

# MAT205a, Fall 2019 Part I: Measures

## Lecture 1, Following Folland, ch1.1-1.3

### PROBLEM OF MEASURING THE LENGTH AND VOLUME

We want to assign a measure to each subset  $E$  of  $\mathbb{R}^d$  in such a way that

- (i) the measure is countably additive,  $\mu(\cup_j E_j) = \sum \mu(E_j)$  for disjoint sets  $E_j$ ,
- (ii) it is invariant under rigid motions, if  $E$  can be transformed to  $F$  by translation, rotation or reflection, then  $\mu(E) = \mu(F)$ ,
- (iii)  $\mu(Q) = 1$ , where  $Q = [0, 1]^d$  is the unit cube in  $\mathbb{R}^d$ .

We will show that there is no such function in dimension  $d = 1$ . Consider  $I = [0, 1]$  and introduce an equivalence relation for elements of  $I$ ,  $x \sim y$  if  $x - y \in \mathbb{Q}$ . Let  $N \subset I$  be a subset that contain exactly one element from each equivalence class (we use the axiom of choice to define  $N$ ). For each rational number  $r \in \mathbb{Q} \cap [0, 1)$  define

$$N_{1,r} = N \cap [0, r), \quad N_{2,r} = N \cap [r, 1).$$

Such that  $N = N_{1,r} \cup N_{2,r}$  and  $\mu(N) = \mu(N_{1,r}) + \mu(N_{2,r})$ . Next, let

$$N_r = (N_{1,r} + 1 - r) \cup (N_{2,r} - r).$$

Then  $\mu(N_r) = \mu(N)$  and  $\cup_r N_r = I$ . By the countable additivity we should have

$$\sum_r \mu(N_r) = \mu(I) = 1.$$

Which is impossible since all summands on the left hand side are equal.

In higher dimensions  $d \geq 3$  a more involved construction shows that even if the condition (i) is replaced by finite additivity, there is still no function  $\mu$  satisfying the properties (i)-(iii).

### 1. $\sigma$ -ALGEBRAS

**1.1. Definition and examples.** One way to resolve the measurability problem is to define a measure only on a family of subsets of  $\mathbb{R}^d$ .

**Definition 1.1.** Let  $X$  be a non-empty set and  $\mathcal{A}$  be a family of sub sets of  $X$ ,  $\mathcal{A} \subset \mathcal{P}(X)$ .

(i)  $\mathcal{A}$  is called an **algebra** if for any finite number of sets in  $\mathcal{A}$ ,  $E_1, \dots, E_n \in \mathcal{A}$ , their union is in  $\mathcal{A}$ ,  $\cup_1^n E_j \in \mathcal{A}$ , and for any set in  $\mathcal{A}$ ,  $E \in \mathcal{A}$ , its complement is also in  $\mathcal{A}$ ,  $E^c = X \setminus E \in \mathcal{A}$ .

(ii)  $\mathcal{A}$  is called a  **$\sigma$ -algebra** if it is an algebra closed under countable unions,  $E_j \in \mathcal{A}$  implies  $\cup_j E_j \in \mathcal{A}$ .

Clearly, if  $\mathcal{A}$  is a  $\sigma$ -algebra, then  $\emptyset, X \in \mathcal{A}$ .

*Example 1.1.* Trivial  $\sigma$ -algebras are  $\mathcal{A}_0 = \{\emptyset, X\}$ ,  $\mathcal{A}_1 = \mathcal{P}(X)$ .

One useful proposition (you might want to refer to it in your homework assignment).

**Proposition 1.1.** *Suppose that  $\mathcal{E}$  is a collection of subsets of a set  $X$  that satisfies: (i)  $\emptyset \in \mathcal{E}$ , (ii) If  $E, F \in \mathcal{E}$  then  $E \cap F \in \mathcal{E}$ , (iii) if  $E \in \mathcal{E}$  then  $E^c$  is a finite disjoint union of elements of  $\mathcal{E}$ . Then the collection  $\mathcal{A}$  of finite disjoint unions of elements of  $\mathcal{E}$  is an algebra.*

*Proof.* We should check that  $\mathcal{A}$  is closed under union and complement. For the former we need to check that the union of elements (not necessarily disjoint) is an element in  $\mathcal{A}$ . We do it by induction. If  $A, B \in \mathcal{E}$  and  $E^c = \cup_j C_j$ , where  $C_j \in \mathcal{E}$  are disjoint, then

$$A \cup B = (A \cap B^c) \cup B = B \cup \bigcup_j (A \cap C_j),$$

all sets in this union are in  $\mathcal{E}$  and are disjoint. For the induction step, assume that  $A = \cup_j A_j$  is the union of disjoint elements of  $\mathcal{E}$  then

$$A \cup B = \left( \bigcup_j A_j \cap B^c \right) \cup B,$$

as above this is union of disjoint elements of  $\mathcal{E}$ .

Suppose that  $E \in \mathcal{A}$ ,  $E = \cup_j A_j$  each  $A_j \in \mathcal{E}$ . Then  $A_j^c = \cup_{k \neq j} B_{j,k}$  with  $B_{j,k} \in \mathcal{E}$  disjoint. We have

$$E^c = \cap_j A_j^c = \cup_{k_1, \dots, k_J} \cap_j B_{j, k_j}.$$

□

*Example 1.2.* It is easy to see that intersection of any family of  $\sigma$ -algebras is a  $\sigma$ -algebra. Then given a subset  $\mathcal{E}$  of  $\mathcal{P}(X)$  we can consider the smallest  $\sigma$ -algebra that contains  $\mathcal{E}$ . We say that this  $\sigma$ -algebra is generated by  $\mathcal{E}$  and denote it by  $\mathcal{M}(\mathcal{E})$ .

## 1.2. Borel $\sigma$ -algebra.

*Example 1.3.* If  $X$  is a topological space, then the  $\sigma$ -algebra generated by open sets is called the Borel  $\sigma$ -algebra of  $X$  and is denoted by  $\mathcal{B}_X$ . Elements of the Borel  $\sigma$ -algebra are called Borel sets.

**Proposition 1.2.** *The Borel  $\sigma$ -algebra on  $\mathbb{R}$  is generated by each of the following families:*

- (i)  $\mathcal{E}_1 = \{(a, b) : a < b\}$ , (ii)  $\mathcal{E}_2 = \{[a, b] : a < b\}$ , (iii)  $\mathcal{E}_3 = \{(a, b] : a < b\}$ ,
- (iv)  $\mathcal{E}_4 = \{[a, b) : a < b\}$ , (v)  $\mathcal{E}_5 = \{(a, +\infty) : a \in \mathbb{R}\}$ , (vi)  $\mathcal{E}_6 = \{[a, +\infty) : a \in \mathbb{R}\}$ .

**1.3. Product  $\sigma$ -algebra.** We did not have time to discuss this section in the class, we will go back to product  $\sigma$ -algebras later in the course.

