MAT205A, FALL 2019 HOMEWORK

ASSIGNMENT 7, DUE NOVEMBER 21

Problem 1. (Folland part of 6.35) Prove that if f, g are measurable functions on $(X, \mathcal{M}, \mu)$ and $1 \leq p < \infty$ then $[cf]_p = |c|[f]_p$ and $[f+g]_p \leq 2([f]_p + [g]_p)$.

Problem 2. (Folland 6.36) (i) Suppose that f is in weak $L^p(\mu)$ for $1 \leq p < \infty$, $\mu(\{f \neq 0\}) < \infty$ and $q < p$. Show that $f \in L^q(\mu)$.
(ii) Suppose that $f \in (\text{weak } L^p(\mu)) \cap L^\infty(\mu)$ for some $p \in [1, \infty)$ and $q > p$. Prove that $f \in L^q(\mu)$.

Problem 3. (Folland 6.39) Suppose that $f \in L^p(\mu)$ and λ_f is the distribution function of f. Prove that

$$\lim_{t \to 0} t^p \lambda_f(t) = \lim_{t \to \infty} t^p \lambda_f(t) = 0.$$

Problem 4. (Folland 6.41) Let $1 < p \leq \infty$ and $1/p + 1/q = 1$. Suppose that T is a bounded operator from $L^p(\mu)$ to $L^p(\mu)$ such that $\int (Tf) g \, d\mu = \int f (Tg) \, d\mu$ for any $f, g \in L^p(\mu) \cap L^q(\mu)$. Show that T extends uniquely to a bounded operator from $L^q(\mu)$ to $L^q(\mu)$ and then also from $L^r(\mu)$ to $L^r(\mu)$ for $r \in (p, q)$ (or $r \in (q, p)$).

Problem 5. (Folland 6.42) Prove the Marcinkiewicz interpolation theorem for the case $p_0 = p_1 = p$ and $q_0 \neq q_1$. (Hint: the assumption implies $\kappa_{Tf}(t) \leq (t^{-1} C_j \|f\|_p)^{q_j}$ for $j = 0, 1$; for each t use the best of those two estimates and bound $\int q t^{q-1} \kappa_{Tf}(t) dt$.)

Problem 6. (Folland 6.43) Let $h = \chi_{[0,1]}$ (then clearly $h \in L^p(m)$ and $\|h\|_p = 1$ for any $p \in [1, \infty]$). Let $H = Mh$ be the Hardy–Littlewood maximal function of h. Compute H explicitly and show that $H \in L^p$ for $p > 1$, H is in weak $L^1(m)$, but $H \notin L^1(m)$. Show also that $\|H\|_p \geq c(p-1)^{-1}$.

Problem 7. (Folland 2.61 & 6.44) Consider functions on $[0, \infty)$. Let $\alpha > 0$, we define

$$J_\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t) dt,$$

where $\Gamma(\alpha)$ is the Gamma function, $\Gamma(n) = (n-1)!$.
(a) Show that if $a = n \in \mathbb{N}$ and $f \in L^1((0, \infty), m)$ then $J_n f$ is an n-th order antiderivative of f.
(b) Prove that if $0 < \alpha < 1$ then J_α is from $L^1((0, \infty))$ to weak $L^q((0, \infty))$, and $[J_\alpha f]_q \leq C \|f\|_1$, where $q = (1-\alpha)^{-1}$.
(c) Suppose that $0 < \alpha < 1$, $p, r > 1$, and $r^{-1} = p^{-1} - \alpha$. Prove that $\|J_\alpha f\|_r \leq C \|f\|_p$.