MAT205a, Fall 2019 Part IV: Lebesgue spaces

Lecture 11, Following Folland, ch 5.1, 6.1

1. Basic theory of Lebesge spaces

1.1. **Banach spaces.** Let V be a vector space over $\mathbb{R}$ (or $\mathbb{C}$).

Definition 1.1. *A function from V to $[0, +\infty)$, such that $x \mapsto \|x\|$ is called a* **norm** *on V if*
(i) $\|x\| = 0$ implies that $x = 0$,
(ii) $\|cx\| = |c|\|x\|$, for all $x \in V$ and $c \in \mathbb{R}$ (or $\mathbb{C}$),
(iii)$\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in V$.

A vector space V with a norm $\|\cdot\|$ is called a **normed space**. A norm defines a metric on the space V.

Definition 1.2. *A normed space $(V, \|\cdot\|)$ is called a* **Banach space** *if it is complete with respect to the metric defined by the norm.*

It means that V is a Banach space if any sequence of vectors $\{v_j\}$ that is a Cauchy sequence, i.e., for any $\varepsilon > 0$ there is $n(\varepsilon)$ such that $\|v_n - v_m\| < \varepsilon$ for $n, m > n(\varepsilon)$, converges in V to some $w \in V$.

Lemma 1.1. *A normed space $(V, \|\cdot\|)$ is a Banach space if and only if for any sequence $\{v_j\}_j$ in V with $\sum_j \|v_j\| < \infty$ the series $\sum_j v_j$ converges in V.*

Proof. If V is a Bancah space and $\sum_j \|v_j\| < \infty$, we define $S_n = \sum_1^n v_j$. Then $\|S_n - S_m\| \leq \sum_{n<j\leq m} \|v_j\|$ and $\{S_n\}$ is a Cauchy sequence. It has a limit, so $\sum_j v_j$ converges in V.

Assume now that each absolutely convergent series converges in V. We want to prove that V is a Banach space. Let $\{x_j\}_j$ be a Cauchy sequence in V. We can find a subsequence $\{x_{j_k}\}$ such that $\|x_{j_k} - x_{j_l}\| < 2^{-m}$ when $k, l > m$. Define $v_n = x_{j_{n+1}} - x_{j_n}$, then $\sum_n \|v_n\| < \infty$. Thus x_{j_n} converges in V to some x_*, and since $\{x_j\}$ is a Cauchy sequence, the whole sequence converges to x^*. ☐

1.2. **Definition of the Lebesgue spaces.** We assume that $(X, \mathcal{M}, \mu)$ is a measure space and that $1 \leq p < \infty$on the set of all measurable functions, we define

$$\|f\|_p = \left(\int_X |f|^p \, d\mu\right)^{1/p}.$$

The Lebesgue space L^p is defined as

$$L^p(\mu) = \{f : f \text{ measurable}, \ \|f\|_p < \infty\}.$$

Where we identify functions that differ μ-almost everywhere.

One can define the spaces for $p \in (0,1)$, but we will work with the case $p \geq 1$.

Our first aim is to show that $(L^p(\mu), \|\cdot\|_p)$ is a normed space. We know that $\|f\|_p = 0$ implies that $f = 0$ μ-a.e. and it is identified with the zero element in L^p, also $\|cf\|_p = |c|\|f\|_p$ since the integral is linear. We can also check that if $f, g \in L^p$ then $f + g \in L^p$ since $|f+g|^p \leq (2\max|f|,|g|)^p \leq 2^p(|f|^p + |g|^p)$ point-wise, and integrating this inequality we obtain

$$\|f+g\|_p \leq 2(\|f\|^p + \|g\|^p)^{1/p} \leq 2(\|f\|_p + \|g\|_p).$$

We want to prove the inequality without the extra factor 2 on the right hand side.

1.3. **Useful inequalities.** We start with an inequality for real numbers.

Lemma 1.2. *Let $a, b \geq 0$ and $\lambda \in (0,1)$, then*

$$a^\lambda b^{1-\lambda} \leq \lambda a + (1-\lambda)b, \tag{1}$$

and the equality holds if and only if $a = b$.