Let  $\{X_\alpha\}_{\alpha \in A}$  be a family of sets, let  $X = \prod_{\alpha \in A} X_\alpha$  be the product of this sets and  $\pi_\alpha : X \rightarrow X_\alpha$  be the projection map.

**Definition 1.2.** Suppose that for each  $\alpha$ ,  $\mathcal{M}_\alpha$  is a  $\sigma$ -algebra on  $X_\alpha$ . The **product  $\sigma$ -algebra** on  $X$  is the  $\sigma$ -algebra generated by  $\mathcal{E} = \{\pi_\alpha^{-1}(E_\alpha) : E_\alpha \in \mathcal{M}_\alpha, \alpha \in A\}$ . It is denoted by  $\otimes_{\alpha \in A} \mathcal{M}_\alpha$ .

**Proposition 1.3.** Assume that  $A$  is countable, then  $\otimes_{\alpha \in A} \mathcal{M}_\alpha$  is generated by

$$\mathcal{E}_1 = \left\{ \prod_{\alpha \in A} E_\alpha : E_\alpha \in \mathcal{M}_\alpha \right\}.$$

*Proof.* Let  $E \in \mathcal{E}$  then  $E = \prod_{\beta \in A} E_\beta \in \mathcal{E}_1$ , where  $E_\beta = X_\beta$  when  $\beta \neq \alpha$ . Thus  $\mathcal{M}(\mathcal{E}) \subset \mathcal{M}(\mathcal{E}_1)$ . On the other hand, if  $E \in \mathcal{E}_1$ ,  $E = \prod_{\alpha \in A} E_\alpha$  then  $E = \bigcap_{\alpha \in A} \pi_\alpha^{-1}(E_\alpha)$ . Since  $A$  is countable the last intersection is a countable intersection of the elements in  $\mathcal{M}(\mathcal{E})$  and thus  $E \in \mathcal{M}(\mathcal{E})$ . Then  $\mathcal{M}(\mathcal{E}_1) \subset \mathcal{M}(\mathcal{E})$ , and the  $\sigma$ -algebra coincide.  $\square$

**Proposition 1.4.** Suppose that  $\mathcal{M}_\alpha$  is generated by  $\mathcal{E}_\alpha$  for each  $\alpha$ . Then  $\mathcal{M} = \otimes_{\alpha \in A} \mathcal{M}_\alpha$  is generated by  $\mathcal{F}_1 = \{\pi_\alpha^{-1}(E_\alpha) : E_\alpha \in \mathcal{E}_\alpha\}$ . Moreover, if  $A$  is countable then  $\mathcal{M}$  is generated by  $\mathcal{F}_2 = \{\prod_{\alpha \in A} E_\alpha : E_\alpha \in \mathcal{E}_\alpha\}$ .

*Proof.* Since  $\mathcal{F}_1 \subset \mathcal{M}$ , we have  $\mathcal{M}(\mathcal{F}_1) \subset \mathcal{M}$ . Now consider  $\tilde{\mathcal{M}}_\alpha = \{E \in X_\alpha : \pi_\alpha^{-1}(E) \in \mathcal{M}(\mathcal{F}_1)\}$ . It is a  $\sigma$ -algebra since  $\mathcal{M}(\mathcal{F}_1)$  is a  $\sigma$ -algebra and  $\mathcal{E}_\alpha \subset \tilde{\mathcal{M}}_\alpha$ . Thus  $\tilde{\mathcal{M}}_\alpha \supset \mathcal{M}_\alpha$  and  $\mathcal{M} = \otimes_{\alpha \in A} \mathcal{M}_\alpha \subset \mathcal{M}(\mathcal{F}_1)$ .

For the case when  $A$  is countable,  $\mathcal{F}_1 \subset \mathcal{F}_2$  and each element of  $\mathcal{F}_2$  is a countable intersection of elements in  $\mathcal{F}_1$  and thus  $\mathcal{F}_2 \subset \mathcal{M}(\mathcal{F}_1)$ . Therefore  $\mathcal{M}(\mathcal{F}_1) = \mathcal{M}(\mathcal{F}_2)$ .  $\square$

**Proposition 1.5.** Let  $X_1, \dots, X_n$  be metric spaces and  $X = \prod_1^n X_j$  be the product space equipped with the product metric. Then  $\otimes_1^n \mathcal{B}_{X_j} \subset \mathcal{B}_X$ . If the spaces  $X_j$  are separable, then  $\otimes_1^n \mathcal{B}_{X_j} = \mathcal{B}_X$ .

Recall that a metric space is called separable if it has a countable dense subset. The finite dimensional Euclidean space is separable since points with rational coordinates form a countable dense subset.

*Proof.* By the Proposition 1.4,  $\otimes_1^n \mathcal{B}_{X_j}$  is generated by sets  $\pi^{-1}(U_j)$ , where  $u_j$  is an open subset of  $X_j$ . Clearly  $\pi^{-1}(U_j)$  is open in the product topology and thus  $\otimes_1^n \mathcal{B}_{X_j} \subset \mathcal{B}_X$ .

Now suppose that  $C$  is a countable dense subset of  $Y$  and let  $\mathcal{E}(C)$  be the family of all open balls in  $Y$  with centers in  $C$  and rational radii. Then all open sets in  $Y$  are countable unions of elements in  $\mathcal{E}(C)$ .

If  $C_j$  is a countable dense subset of  $X_j$  for each  $j$  then  $C = \{\{x_j\}_1^n, x_j \in C_j\}$  is a countable dense subset of  $X$  and by Proposition 1.4  $\otimes_1^n \mathcal{B}_{X_j}$  is generated by  $\{\prod_1^n E_j : E_j \in \mathcal{E}(C_j)\}$ . But products of the balls are "cubes" in  $X$ . There are countable union of such cubes and any open set in  $X$  is a countable union of them thus  $\mathcal{B}_X \subset \otimes_1^n \mathcal{B}_{X_j}$ .  $\square$

**Corollary 1.1.**  $\mathcal{B}_{\mathbb{R}^d} = \otimes_1^d \mathcal{B}_{\mathbb{R}}$ .

## 2. MEASURES

### 2.1. Definition and examples.

**Definition 2.1.** Let  $X$  be a set and  $\mathcal{M}$  be a  $\sigma$ -algebra of its subsets. A **measure** on  $(X, \mathcal{M})$  is a function  $\mu : \mathcal{M} \rightarrow [0, +\infty]$  such that  $\mu(\emptyset) = 0$  and  $\mu(\cup_j E_j) = \sum_j \mu(E_j)$  for any sequence of disjoint sets  $\{E_j\} \subset \mathcal{M}$ .

If  $X$  is a set and  $\mathcal{M} \subset \mathcal{P}(X)$  is a  $\sigma$ -algebra, then  $(X, \mathcal{M})$  is called a measurable space. If  $\mu$  is a measure on  $(X, \mathcal{M})$  then the triple  $(X, \mathcal{M}, \mu)$  is called a measure space.

