MAT205A, FALL 2019 HOMEWORK

ASSIGNMENT 8, DUE DECEMBER 5

Problem 1. (Folland, a version of 7.2) Suppose that μ is a Radon measure on $\mathbb{R}^n$. Define O_μ as the union of all open U such that $\mu(U) = 0$.
(i) Show that $\mu(O) = 0$. The set $S = O^c$ is called the support of the measure μ. (Hint: open sets are unions of balls $B(x, r)$, where x has rational coordinated and r is rational.)
(ii) Prove that $x \in S$ if and only if $\int f d\mu > 0$ for any $f \in C_0(\mathbb{R}^n)$, $f \geq 0$, and $f(x) > 0$.

Problem 2. (Folland, a version of 7.4, part) (a) Let $f \in C_0(\mathbb{R}^n)$. Show that the set $E_a = \{|f| \geq a\}$ is a compact set.
(b) Suppose that $K \subset \mathbb{R}^n$ is a compact set. Show that there exists $f : \mathbb{R}^n \to [0, 1]$ such that $f \in C_0(\mathbb{R}^n)$ and $K = \{f = 1\}$.

Problem 3. (Folland 7.22) Let $\{f_n\}$ be a sequence of functions in $C_0(\mathbb{R}^n)$. Show that $\{f_n\}$ converges weakly to f if and only if $\sup_n \|f_n\|_\infty < \infty$ and $f_n \to f$ pointwise. (Remind that $\{f_n\}$ converges weakly to $f \in C_0(\mathbb{R}^n)$ means that $\int f_n d\mu \to \int f d\mu$ for any finite Radon measure μ.)

Problem 4. (Folland 7.24) A sequence of signed measures $\{\mu_n\}$ on $\mathbb{R}$ is said to converge to a signed measure μ vaguely if $\int f d\mu_n \to \int f d\mu$ for any $f \in C_0(\mathbb{R})$.
(a) Give an example of a sequence of measures μ_n such that $\mu_n \to 0$ vaguely, but $\|\mu_n\| = |\mu_n|(\mathbb{R}) \not\to 0$.
(b) Give an example of a sequences of signed measures μ_n such that $\mu_n \to 0$ vaguely but there exists a bounded function g with compact support such that $\int g \, d\mu_n \not\to 0$.

Problem 5. (Folland 7.26) Suppose that μ_n and μ are Radon measures on $\mathbb{R}^m$ and $\mu_n \to \mu$ vaguely and $\|\mu_n\| \to \|\mu\| < \infty$. Show that then for any bounded continuous function f we have $\int f \, d\mu_n \to \int f \, d\mu$.

Problem 6. (Qual) (a) Show, including the explicit constant, that if $\phi \in C_0^\infty(\mathbb{R})$ then $\|\phi\|_\infty \leq \frac{1}{2} \int_{-\infty}^{\infty} |\phi'(t) \, dm(t)$.
(b) Suppose that there is $C > 0$ such that for all functions $\phi \in C_0^\infty(\mathbb{R}^n)$ there is an inequality of the form $\|\phi\|_q \leq C \|\nabla\phi\|_p$. Show that then necessarily $\frac{1}{q} = \frac{1}{p} - \frac{1}{n}$.(Hint: consider the functions ϕ_t defined by $\phi_t(x) = \phi(tx)$.)
(c) When $n = 2$, part b) suggests that one might have an inequality of the form $\|\phi\|_\infty \leq C\|\nabla\phi\|_2$ Show that there is no $C > 0$ such that this inequality holds for all $\phi \in C_0^\infty(\mathbb{R}^n)$.

Problem 7. (Qual) Let $C \subset [0, 1]$ be the middle third Cantor set $C = \cap_n C_n$, where $C_0 = [0, 1]$ and C_n is obtained from $C_{n-1}$by removing the middle third of each component interval. Show that if $f \in C([0, 1])$, then $\lim_{n\to\infty} m(C_n)^{-1} \int_{C_n} f \, dm$ exists.

Show moreover that there is a Borel measure μ on $[0, 1]$ such that

$$\lim_{n\to\infty} m(C_n)^{-1} \int_{C_n} f \, dm = \int_{[0,1]} f \, d\mu.$$