Proof. If $b = 0$ the inequality holds and equality is true only for $a = 0$, If $b \neq 0$, we divide both sides by b and denote $a/b = t$, then the inequality reads

$$t^\lambda \leq \lambda t + (1-\lambda).$$

It becomes an equality when $t = 1$ and for $f(t) = t^\lambda - \lambda t$ we have $f'(t) = \lambda(t^{\lambda-1} - 1)$. Since $\lambda - 1 < 0$, f increases when $t \in (0,1)$ and decreases when $t \in (1,\infty)$. Thus $f(t) \leq f(1) = 1 - \lambda$. □

Theorem 1.1 (Hölder's inequality). *Let $1p, q < \infty$ be such that $1/p + 1/q = 1$. If f, g are measurable functions on $(X, \mathcal{M}, \mu)$ then $\|fg\|_1 \leq \|f\|_p \|g\|_q$.*

Proof. The inequality holds if $\|f\|_p = 0$ and $f = 0$ almost everywhere or when $\|f\|_p = \infty$, similarly for g. Now if $\|f\|_p \neq 0, +\infty$ and $\|g\|_q \neq 0, \infty$, we may assume that $\|f\|_p = 1$ and $\|g$

$_q = 1$ by multiplying f and g by suitable constants. Consider $x \in X$, we apply inequality (1) to $a = |f(x)|^p$, $b = |g(x)|^q$ and $\lambda = 1/p$. Then $1 - \lambda = 1/q$, and we obtain

$$|f(x)g(x)| \leq p^{-1}|f(x)|^p + q^{-1}|g(x)|^q.$$

Now, integrating this inequality over X with respect to the measure μ, we get

$$\|fg\|_1 \leq p^{-1}\|f\|_p^p + q^{-1}\|g\|_q^q = 1 = \|f\|_p\|g\|_q.$$

□

Corollary 1.1 (Cauchy-Schwarz inequality)**.** *When $p = q = 2$ we obtain the following*

$$\int |fg|\, d\mu \leq \left(\int |f|^2\, d\mu \right)^{1/2} \left(\int |g|^2\, d\mu \right)^{1/2}$$

It also follows from Theorem that in $f \in L^p$ and $g \in L^q$ then $fg \in L^1$. The exponent q such that $p^{-1} + q^{-1} = 1$ is called the conjugate exponent to p.

Theorem 1.2 (Minkowski's inequality)**.** *If $1 \leq p < \infty$ and $f, g \in L^p$ then*

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p.$$

Proof. When $p = 1$ the inequality follows from the properties of integral. Assume that $p > 1$. We have

$$|f + g|^p \leq (|f| + |g|)|f + g|^{p-1} = |f||f + g|^{p-1} + |g||f + g|^{p-1}.$$

Now we apply the Hölder inequality twice,

$$\int |f + g|^p\, d\mu \leq \|f\|_p \|(f + g)^{p-1}\|_q + \|g\|_p \|(f + g)^{p-1}\|_q.$$

We know that $q = p/(p - 1)$ and the inequality becomes

$$\|f + g\|_p^p \leq (\|f\|_p + \|g\|_p)\|f + g\|_p^{p-1}.$$

Now if $\||f + g\|_p \neq 0$ we divide both sides by $\|f + g\|_p^{p-1}$ and obtain the required inequality. When $\|f + g\|_p = 0$ the inequality also holds. □

Theorem 1.3. *For $1 \leq p < \infty$ the Lebesgue space $L^p(\mu)$ is a Banach space.*

Proof. We want to show that an absolutely convergent series converges, using Lemma 1.1. Let $\{f_j\}_j$ be a sequence in $L^p(\mu)$ and $\sum_j \|f_j\|_p = B < \infty$. We define $G_n = \sum_1^n |f_j|$ and $F_n = \sum_1^n f_j$. Then $\|F_n\|_p \leq \|G_n\|_p \leq \sum_1^n \|f_j\|_p$. Let $G = \sum_1^\infty |f_j|$. By the monotone convergence theorem $\int G^p \leq B^p$. Thus $|G| < \infty$ a.e. and we can define $F = \sum f_j$ a.e. We also have $|F| \leq G$ and $F \in L^p(\mu)$. Using the dominated convergence theorem we see that

$$\|F - \sum_1^n f_j\|_p = \int |F - \sum_1^n f_k|^p \to 0$$

since $|F - \sum_1^n f_j|^p \leq (2G)^p \in L^1(\mu)$, □

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Lecture 12, Following Folland, ch 6.2

2. The dual of $L^p(\mu)$

2.1. **The dual of a normed space.** We remind that if V is a normed space that its dual, denoted by V^*, is the space of of continuous linear functionals on V,

$$V^* = \{T : V \to \mathbb{C} \text{ linear}, \ |T(x)| \le C\|x\|.\}.$$

The norm of the functional T is defined as $\|T\| = \sup\{|Tx| : \|x\| = 1\}$. Then V^* is a normed space. Clearly $\|T\| = 0$ if and only if $T = 0$ and $\|Tc\| = |c|\|T\|$ since T is linear. The triangle inequality follows from the linearity as well,

$$|T_1(x) + T_2(x)| \le |T_1(x)| + |T_2(x)| \le \|T_1 + \|T\|_2,$$

for any $x \in V$ with $\|x\| = 1$. Then, taking the supremum, we obtain $\|T_1 + T_2\| \le \|T_1\| + \|T_2\|$.