*Example 2.1.* Two simple examples of measures are:

- (1) the counting measure  $\mu(E)$  equals the number of elements in  $E$  if  $E$  is finite and  $\mu(E) = \infty$  otherwise,
- (2) the Dirac measure, if  $x_0 \in X$  is fixed then  $\mu(E) = 0$  when  $x_0 \notin E$  and  $\mu(E) = 1$  when  $x_0 \in E$  is a measure.

We do not have many interesting examples of measures so far. One of the aims of the next lecture is to expand our list of examples.

A measure  $\mu$  is called **finite** if  $\mu(X) < \infty$ , it is called  **$\sigma$ -finite** if  $X = \cup_j E_j$  is the countable union of sets,  $E_j \in \mathcal{M}$  and  $\mu(E_j) < \infty$  for each  $j$ . A measure is called **semifinite** if for any  $E \in \mathcal{M}$  such that  $\mu(E) = +\infty$  there exists  $F \subset E$  such that  $0 < \mu(F) < \infty$ .

### 2.2. Basic properties.

**Theorem 2.1.** Let  $(X, \mathcal{M}, \mu)$  be a measure space. Then

- (i) If  $E, F \in \mathcal{M}$  and  $E \subset F$  then  $\mu(E) \leq \mu(F)$ ,
- (ii) If  $\{E_j\}_j \subset \mathcal{M}$  then  $\mu(\cup_j E_j) \leq \sum_1^\infty \mu(E_j)$ ,
- (iii) If  $\{E_j\}_j \subset \mathcal{M}$  and  $E_1 \subset E_2 \subset \dots$  then  $\mu(\cup_1^\infty E_j) = \lim_{j \rightarrow \infty} \mu(E_j)$ ,
- (iv) If  $\{E_j\}_j \subset \mathcal{M}$  and  $E_1 \supset E_2 \supset \dots$  and  $\mu(E_1) < \infty$  then  $\mu(\cap_1^\infty E_j) = \lim_{j \rightarrow \infty} \mu(E_j)$ .

*Proof.* (i) Follows from the additivity of the measure,  $F \setminus E \in \mathcal{M}$  and

$$\mu(F) = \mu(E) + \mu(F \setminus E) \geq \mu(E).$$

(ii) Let  $F_1 = E_1$  and  $F_k = E_k \setminus (\cap_1^{k-1} E_j)$  then  $\{F_k\}$  are disjoint and  $\cup_1^\infty F_j = \cup_1^\infty E_j$ . For  $F_j$  we can apply the additivity property of  $\mu$  and obtain

$$\mu(\cup_j E_j) = \mu(\cup_j F_j) = \sum_j \mu(F_j) \leq \sum_j \mu(E_j).$$

(iii) Let  $E_0 = \emptyset$ , then  $\cup_j E_j = \sup_j (E_j \setminus E_{j-1})$  and the difference sets are disjoint. Then

$$\mu(\cup_j E_j) = \sum_j \mu(E_j \setminus E_{j-1}) = \lim_{k \rightarrow \infty} \sum_1^k \mu(E_j \setminus E_{j-1}) = \lim_{k \rightarrow \infty} \mu(E_k).$$

(iv) Let  $F_j = E_1 \setminus E_j$ , then  $F_1 \subset F_2 \subset \dots$ . Thus by (iii)

$$\mu(E_1 \setminus \cap_j E_j) = \mu(\cup_j F_j) = \lim_{n \rightarrow \infty} \mu(F_n) = \lim_{n \rightarrow \infty} (\mu(E_1) - \mu(E_n)),$$

here we used the fact that  $\mu(E_1) < \infty$ . Now subtracting both sides from  $\mu(E_1)$  we obtain the desired identity.  $\square$

**2.3. Sets of measure zero and complete measures.** If  $(X, \mathcal{M}, \mu)$  is a measure space and  $E \in \mathcal{M}$  is such that  $\mu(E) = 0$  then we say that  $E$  is a set of zero measure, or a **null** set. If some statement holds for all  $x \in X$  except for a set of zero measure, we say that it holds **almost everywhere**.

If  $\mu(E) = 0$  and  $F \subset E$  is also an element of  $\mathcal{M}$  then clearly  $\mu(F) = 0$ .

**Definition 2.2.** We say that a measure  $\mu$  on a measure space  $(X, \mathcal{M}, \mu)$  is **complete** if for any  $E \in \mathcal{M}$  such that  $\mu(E) = 0$  and any  $F \subset E$ ,  $F \in \mathcal{M}$ .

**Theorem 2.2.** Suppose that  $(X, \mathcal{M}, \mu)$  is a measure space,  $\mathcal{N} = \{N \in \mathcal{M} : \mu(N) = 0\}$ . Define  $\overline{\mathcal{M}} = \{E \cup F : E \in \mathcal{M}, F \subset N, N \in \mathcal{N}\}$ . Then  $\overline{\mathcal{M}}$  is a  $\sigma$ -algebra and there is a unique extension  $\bar{\mu}$  of the measure  $\mu$  to  $\overline{\mathcal{M}}$ ,  $\bar{\mu}$  is a complete measure on  $\overline{\mathcal{M}}$ .

*Proof.* First, since  $\mathcal{M}$  and  $\mathcal{N}$  are closed under countable unions so is  $\overline{\mathcal{M}}$ . Now suppose that  $S = E \cup F \in \overline{\mathcal{M}}$ ,  $F \subset N$ . We may assume that  $E$  and  $N$  are disjoint by replacing  $F$  by  $F \setminus E$  and  $N$  by  $N \setminus E$  if necessary. Then  $S^c = (E \cup F)^c = (E \cup N)^c \cup (N \setminus F)$ , clearly  $(E \cup N)^c \in \mathcal{M}$  and  $N \setminus F \subset N$ . Thus  $S^c \in \overline{\mathcal{M}}$  and  $\overline{\mathcal{M}}$  is a  $\sigma$ -algebra.

For  $S = E \cup F$  define  $\bar{\mu}(S) = \mu(E)$ . Clearly it is well defined and any extension of  $\mu$  to  $\overline{\mathcal{M}}$  should satisfy that since  $\mu(E) = \mu(E \cup N)$ . To show that  $\bar{\mu}$  is complete we assume that  $\bar{\mu}(S) = 0$ , then  $S = E \cup F \subset E \cup N$  and  $\mu(E \cup N) = 0$  thus  $S \subset S' \in \mathcal{N}$ . Then any subset of  $S$  is also an element of  $\overline{\mathcal{M}}$ .  $\square$

## MAT205a, Fall 2019 Part I: Measures

### Lecture 2, Following Folland, ch1.4-1.5

#### 3. OUTER MEASURE

##### 3.1. Definition and main example.

**Definition 3.1.** A function  $\mu^* : \mathcal{P}(X) \rightarrow [0, +\infty]$  is called an **outer measure** if  $\mu^*(\emptyset) = 0$ ,  $\mu^*(A) \leq \mu^*(B)$  when  $A \subset B$  and  $\mu^*(\cup_j A_j) \leq \sum_j \mu^*(A_j)$ .

