space of uniformly continuous bounded functions on $(0, \infty)$ with the norm $\|h\| = \sup_x |h(x)|$.