2.2. **The space L^∞.** Let $(X, \mathcal{M}, \mu)$ be a measure space and let f be a measurable function, we say that $f \in L^\infty(X)$ if there exists $C < \infty$ such that $|f| \le C$ μ-a.e. And we define

$$\|f\|_\infty = \inf\{C \ge 0 : \mu(\{|f| > C\}) = 0\}.$$

Note also that if $\|f\|_\infty = C$ then $\mu(\{|f| > C\}) = \lim_{n\to\infty} \mu(\{|f| > C + 1/n\}) = 0$. Clearly $L^\infty(\mu)$ is a linear space and $\|f + g\|_\infty \le \|f\|_\infty + \|g\|_\infty$.

When $\mu(X) < \infty$ we see that $L^\infty(\mu) \subset L^p(\mu)$ for any $p \ge 1$. Without the assumption that μ is finite the conclusion does not hold. The space L^∞ is also a Banach space (see your problem set).

The following version of the Hölder inequality holds. If $f \in L^\infty(\mu)$ and $g \in L^1(\mu)$ then $fg \in L^1(\mu)$ and $\|fg\|_1 \le \|f\|_\infty \|g\|_1$.

2.3. **Functionals on $L^p(\mu)$.** Let $1/p + 1/q = 1$, where $1 \le p, q \le \infty$, for each $g \in L^q(\mu)$ we define the functional

$$T_g : L^p(\mu) \to \mathbb{C}, \quad T_g(f) = \int_X fg d\mu.$$

Clearly T_g is linear and by Hölder's inequality

$$|T_g| \le \|f\|_p \|g\|_q, \quad \text{i.e.,} \quad \|T_q\| \le \|g\|_q.$$

Proposition 2.1. *Suppose that $1 \le q < \infty$ then $\|T_g\| = \|g\|_q$.*
If $q = \infty$ and μ is semifinite, then $\|T_g\| = \|g\|_\infty$.

Proof. Suppose first that $1 < q < \infty$. We already know that $\|T_g\| \leq \|g\|_q$, now we consider the function $f(x) = |g(x)|^{q-2}\overline{g(x)}$, such that $|f(x)| = |g(x)|^{q-1}$. Then $\|f\|_p^p = \int |g|^{pq-p}\, d\mu = \|g\|_q^q$. We also have

$$|T_g f| = \int_X |g|^q d\mu = \|g\|_q^q.$$

Therefore

$$\|T_g\| \geq \|g\|_q^q \|f\|_p^{-1} = \|g\|_q^{q-q/p} = \|g\|_q.$$

If $q = 1$ we let $f(x) = \overline{g(x)}|g(x)|^{-1}$ such that $|f(x)| = 1$ and $f(x)g(x) = |g(x)|$, then $|T_g f| = \inf_X |g(x)|\, d\mu = \|g\|_1$ and $\|f\|_\infty = 1$.

Finally, when $q = \infty$, consider a set $E \subset X$ such that $0 < \mu(E) < \infty$ and take $f(x) = \overline{g(x)}|g(x)|^{-1}\chi_E(x)$. So that $|f(x)| = 1$ when $x \in E$ and $|f(x)| = 0$ otherwise, then $\|f\|_1 = \mu(E)$. Moreover,

$$|T_g f| = \int_E |g(x)|\, d\mu \geq \min_E |g(x)|\mu(E).$$

Now since $g \in L^\infty(\mu)$ we have that $\mu(\{|g(x)| > \|g\|_\infty - \varepsilon\}) > 0$ and since the measure is semifinal, there exists $E \subset \{|g(x)| > \|g\|_\infty - \varepsilon\}$ with $0 < \mu(E) < \infty$. We obtain

$$\|T_g\| \geq \|g\|_\infty - \varepsilon.$$

for any positive ε, and then by taking $\varepsilon \to 0$ the conclusion of the proposition follows. $\square$

We want to show that all functionals on $L^p(\mu)$, when $1 < p < \infty$ are defined in that way. First we prove a weaker statement. Let $\mathcal{S}_0(\mu)$ denote the family of simple functions ϕ on $(X, \mathcal{M}, \mu)$ such that $\mu(\{\phi \neq 0\}) < \infty$.