A general way to construct an outer measure is given by the following proposition.

**Proposition 3.1.** Let  $\mathcal{E} \subset \mathcal{P}(X)$  be a family of sets such that  $\emptyset, X \in \mathcal{E}$ . Let also  $\rho : \mathcal{E} \rightarrow [0, +\infty]$  be any function such that  $\rho(\emptyset) = 0$ . For any  $A \subset X$  define

$$(1) \quad \mu^*(A) = \inf \left\{ \sum_j \rho(E_j) : E_j \in \mathcal{E}, A \subset \bigcup_j E_j \right\}.$$

Then  $\mu^*$  is an outer measure on  $X$ .

*Proof.* Since  $\emptyset \in \mathcal{E}$  and  $\rho(\emptyset) = 0$  it is clear that  $\mu^*(\emptyset) = 0$ . If  $A \subset B$  then the infimum in the definition of  $\mu^*(A)$  is taken over a larger family of sequences  $\{E_j\}$  than is the definition of  $\mu^*(B)$ . Thus  $\mu^*(A) \leq \mu^*(B)$ .

Finally, let  $A = \cup_j A_j$  and  $\varepsilon > 0$ . Then for each  $A_j$  there exists a covering of  $A_j$ ,  $A_j \subset \cup_k E_{j,k}$  such that  $\sum_k \rho(E_{j,k}) \leq \mu^*(A_j) + \varepsilon/2^j$ . Then  $A \subset \cup_j \cup_k E_{j,k}$  and  $\mu^*(A) \leq \sum_j \sum_k \rho(E_{j,k}) \leq \sum_j \mu^*(A_j) + \varepsilon$ . The last inequality holds for each positive  $\varepsilon$  and therefore  $\mu^*(A) \leq \sum_j \mu^*(A_j)$ .  $\square$

For example on the real line  $\mathbb{R}$  we can take  $\mathcal{E}$  to be family of all (open) intervals and choose  $\rho$  to be the length. Then the proposition above describes how to define an outer measure on  $\mathbb{R}$ . The next step is to construct a measure from an outer measure.

##### 3.2. Measurable sets.

**Definition 3.2.** Suppose that  $\mu^*$  is an outer measure on  $X$ . A set  $A \subset X$  is called  **$\mu^*$ -measurable** if for any  $E \subset X$ ,  $\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c)$ .

Note that one inequality always holds, by the definition of an outer measure,

$$\mu^*(E) \leq \mu^*(E \cap A) + \mu^*(E \cap A^c).$$

**Lemma 3.1.** If  $\mu^*$  is an outer measure on  $X$  then the collection  $\mathcal{M}$  of all  $\mu^*$ -measurable subsets of  $X$  is a  $\sigma$ -algebra.

*Proof.* Clearly  $\emptyset \in \mathcal{M}$  and if  $A \in \mathcal{M}$  then  $A^c \in \mathcal{M}$ . We want to prove that  $\mathcal{M}$  is closed under countable unions. First we show that finite unions of sets in  $\mathcal{M}$  are also in  $\mathcal{M}$ . Let  $C = A \cup B$ , then  $C = (A \cap B) \cup (A \cap B^c) \cup (A^c \cap B)$  and  $C^c = A^c \cap B^c$ . The measurability of  $A$  and  $B$  implies

$$\mu^*(E) = \mu^*(E \cap A \cap B) + \mu^*(E \cap A \cap B^c) + \mu^*(E \cap A^c \cap B) + \mu^*(E \cap A^c \cap B^c).$$

By the subadditivity of  $\mu^*$  the sum of the first three terms is  $\geq \mu^*(E \cap C)$ , thus

$$\mu^*(E) \geq \mu^*(E \cap C) + \mu^*(E \cap C^c).$$

Since the opposite inequality always holds, we conclude that  $C \in \mathcal{M}$ . Therefore finite unions and intersections of elements in  $\mathcal{M}$  are also in  $\mathcal{M}$ .

Now suppose that  $C = \cup_j A_j = \cup_j A_j \cap \left(\bigcup_1^{j-1} A_k\right)^c = \cup_j B_j$ , where  $B_j$  are disjoint and  $B_j \in \mathcal{M}$ . Let  $C_n = \cup_1^n B_j$ , then

$$\mu^*(E \cap C_n) = \mu^*(E \cap B_n) + \mu^*(E \cap C_{n-1}) = \dots = \sum_1^n \mu^*(E \cap B_k).$$

Furthermore, since  $C_n$  is  $\mu^*$ -measurable,

$$\mu^*(E) = \mu^*(E \cap C_n) + \mu^*(E \cap C_n^c) \geq \sum_1^n \mu^*(E \cap B_k) + \mu^*(E \cap C_n^c).$$

Now, taking the limit as  $n \rightarrow \infty$  and using the fact that  $\mu^*(E \cap C) \leq \sum_k \mu^*(E \cap B_k)$ , we conclude

$$\mu^*(E) \geq \mu^*(E \cap C) + \mu^*(E \cap C^c)$$

and thus  $C$  is  $\mu^*$ -measurable. □

### 3.3. Carathéodory construction.

**Theorem 3.1** (Carathéodory). *Let  $\mu^*$  be an outer measure on  $X$  and  $\mathcal{M}$  be the collection of  $\mu^*$  measurable subsets in  $X$ . Then  $(X, \mathcal{M}, \mu^*)$  is a measure space and the measure  $\mu^*$  is complete on  $\mathcal{M}$ .*

*Proof.* We want to show that  $\mu^*$  is a measure on  $\mathcal{M}$ . Clearly,  $\mu^*(\emptyset) = 0$ . If  $A, b \in \mathcal{M}$  and are disjoint then by definition of the  $\mu^*$ -measurable sets,

$$\mu^*(A \cup B) = \mu^*((A \cup B) \cap A) + \mu^*((A \cup B) \cap A^c) = \mu^*(A) + \mu^*(B).$$

For countable union of disjoint  $\mu^*$ -measurable sets,  $A = \cup_j A_j$ ,

$$\mu(A) = \mu(\cup_j A_j) \geq \mu(\cup_1^n A_j) = \sum_1^n \mu^*(A_j)$$

for any  $n$ , therefore

$$\mu^*(A) \geq \sum_j \mu(A_j),$$

the opposite inequality holds since  $\mu^*$  is an outer measure. Thus  $\mu^*$  is countably additive on  $\mathcal{M}$ .