Theorem 2.1. *Let $1/p + 1/q = 1$, $1 \leq p, q \leq \infty$. Suppose that g is a measurable function on $(X, \mathcal{M}, \mu)$ such that $E_g = \{g \neq 0\}$ is (μ) σ-finite and $g\phi \in L^1(\mu)$ for any $\phi \in \mathcal{S}_0(\mu)$ with*

$$M_q(g) = \sup\left\{\left|\int \phi g\, d\mu\right|,\ \phi \in \mathcal{S}_0,\ \|\phi\|_p = 1\right\} < \infty.$$

Then $g \in L^q(\mu)$ and $M_g = \|g\|_q$.

Proof. Suppose that f is a measurable function with $\mu(\{f \neq 0\}) < \infty$ and $f \in L^p(\mu)$. We claim that $fg \in L^1$ and $\|fg\|_1 \leq M_q(g)\|f\|_p$. Indeed, there is a sequence of simple functions ϕ_n such that $\phi_n \to f$ pointwise and $|\phi_n| \leq |f|$. Moreover, $|\phi_n g| \leq |\chi_E g|$, where $E = \{f \neq 0\}$ and $\chi_E g \in L^1(\mu)$. Then by the dominated convergence theorem

$$\left|\int fg\, d\mu\right| = \left|\lim_{n\to\infty} \int \phi_n g\, d\mu\right| \leq M_q(g) \lim_{n\to\infty} \|\phi_n\|_p \leq M_q(g)\|f\|_p. \tag{2}$$

We want to show that $g \in L^q(\mu)$. Assume first that $1 < q < \infty$ and let $E_g = \cup_m E_m$ such that $E_m \subset E_{m+1}$ and $\mu(E_m) < \infty$. There exists a sequence of simple functions ψ_m that converges to g and such that $|\psi_m| \le |g|$, we define $g_m = \psi_m \chi_{E_m}$. Then g_m is also a sequence of simple functions converging to g, $|g_m| \le |g|$, and $\mu(\{g_m \neq 0\}) < \infty$. Define $\phi_m = |g_m|^{q-2}\overline{g_m}\|g_m\|_q^{1-q}$ then $\phi_m \in \mathcal{S}_0$. Moreover $|\phi_m| = |g_m|^{q-1}\|g_m\|^{1-q}$ and $|\phi_m|^p = |g_m|^q \|g_m\|_q^{-p}$. Hence $\|\phi_m\|_p = 1$. By the Fatou lemma

$$\left(\int |g|^q \, d\mu\right)^{1/q} \le \liminf_{m\to\infty} \|g_m\|_q = \liminf_{m\to\infty} \int |\phi_m g_m| d\mu \le \liminf_{m\to\infty} \int |\phi_m g| d\mu \le M_q(g).$$

In the last inequality we used that $|\phi_m g| = fg$, where f satisfies $|f| = |\phi|$ and therefore $f \in L^p$, $\|f\|_p = \|\phi_m\|_p = 1$, so we can use the first part of the proof. The last inequality implies that $g \in L^q(\mu)$ and $\|g\|_q \le M_q(g)$. We know also that $M_q(g) \le \|g\|_q$ by the Hölder inequality. Thus $\|g\|_q = M_q(g)$. For $q = 1$ we repeat the argument taking $\phi_m = \overline{g_m}/|g_m|$, clearly $\|\phi_m\|_\infty = 1$.

For the case $q = \infty$, assume that $\|g\|_\infty > M_\infty(g)$ then $\mu(\{|g| > M_\infty(g) + \varepsilon\}) > 0$ for some $\varepsilon > 0$. Since E_g is σ-finite, there exists $B \subset \{|g| > M_\infty(g) + \varepsilon\}$ with $0 < \mu(B) < \infty$. Let $f = \overline{g}|g|^{-1}\chi_B$, then $\|f\|_1 = \mu(B)$ and

$$\int fg \, d\mu = \int_B |g| > (M_\infty(g) + \varepsilon)\|f\|_\infty.$$

This contradicts the inequality (2) above. Thus $g \in L^\infty$ and $\|g\|_\infty \le M_\infty(g)$. On the other hand, clearly $M_\infty(g) \le \|g\|_\infty$. □

Theorem 2.2. *Let $1 < p < \infty$ and $1/p + 1/q = 1$. Then for any $T \in (L^p(\mu))^*$ there is $g \in L^q(\mu)$ such that $T(f) = T_g(f) = \int fg \, d\mu$.*

If μ is σ-finite then for any $T \in (L^1(\mu))^*$ there is $g \in L^\infty(\mu)$ such that $T(f) = T_g(f) = \int fg \, d\mu$.

Proof. First assume that $\mu(X) < \infty$. Then for any measurable E we have $\chi_E \in L^p(d\mu)$. Define $\nu(E) = T(\chi_E)$. We claim that it is a measure and $\nu \ll \mu$. To show that ν is countably additive consider a sequence of disjoint sets $\{E_j\}$. We claim that if $\phi_n = \sum_1^n \chi_{E_j}$ and $\phi = \chi_{\cup_j E_j}$ then $\phi_n \to \phi$ in L^p and then $\nu(\cup_j E_j) = \sum \nu(E_j)$. Then applying the Radon-Nikodym theorem we get a function g such that $\nu(E) = \int_E g \, d\mu$. Then we see that for all simple functions ϕ we have $t(\phi) = \int \phi g \, d\mu$ and $\left|\int \phi g \, d\mu\right| \le \|T\|\|\phi\|_p$. Applying the previous theorem we see that $g \in L^q(\mu)$.

Next, we assume that μ is σ-finite.Then $X = \cup_n X_n$ with $\mu(X_n) < \infty$ and $X_n \subset X_{n+1}$. We know that for each n there exists $g_n \in L^q(X_n, \mu)$ such that $T(f) = \int fg \, d\mu$ when $f \in L^p(X, \mu)$, $f = 0$ on X_n^c and $\|g_n\|_q \le \|T\|$. Moreover g_n is unique. If g_n and g_n' define the same functional on $L^p(X_n, \mu)$ then we know that $\|g_n - g_n'\|_q = 0$. Thus $g_n = g_m$ μ-a.e. on X_n when $m > n$ and we can define function g on X such that $g = g_n$ on X_n. By the monotone convergence theorem $\|g\|_q \leq \|T\|$. For any $f \in L^p(\mu)$ we have $f\chi_{X_n} \to f$ in L^p by the dominated convergence and

$$T(f) = \lim_{n\to\infty} \int_{X_n} fg\, d\mu = \int_X fg\, d\mu.$$

Finally, assume that μ is arbitrary and $1 < p < \infty$. For each set $E \subset X$ that has a σ-finite measure we can find $g_E \in L^q(E, \mu)$ such that $T(f) = \int fg\, d\mu$ when $f = 0$ on E^c and $\|g_E\|_q \leq \|T\|$. Let

$$M = \sup\{\|g_E\|_q : E \text{ is } \sigma - \text{finite}\}.$$

There is a sequence E_n such that $\|g_{E_n}\|_q \to M$, let $E = \cup_n E_n$ then $g_E = g_{E_n}$ a.e. on E_n and thus $\|g_E\|_q = M$. Then for any σ-finite set B such that $E \cap B = \emptyset$ we have $g_B = 0$, here we use that $q < \infty$. Let $f \in L^p(\mu)$ then $F = \{f \neq 0\}$ is a σ-finite set. We have

$$T(f) = \int g_{F\cup E} f\, d\mu = \int g_E f\, d\mu + \int g_{F\setminus E} f\, d\mu = \int g_E f\, d\mu.$$

Thus $T(f) = \int g_E f\, d\mu$. $\square$

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Lecture 13, Following Folland, ch 6.3, 6.4

3. Some operators between L^p spaces

3.1. **Norm of an operator.** Now we will consider linear operators between two normed spaces $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$. Let $T : V \to W$, T is linear, i.e.,

$$T(av_1 + bv_2) = aT(v_1) + bT(v_2),$$

we say that T is bounded if

$$\|T\| = \sup_{\|v\|_V=1} \|Tv\|_W = \sup_{v\neq 0} \frac{\|Tv\|_W}{\|v\|_V} < \infty.$$

We will study linear operators between some L^p spaces.