To prove that  $\mu^*$  is complete, notice that if  $\mu^*(N) = 0$  and  $N$  is measurable, then any  $F \subset N$  is measurable, since

$$\mu^*(E) = \mu^*(E \cap N) + \mu^*(E \cap N^c) = \mu^*(E \cap N^c) \leq \mu^*(E \cap F^c)$$

and the opposite inequality holds, since the outer measure is monotone. Thus  $\mu^*(E) = \mu^*(E \cap F^c) = \mu^*(E \cap F) + \mu^*(E \cap F^c)$ .  $\square$

The Carathéodory theorem allows us to extend our list of measures. We know that any function  $\rho$  on a subset of  $\mathcal{P}(X)$  defines an outer measure and outer measure is a measure on the set of its measurable sets. We want to specify this construction further.

**Definition 3.3.** Let  $\mathcal{A} \in \mathcal{P}(X)$  be an algebra of sets. A function  $\mu_0$  is called a **premeasure** if  $\mu_0(\emptyset) = 0$  and  $\mu_0(\cup_j A_j) = \sum_j \mu_0(A_j)$  for disjoint sets  $A_j \in \mathcal{A}$  such that  $\cup_j A_j \in \mathcal{A}$ .

It follows from the definition that if  $A, B \in \mathcal{A}$  and  $A \subset B$  then  $B \cap A^c \in \mathcal{A}$  and  $\mu_0(B) = \mu_0(A) + \mu_0(B \cap A^c) \geq \mu_0(A)$ .

**Proposition 3.2.** Let  $\mu_0$  be a premeasure on  $\mathcal{A}$  and define  $\mu^*$  by (1) with  $\rho = \mu_0$  and  $\mathcal{E} = \mathcal{A}$ . Then for any  $A \in \mathcal{A}$ ,  $A$  is  $\mu^*$ -measurable and  $\mu^*(A) = \mu_0(A)$ .

*Proof.* Let  $A \in \mathcal{A}$  and  $E \subset X$ , we want to estimate  $\mu^*(E)$ . Let  $B_j \in \mathcal{A}$  be such that  $E \subset \cup_j B_j = B$  and  $\mu^*(E) \geq \sum_j \mu_0(B_j) - \varepsilon$ . Then

$$\mu^*(E) + \varepsilon \geq \sum_j \mu_0(B_j) = \sum_j \mu_0(B_j \cap A) + \sum_j \mu_0(B_j \cap A^c) = \mu_0(B \cap A) + \mu_0(B \cap A^c),$$

Thus  $\mu^*(E) \geq \mu^*(E \cap A) + \mu^*(E \cap A^c)$  and  $A$  is  $\mu^*$ -measurable.

To compute  $\mu^*(A)$  first note that  $\mu^*(A) \leq \mu_0(A)$ . Assume now that  $A \subset \cup_j A_j$  and  $A_j \in \mathcal{A}$ . Since  $\mathcal{A}$  is an algebra, we may assume that  $A_j$  are disjoint (replacing them if necessary by the sets  $\tilde{A}_j = A_j \setminus \cup_1^{j-1} A_k \in \mathcal{A}$ ). Then  $A = \cup_j (A \cap A_j)$  and  $A \cap A_j \in \mathcal{A}$ . Thus by the definition of premeasure

$$\sum_j \mu_0(A_j) \geq \sum_j \mu_0(A \cap A_j) = \mu_0(A).$$

It implies that  $\mu^*(A) \geq \mu_0(A)$ .  $\square$

We summarize the Carathéodory construction in the next theorem.

**Theorem 3.2.** *Let  $\mathcal{A}$  be an algebra of subsets of  $X$  and  $\mu_0$  be a premeasure on  $\mathcal{A}$ . Let further  $\mathcal{M} = \mathcal{M}(\mathcal{A})$  be the  $\sigma$ -algebra generated by  $\mathcal{A}$ . Then there exists a measure  $\mu$  on  $\mathcal{M}$  such that  $\mu(A) = \mu_0(A)$  for any  $A \in \mathcal{A}$ ,  $\mu$  is the restriction of the outer measure  $\mu^*$  onto  $\mathcal{M}$ , where the outer measure is defined by (1) with  $\rho = \mu_0$  and  $\mathcal{E} = \mathcal{A}$ .*

*If  $\nu$  is another measure on  $\mathcal{M}$  that coincides with  $\mu_0$  on  $\mathcal{A}$  then  $\nu(E) \leq \mu(E)$  for any  $E \in \mathcal{M}$ , with  $\nu(E) = \mu(E)$  when  $\mu(E) < +\infty$ .*

The first statement follows from the propositions above. The second statement is left for the reader as an exercise.

#### 4. MEASURES ON THE REAL LINE

**4.1. Borel measures on  $\mathbb{R}$ .** Our aim is to construct the Lebesgue measure on the real line, but we start first with a more general construction. Let  $\mathcal{B}$  be the Borel  $\sigma$ -algebra on  $\mathbb{R}$ . Suppose the  $\mu$  is a finite Borel measure, we associate the following function  $F$  with  $\mu$ ,

$$F(x) = \mu((-\infty, x]),$$

$f$  is called the **distribution function** of  $\mu$ . Then  $F$  is increasing and  $F(x) = \lim_{x_n \rightarrow x+} F(x_n)$ ,  $F$  is right continuous, by Theorem 2.1, since  $(-\infty, x] = \cup_n (-\infty, x_n]$ .

We will use the construction of the previous section to start with a function  $F$  and define a measure  $\mu$  with this distribution function. First there is a standard algebra that generates the Borel  $\sigma$ -algebra. Let  $\mathcal{A}$  be the collection of finite disjoint unions of half-open intervals  $(a, b]$ , where  $-\infty \leq a < b \leq \infty$ , when  $b = \infty$  we take intervals  $(a, +\infty)$ . By Proposition 1.1  $\mathcal{A}$  is an algebra and by Proposition 1.2 it generates the Borel  $\sigma$ -algebra.