3.2. **Boundedness of integral operators.** Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be two measure spaces with σ-finite measures. Suppose that $K : X \times Y \to \mathbb{C}$ is a $\mathcal{M} \otimes \mathcal{N}$ measurable function. Under some additional assumption K we want to define the integral operator

$$Tf(x) = \int_Y K(x, y)f(y)d\nu(y).$$

Theorem 3.1. *Suppose that there exists C such that $\int |K(x, y)|\, d\mu(x) \le C$ for ν-a.e. $y \in Y$ and $\int |K(x, y)|\, d\nu(y) \le C$ for μ-a.e. $x \in X$. Then for any $f \in L^p(\nu)$, $1 \le p \le \infty$ the integral $Tf(x)$ converges absolutely for μ-a.e. $x \in X$. Moreover $Tf \in L^p(\mu)$ and $\|Tf\|_p \le C\|f\|_p$.*

Proof. Consider first $p \in (1, \infty)$ and let q be such that $1/p + 1/q = 1$. We write $|K(x, y)f(y)| = |K(x, y)|^{1/q}(|K(x, y)|^{1/p}|f(y)|)$ and apply the Hölder inequality:

$$\int_Y |K(x, y)f(y)|\, d\nu(y) \le \left(\int_Y |K(x, y)|\, d\nu(y)\right)^{1/q} \left(\int_y |K(x, y)||f(y)|^p d\nu(y)\right)^{1/p},$$

where the first factor is bounded by $C^{1/q}$ for a.e $x \in X$. Now we apply the Tonelli theorem

$$\int_X \left(\int_Y |K(x, y)f(y)|\, d\nu(y)\right)^p d\mu(x) \le$$

$$C^{p/q} \int_Y \int_X |K(x, y)||f(y)|^p d\mu(x)d\nu(y) \le C^{p/q+1}\|f\|_p^p.$$

Then we know that $K(x, y)f(y) \in L^1(\nu)$ for a.e. x, thus $Tf(x)$ is well-defined for a.e. $x \in X$ and $\|Tf\|_f \le C^{1/q+1/p}\|f\|_p = C\|f\|_p$.

For $p = 1$ we have by the Tonelli theorem

$$\int_X \int_Y |K(x,y)f(y)| d\nu(y) d\mu(x) \le C \int_Y \|f\| d\nu(y) = C\|f\|_1,$$

and the rest of the proof is the same as above. In this case we used only one condition, $\int |K(x,y)| d\mu(x) \le C$.

Similarly, for $p = \infty$, $\int |K(x,y)f(y)| d\nu(y) \le \|f\|_\infty \int_Y |K(x,y)| d\nu(y) \le C\|f\|_\infty$ and $\|Tf\|_\infty \le C\|f\|_\infty$, and we used only one bound on K, $\int |K(x,y)| d\nu(y) \le C$. □

3.3. **Minkowski inequality.** Next, we generalize the triangle inequality $\|f_1 + f_2\|_p \le \|f_1\|_p + \|f_2\|_p$ by replacing the sum by the integral.

Theorem 3.2. *Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be two measure measure spaces with σ-finite measures and let $f : X \times Y \to \mathbb{C}$ be a $\mathcal{M} \otimes \mathcal{N}$ measurable function, $f \ge 0$. Then for $1 \le p < \infty$*

$$\left(\int_X \left(\int_Y f(x,y)\, d\nu(y) \right)^p d\mu(x) \right)^{1/p} \le \int_Y \left(\int_X f(x,y)^p d\mu(x) \right)^{1/p} d\nu(y).$$

Proof. For $p = 1$ this is the Tonelli theorem. Let $1 < p < \infty$ and let $1/p + 1/q = 1$. We define

$$F(x) = \int_Y f(x,y) d\nu(y)$$

and we want to estimate $\|F\|_p$ in $L^p(\mu)$. By the result of the previous lecture (Theorem 2.1), it is equivalent to estimating the norm of the corresponding functional on $L^q(\mu)$. Let $g \in L^q(\mu)$, applying the Tonelli theorem and then the Hölder inequality, we obtain

$$\left| \int_X F(x) g(x)\, d\mu(x) \right| \le \int_X \int_Y f(x,y) |g(x)|\, d\nu(x) d\mu(y) \le$$

$$\int_Y \left(\int_X f(x,y)^p d\mu(x) \right)^{1/p} \left(\int_X g(x)^q\, d\mu(x) \right)^{1/q} d\nu(y) =$$

$$\int_Y \left(\int_X f(x,y)^p d\mu(x) \right)^{1/p} d\nu(y) \|g\|_q.$$

the required inequality follows. □

Corollary 3.1. *Let $1 \le p \le \infty$, $f(\cdot, y) \in L^p(\mu)$ for ν-a.e. $y \in Y$ and the function $y \mapsto \|f(\cdot, y)\|_p$ is in $L^1(\nu)$ then $f(x, \cdot) \in L^1(\nu)$ for μ-a.e. $x \in X$, the function $F(x) = \int_Y f(x,y)\, d\nu(y)$ is in $L^p(\mu)$ and*

$$\|F\|_p \le \int \|f(\cdot, y)\|_p\, d\nu(y).$$

If we replace f by $|f|$, the corollary (for $p < \infty$) follows from the theorem. It is clear when $p = \infty$.