**Proposition 4.1.** *Let  $F : \mathbb{R} \rightarrow \mathbb{R}$  be increasing and right continuous. If  $\{(a_j, b_j]\}_j$  are disjoint intervals, define*

$$\mu_0(\cup_j (a_j, b_j]) = \sum_j F(b_j) - F(a_j),$$

*then  $\mu_0$  is a premeasure on  $\mathcal{A}$ .*

*Proof.* First, we check that  $\mu_0$  is well defined. If  $\cup_1^J I_j = \cup_1^J (a_j, b_j] = \cup_1^K (c_k, d_k] = \cup_1^K J_k$  then

$$\sum_1^J F(b_j) - F(a_j) = \sum_1^K F(d_k) - F(c_k).$$

To see that consider the third family of intervals,  $L_{j,k} = I_j \cap J_k$ , clearly  $L_{j,k} = (\alpha_{j,k}, \beta_{j,k}]$  or  $L_{j,k} = \emptyset$ . Note that each interval  $I_j = (a_j, b_j]$  is a finite disjoint union of

intervals  $L_{j,k}$ . Then we can enumerate those intervals such that  $\cup L_{j,k} = \cup_1^M (e_m, f_m]$  with  $a_j = e_1 < f_1 = e_2 < \dots < f_M = b_j$  and

$$F(b_j) - F(a_j) = \sum_1^M F(f_m) - F(e_m).$$

Summing over all  $j$  we obtain

$$\sum_j F(b_j) - F(a_j) = \sum_{j,k} F(\beta_{j,k}) - F(\alpha_{j,k}) = \sum_k F(d_k) - F(c_k).$$

Now suppose that  $A \in \mathcal{A}$  and  $A = \cup_j (a_j, b_j]$  is a countable union of disjoint intervals. We want to show that  $\mu_0(A) = \sum_j \mu_0((a_j, b_j])$ . Since  $A$  is a finite disjoint union of intervals, it suffices to consider the case when  $A = (a, b]$ . For a finite subset of the intervals we have (since  $F$  is increasing)  $\sum_1^J \mu_0((a_j, b_j]) \leq \mu_0(A)$ . Thus  $\sum_j \mu_0((a_j, b_j]) \leq \mu_0(A)$ . To prove the opposite inequality choose  $\varepsilon > 0$  and  $\delta, \delta_j > 0$  such that  $F(a + \delta) - F(a) < \varepsilon$  and  $F(b_j + \delta_j) - F(b_j) < \varepsilon/2^j$ , this can be done since  $F$  is right continuous. Then  $[a + \delta, b]$  is covered by open intervals  $(a_j, b_j + \delta_j)$ . There is a finite subcover since  $[a + \delta, b]$  is compact. Thus by monotonicity of  $F$

$$F(b) - F(a + \delta) \leq \sum_j F(b_j + \delta_j) - F(a_j), \quad \text{and} \quad F(b) - F(a) \leq \sum_j F(b_j) - F(a_j) + 2\varepsilon.$$

Therefore  $\mu_0(A) \leq \sum_j \mu_0((a_j, b_j])$  and  $\mu_0$  is a premeasure.  $\square$

The proposition above combined with the results in the previous section imply that for any increasing right continuous function  $F$  there is a complete measure  $\mu$  on a  $\sigma$ -algebra  $\mathcal{M}$  which contains the Borel  $\sigma$ -algebra  $\mathcal{B}_{\mathbb{R}}$ . This measure is called the **Lebesgue-Stieltjes measure** associated to  $F$ .

**4.2. Lebesgue measure.** The most important measure on  $\mathbb{R}$  is the **Lebesgue measure**. This is the complete measure associated to the function  $F(x) = x$ , we denote it by  $m$ . The  $\sigma$ -algebra on which the Lebesgue measure is complete is called the Lebesgue  $\sigma$ -algebra,  $\mathcal{L}$ , and its elements are Lebesgue measurable sets. Since  $F(x) = x$  is continuous,  $m(\{x\}) = 0$  for any  $x \in \mathbb{R}$ . then by countable additivity,  $m(E) = 0$  for any countable set  $E$ .

**Theorem 4.1.** *If  $E \in \mathcal{L}$  then  $E + s \in \mathcal{L}$  and  $rE \in \mathcal{L}$  for all  $s, r \in \mathbb{R}$ . Moreover,  $m(E + s) = m(E)$  and  $m(rE) = |r|m(E)$ .*

The theorem follows from the facts that the for  $F(x) = x$  we have

$$F(x + s) - F(y + s) = F(x) - F(y) \quad \text{and} \quad F(rx) = rF(x).$$

The example in the beginning of Lecture 1 shows that there exists a set which is not Lebesgue measurable.

**4.3. Cantor set.** The classical example of an uncountable set of zero measure is the so-called Cantor set. Let  $C_0 = [0, 1]$ , given  $C_{n-1} = \cup_1^J [a_j, b_j]$  we define  $C_n$  as the union of  $2J$  closed intervals,  $C_n = \cup_1^J [a_j, (2a_j + b_j)/3] \cup [(a_j + 2b_j)/3, b_j]$  which is obtained from  $C_{n-1}$  by deleting the middle open third of each interval. Then  $C = \cap_n C_n$  is a set of cardinality continuum and  $m(C) = 0$ .

It is easy to check that  $x \in C$  if and only if can be written as  $x = \sum_j a_j 3^{-j}$  where  $a_j \in \{0, 2\}$ . We define a function  $f$  on  $C$  by  $f(x) = f(\sum a_j 3^{-j}) = \sum_j a_j 2^{-j-1}$ ,  $f$  is an increasing function on  $C$  and  $f(C) = [0, 1]$ . There is a unique way to extend  $f$  to the interval  $[0, 1]$  such that it is still non-decreasing and then  $f$  is continuous,  $f$  is called the Cantor (staircase) function.

The construction of the Cantor set can be modified for example by deleting distinct middle part of the intervals on each step. These generalized Cantor sets provide a number of useful examples.

## MAT205a, Fall 2019 Part I: Measures

*Lecture 3, Following Folland, parts of ch 1.5, 2.5-2.6*

### 5. BOREL MEASURES ON $\mathbb{R}$

**5.1. Borel and Lebesgue-Stieltjes measures.** We say that  $\mu$  on  $\mathbb{R}^n$  is a **Borel measure** if  $\mu$  is defined on  $(\mathbb{R}^n, \mathcal{M})$  and  $\mathcal{B}_{\mathbb{R}^n} \subset \mathcal{M}$ , i.e., all Borel sets are measurable. On the previous lecture we showed that any non-decreasing right continuous function defines a premeasure on the algebra  $\mathcal{A}$  of half-open intervals on  $\mathbb{R}$ , this premeasure is then extended to a measure on the  $\sigma$ -algebra  $\mathcal{M}_F \supset \mathcal{B}_{\mathbb{R}}$ . We want to show that this extension is unique.

**Lemma 5.1.** *Suppose that  $\mu_0$  is a premeasure on an algebra  $\mathcal{A} \subset \mathcal{P}(X)$  and  $\mu_0$  is  $\sigma$ -finite. There is a unique extension of  $\mu_0$  onto the  $\sigma$ -algebra generated by  $\mathcal{A}$ .*

*Proof.* We learned on the last lecture how to construct one extension  $\mu$  by first defining the outer measure  $\mu^*$ . Suppose that  $\nu$  is also an extension of  $\mu_0$  to  $\mathcal{M}$ . Suppose that  $E \in \mathcal{M}$  and  $E \subset \cup_j A_j$  for some  $A_j \in \mathcal{A}$  then

$$\nu(E) \leq \nu(\cup_j A_j) \leq \sum_j \nu(A_j) = \sum_j \mu_0(A_j).$$

Then by the definition of  $\mu^*$ ,  $\nu(E) \leq \mu(E)$ . Furthermore, if  $\mu(E)$  is finite then for any  $\varepsilon > 0$  there exists  $A = \cup_j A_j$  such that  $E \subset A$  and  $\mu(E) < \mu(A) + \varepsilon$ . on the other hand  $\mu(A) = \lim_{n \rightarrow \infty} \mu_0(\cup_1^n A_j) = \nu(A)$ . Then we obtain

$$\mu(E) \leq \mu(A) = \nu(A) = \nu(E) + \nu(A \setminus E) \leq \nu(E) + \mu(A \setminus E) \leq \nu(E) + \varepsilon.$$

□

**Proposition 5.1.** *Suppose that  $\mu$  is a Borel measure on  $\mathbb{R}$  such that  $\mu(K) < \infty$  for any compact set  $K$ . Then we define the function*

$$F(x) = \begin{cases} -\mu((x, 0]), & x < 0, \\ 0, & x = 0 \\ \mu((0, x]), & x > 0 \end{cases}.$$

*Then  $F$  is a non-decreasing right-continuous function and  $\mu$  is the Lebesgue-Stieltjes measure associated to  $F$ ,  $\mu = \mu_F$ .*

*Proof.* It is clear that  $F(x) \geq F(y)$  when  $x \geq y$  (we may consider separately two cases  $x \geq y \geq 0$  and  $0 \geq x \geq y$ ).

To check that  $F$  is right continuous at point  $x$  assume first that  $x \geq 0$  then  $F(y) = \mu((0, y])$  for  $y > x \geq 0$  and  $\lim_{y \rightarrow x+} F(y) = \lim_{y \rightarrow x+} \mu((0, y])$ , we can take a countable sequence and use the property of measures

$$\lim_{y \rightarrow x+} \mu((0, y]) = \mu(\cap_{y > x} (0, y]) = \mu((0, x]).$$

For  $x < 0$  and  $y \in (x, 0)$  we have  $F(y) = -\mu((y, 0])$  and

$$\lim_{y \rightarrow x+} F(y) = -\mu(\cap_{y > x} (y, 0]) = -\mu((x, 0]) = F(x).$$

Moreover,  $\mu((a, b]) = F(b) - F(a)$  for  $b > a \geq 0$  and for  $0 \geq b > a$ . Thus  $\mu = \mu_F$  on half-open intervals and by Lemma 5.1  $\mu = \mu_F$  on  $\mathcal{B}$  ( $\mu$  is  $\sigma$ -finite since  $\mathbb{R} = \cup_{n \in \mathbb{Z}} [n, n+1]$ .)  $\square$

**5.2. Regularity of measures.** Suppose that  $\mu_F$  is a Lebesgue-Stieltjes measure on  $\mathbb{R}$  defined by a function  $F$ . Then for any open interval  $-\infty < a < b < \infty$ ,  $\mu_F(a, b) = \lim_{c \rightarrow b-} F(c) - F(a)$ . The limit exists since  $F$  is a non-decreasing function.

**Lemma 5.2.** *Let  $\mu$  be a Lebesgue-Stieltjes measure on  $(\mathbb{R}, \mathcal{M})$ , then for any  $E \in \mathcal{M}$*

$$\mu(E) = \inf \left\{ \sum_j \mu((a_j, b_j)) : E \subset \cup_j (a_j, b_j) \right\}$$

*Proof.* Each open interval is a countable disjoint union of half-open intervals, if  $(a_j, b_j) = \cup_k (c_{jk}, d_{jk}]$  then

$$\sum_j \mu((a_j, b_j)) = \sum_{j,k} \mu((c_{jk}, d_{jk}]) \geq \mu(E),$$

by the definition of  $\mu$ .

On the other hand for any  $\varepsilon > 0$  there exists a sequence  $\{(a_j, b_j]\}$  such that  $E \subset \cup_j (a_j, b_j]$  and  $\mu(E) + \varepsilon \geq \sum_j \mu((a_j, b_j])$ . Since  $F$  is continuous from the right for each  $j$  there is  $c_j > b_j$  such that  $F(c_j) < F(b_j) + \varepsilon 2^{-j}$ . Then  $E \subset \cup_j (a_j, c_j)$  and  $\sum_j \mu(a_j, c_j) \leq \sum (F(c_j) - F(a_j)) \leq \sum_j \mu(a_j, b_j] + \varepsilon \leq \mu(E) + 2\varepsilon$ .  $\square$

This lemma allows us to prove the following regularity properties of Lebesgue-Stieltjes measures.

**Proposition 5.2.** *Let  $\mu$  be a Lebesgue-Stieltjes measure on  $(\mathbb{R}, \mathcal{M})$  then for any  $E \in \mathcal{M}$ ,  $\mu(E) = \inf\{\mu(U), E \subset U, U \text{ is open}\} = \sup\{\mu(K) : K \subset E, K \text{ is compact}\}$ .*

*Proof.* By Lemma 5.2 there are open sets  $U_n$  such that  $E \subset U_n = \cup_j (a - j, b_j)$  and  $\mu(E) = \inf\{\mu(U_n)\}$  and clearly  $\mu(E) \leq \mu(U)$  is  $E \subset U$ . Thus first equality follows.

For the second one first assume that  $E$  is bounded. Let  $F = \overline{E} \setminus E$ . For any  $\varepsilon > 0$  there is open set  $O \supset F$  such that  $\mu(O) \leq \mu(F) + \varepsilon$ . Let  $K = \overline{E} \setminus O$  then  $K$  is closed and bounded, and  $K \subset E$  since  $O$  contains  $\overline{E} \setminus E$ . Then

$$\mu(K) = \mu(E) - \mu(E \cap O) = \mu(E) - \mu(O) + \mu(O \cap E^c) \geq \mu(E) - \mu(O) + \mu(F) \geq \mu(E) - \varepsilon.$$

□

**Definition 5.1.** A Borel measure  $\mu$  on  $\mathbb{R}^n$  (or on a topological space) is called a **Radon measure** if  $\mu(K) < \infty$  for any compact set  $K$ ,

$$\mu(B) = \inf\{\mu(U) : B \subset U, U \text{ is open}\} \text{ for any Borel set } B \text{ and}$$

$$\mu(O) = \sup\{\mu(K) : K \subset O, K \text{ is compact}\} \text{ for any open set } O.$$

Any Borel measure on  $\mathbb{R}$  which is finite on compact sets is a Lebesgue-Stieltjes measure by Proposition 5.1 and thus it is a Radon measure.

**5.3. The structure of measurable sets.** If  $\mu$  is a Lebesgue-Stieltjes measure on  $\mathbb{R}$  then by the construction all Borel sets are measurable. However the  $\sigma$ -algebra of measurable sets are usually larger than the Borel  $\sigma$ -algebra. For the case of the Lebesgue measure we denote the  $\sigma$ -algebra of measurable sets by  $\mathcal{L}$ . The next proposition describe the measurable sets for Lebesgue-Stieltjes measures.