3.4. **Hardy's inequality.** We first describe another class of bounded integral operators on $L^p(0,\infty)$.

Theorem 3.3. *Let K be a Lebesgue measurable function on $(0,\infty)\times(0,\infty)$ and $1 \le p \le \infty$ such that $K(\lambda x, \lambda y) = \lambda^{-1}K(x,y)$ for all $\lambda > 0$ and*

$$\int_0^\infty |K(x,1)|x^{-1/p}dm(x) = C < \infty.$$

Then the operator T defined by

$$Tf = \int_0^\infty K(x,y)f(x)\,dm(x),$$

is bounded on $L^p(0,\infty)$, $\|Tf\|_p \le C\|f\|_p$.

Proof. Consider $f_t(y) = f(ty)$ for $t > 0$, then $\|f_t\|_p = t^{-1/p}\|f\|_p$. We fix y, and introduce a new variable $t = x/y$, then

$$\int_0^\infty |K(x,y)f(x)|\,dm(x) = \int_0^\infty |K(t,1)f_t(y)|\,dm(t).$$

Then, by Minkowski's inequality, (see the corollary above)

$$\|Tf\|_p \le \int_0^\infty |K(t,1)|\|f_t\|_p\,dm(t) = \|f\|_p \int_0^\infty t^{-1/p}|K(t,1)|\,dm(t) = C\|f\|_p.$$

□

Corollary 3.2 (Hardy's inequality). *Let $1 < p \le \infty$ and*

$$Tf(y) = y^{-1}\int_0^y f(x)dx.$$

Then $\|Tf\|_p \le p(p-1)^{-1}\|f\|_p$.

Proof. We apply the theorem above for $K(x,y) = y^{-1}\chi_{x<y}$. Then $K(\lambda x, \lambda y) = \lambda^{-1}K(x,y)$ and

$$\int_0^\infty |K(x,1)|x^{-1/p}dm(x) = \int_0^1 x^{-1/p}\,dm(x) = \frac{p}{p-1}.$$

□

4. Distribution function

4.1. **Chebyshev inequality and the distribution function.** Suppose that $f \in L^p(\mu)$, $1 \le p < \infty$, and $t > 0$ then

$$\mu(\{|f| > t\}) \le t^{-p}\|f\|_p^p.$$

It follows by integrating the inequality $|f(x)| > t\chi_{E_t}$, where $E_t = \{|f| > t\}$.

For a measurable function f on $(X, \mathcal{M}, \mu)$, we define the distribution function of f by

$$\lambda_f(t) = \mu(\{|f| > t\}.$$

Clearly $\lambda_f = \lambda_{|f|}$, if $|f| \leq |g|$ a.e. then $\lambda_f \leq \lambda_g$.

Lemma 4.1. *The distribution function has the following properties*
(i) λ_f id decreasing and right continuous,
(ii) if $\{f_n\}$ is a sequence of measurable functions such that $\{|f_n|\}$ is increasing and $|f| = \lim_n |f_n|$, then $\lambda_{f_n} \to \lambda_f$.
(iii) $\lambda_{f+g}(t) \leq \lambda_f(t/2) + \lambda_g(t/2)$.

Proof. (i) It is clear that λ is increasing, we also hav that $\{|f| > t\} = \cup_n \{|f| > t + 1/n\}$ and by the properties of measure μ, $\lambda(t) = \lim_{n\to\infty} \lambda(t + 1/n)$. (ii) Here we have $\{|f| > t\} = \cup_n \{|f_n| > t\}$ and the sets on the right side of the equality form an increasing sequence, since $|f_n| \leq |f_{n+1}|$. Thus $\lambda_f(t) = \lim_{n\to\infty} \lambda_{f_n}(t)$.
(iii) We have $\{|f + g| > t\} \subset \{|f| > t/2\} \cup \{|g| > t/2\}$, then computing the measure μ, we get $\lambda_{f+g}(t) \leq \lambda_f(t/2) + \lambda_g(t/2)$. □

4.2. **Distribution function and L^p-norms.** We will show that it is enough to know the distribution function to compute the L^p-norms of the function.