**Proposition 5.3.** Suppose that  $\mu$  is a Lebesgue-Stieltjes measure on  $(\mathbb{R}, \mathcal{M})$ . Then the following statements are equivalent.

(i)  $E \in \mathcal{M}$ , (ii)  $E = A \cup F$ , where  $A \in \mathcal{B}$  and  $\mu(F) = 0$ , (iii)  $E = B \setminus N$ , where  $B \in \mathcal{B}$  and  $\mu(N) = 0$ .

*Proof.* We know that each Lebesgue-Stieltjes measure is complete. Then (ii) and (iii) imply (i).

Assume now that  $E \in \mathcal{M}$  and  $\mu(E) < \infty$ . By the previous proposition there is a sequence of compact sets  $K_n$  such that  $K_n \subset E$  and  $\mu(E) = \lim_n \mu(K_n)$ . Let  $A = \cup_n K_n$  then  $A \in \mathcal{B}$  (we say that  $A$  is an  $F_\sigma$ -set, a countable union of closed sets). Moreover,  $F = E \setminus A$  is a null set. For the case  $\mu(E) = \infty$  we use that the Lebesgue-Stieltjes measure is  $\sigma$ -finite and take  $\mathbb{R} = \cup_j S_j$  with  $\mu(S_j) < \infty$  and  $E = \cup_j (E \cap S_j)$  and construct sets  $A_j$  and  $N_j$  for each  $j$ . Then  $E = (\cup_j A_j) \cup (\cup_j F_j)$  and since the families of Borel sets and null sets are closed under countable unions we get the required decomposition for  $E$ .

Furthermore, if  $\mu(E) < \infty$  we know that there is a sequence of open sets  $U_n$  such that  $\mu(E) = \lim_n \mu(U_n)$ . Define  $B = \cap_n U_n$  then  $E \subset B$  and  $\mu(B \setminus E) = 0$  (we say that  $B$  is a  $G_\delta$  set, as a countable union of open sets). For the general case once again write  $E = \cup_j (E \cap S_j)$  and for each  $E \cap S_j = B_j \setminus N_j$ . We define  $B = \cup_j B_j$  and  $N = B \setminus E$ . Clearly  $E \subset B$  and  $N = B \setminus E \subset \cup_j N_j$  is a null set. □

6. BOREL MEASURES ON  $\mathbb{R}^n$ 

**6.1. Product  $\sigma$ -algebras.** Suppose that  $(X, \mathcal{M})$  and  $(Y, \mathcal{N})$  are two measurable spaces. The  $\sigma$ -algebra in  $X \times Y$  generated by the sets  $E \times F$ , where  $E \in \mathcal{M}$  and  $F \in \mathcal{N}$  is called the product  $\sigma$ -algebra  $\mathcal{M} \times \mathcal{N}$ . If  $\mathcal{M}$  is generated by a set  $\mathcal{E}$  and  $\mathcal{N}$  is generated by a set  $\mathcal{D}$  then  $\mathcal{M} \times \mathcal{N}$  is generated by  $\{E \times D : E \in \mathcal{E}, D \in \mathcal{D}\}$ . By induction one can define finite products of  $\sigma$ -algebras. We will use the fact that  $\mathcal{B}_{\mathbb{R}^n} = \otimes_1^n \mathcal{B}_{\mathbb{R}}$ .

The family of sets  $\mathcal{A}_0 = \{E \times F : E \in \mathcal{M}, F \in \mathcal{N}\}$  satisfies

$$(E \times F) \cap (H \times D) = (E \cap H) \times (F \cap D), \quad (E \times F)^c = (E^c \times F) \cup (E \times F^c),$$

and then the finite disjoint unions of  $\mathcal{A}_0$  form an algebra  $\mathcal{A}$  and  $\mathcal{A}$  generates the  $\sigma$ -algebra  $\mathcal{M} \times \mathcal{N}$ .

**Proposition 6.1.** *Suppose that  $L \in \mathcal{M} \times \mathcal{N}$  for each  $x \in X$  define  $L_x = \{y \in Y : (x, y) \in L\}$ . Then  $L_x \in \mathcal{N}$ . Similarly, for each  $y \in Y$ ,  $L^y = \{x \in X : (x, y) \in L\} \in \mathcal{M}$ .*

*Proof.* Let  $\mathcal{R}$  be the family of sets  $S \subset X \times Y$  such that  $S_x \in \mathcal{N}$  for each  $x \in X$  and  $S^y \in \mathcal{M}$  for each  $y \in Y$ . Clearly if  $E \in \mathcal{M}$  and  $F \in \mathcal{N}$  then  $E \times F \in \mathcal{R}$ . Furthermore,  $(S^c)_x = (S_x)^c$  and  $(\cup_j S_j)_x = \cup_j (S_j)_x$  and since  $\mathcal{M}$  and  $\mathcal{N}$  are  $\sigma$ -algebras,  $\mathcal{R}$  is also a  $\sigma$ -algebra. Thus  $\mathcal{M} \times \mathcal{N} \subset \mathcal{R}$ .  $\square$

**6.2. Product measures.** Assume now that  $(X, \mathcal{M}, \mu)$  and  $(Y, \mathcal{N}, \nu)$  are measure spaces. We construct the product measure  $\mu \times \nu$  on the  $\sigma$ -algebra  $\mathcal{M} \times \mathcal{N}$  by defining a pre-measure on  $\mathcal{A}$  first. On  $\mathcal{A}_0$  we define  $(\mu \times \nu)_0(E \times F) = \mu(E)\nu(F)$  and extend the function to finite disjoint unions of elements in  $\mathcal{A}_0$ . This pre-measure is well-defined and extends to a measure on  $\mathcal{M} \times \mathcal{N}$ , which is called the **product measure**  $\mu \times \nu$ .

We can now define measures on  $\mathbb{R}^n$  as product of measures on the real line. In particular the Lebesgue measure on  $\mathbb{R}^n$  is defined as the product of one-dimensional Lebesgue measures. More carefully, the Lebesgue measure  $m_n$  is the completion of the product measure  $\otimes_1^n m$  on  $\mathcal{L} \times \dots \times \mathcal{L}$ . The Proposition 6.1 shows that there are null sets that are not in the product  $\sigma$ -algebra. The Lebesgue  $\sigma$ -algebra in  $\mathbb{R}^n$  is denoted by  $\mathcal{L}_n$ .

The Lebesgue measure on  $\mathbb{R}^n$  is translation invariant. For any  $E \in \mathcal{L}_n$  and any  $x \in \mathbb{R}^n$  we have  $m_n(E) = m_n(E + x)$ . It is enough to check that this is true on the product sets  $\otimes E_j$ , where  $E_j \in \mathcal{L}$ , for the product sets the statement follows from the fact that one-dimensional Lebesgue measure is translation invariant.