Theorem 4.1. *Suppose that $h : [0, \infty) \to [0, \infty)$ is an increasing continuous function, $h(0) = 0$, and h is absolutely continuous on each bounded interval $[0, T]$. Then for any measurable function f on $(X, \mathcal{M}, \mu)$ we have*

$$\int_X h(|f(x)|)\, d\mu(x) = \int_0^\infty h'(t)\lambda_f(t)\, dm(t).$$

Proof. Assume that $f = a\chi_E$, where $E \in \mathcal{M}$ and $\mu(E) < \infty$. Then $\lambda_f(t) = \mu(E)$ when $0 < t < |a|$ and $\lambda_f(t) = 0$ when $t \geq |a|$. The integral on the left hand side equals $h(|a|)\mu(E)$ and the integral on the right hand side is $\mu(E) \int_0^{|a|} h'(t)dm(t) = \mu(E)h(|a|)$ since h is absolutely continuous function on $[0, |a|]$.

Now let $f = \sum c_j\chi_{E_j}$ is a simple function and assume that $E_j \cap E_k = \emptyset$ when $j \neq k$. Then $\lambda_f(t) = \sum_j \lambda_{f_j}$ with $f_j = a_j\chi_{E_j}$ and

$$\int_X h(|f(x)|)\, d\mu(x) = \sum_j \int_{E_j} h(|f_j(x)|)\, d\mu(x).$$

Thus the equality holds for simple functions by the linearity of both sides.

Finally if f is measurable, there is a sequence f_n of simple functions such that $|f_n| \leq |f_{n+1}$ and $f_n \to f$. Then the sequence λ_{f_n} increases and converges to λ_f and

$h(f_n(x)) \to h(f_n(x))$ since h is continuous, moreover $h(f_n) \le h(f_{n+1})$. We apply the monotone convergence theorem to the both sides and conclude that

$$\int_X h(|f(x)|)\, d\mu(x) = \int_0^\infty h'(t)\lambda_f(t)\, dm(t)$$

for any measurable function f. $\square$

Corollary 4.1. *Let f be a measurable function on $(X, \mathcal{M}, \mu)$ then*

$$\|f\|_p = \int_0^\infty p t^{p-1} \lambda_f(t) dm(t).$$

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Lecture 14, Following Folland, ch 6.5

5. Riesz-Thorin Interpolation theorem

5.1. **A lemma from complex analysis.** We need a result from introductory complex analysis. First we remind that if $F(z)$ is a holomorphic function on a bounded domain Ω such that F is continuous on the closure of Ω then by the maximum principle

$$\sup_{z\in\Omega}|F(z)|\le\max_{z\in\partial\Omega}|F(z)|.$$

This maximum principle implies the following result.

Lemma 5.1 (Three line inequality)**.** *Suppose that F is holomorphic bounded function on the strip $P=\{z:0<\mathrm{Re}(z)<1\}$ and F is continuous on the closure of P. Suppose that $|F(z)|\le M_0$ when $\mathrm{Re}(z)=0$ and $|F(z)|\le M_1$ when $\mathrm{Re}(z)=1$. Then $|F(z)|\le M_0^{1-t}M_1^t$ when $\mathrm{Re}(z)=t$.*

Proof. Let $z\in S$, $z=x+iy$, then $z(z-1)=x(x-1)-y^2+i(2xy-y)$ and $\mathrm{Re}(x)=x(x-1)-y^2<0$ and it goes to $-\infty$ when $|y|\to\infty$. Let $G(z)=F(z)M_0^{z-1}M_1^{-z}\exp(\varepsilon z(z-1))$. Then G is holomorphic in P, continuous on the closure of P, $|G(z)|$ tends to zero when $\mathrm{Im}(z)\to\pm\infty$ and $|G(z)|\le 1$ when $\mathrm{Re}(z)\in\{0,1\}$. Then, applying the maximum principle to truncated strips

$$P_R=\{z:0<\mathrm{Re}(z)<1,\ -R<\mathrm{Im}(z)<R\},$$

we conclude that $|G(z)|\le 1$ for all $z\in P$. Note that if $z=t+is$, then

$$|G(t+is)|=|F(t+is)|M_0^{t-1}M_1^{-t}\exp(\varepsilon(t(t-1)-s^2)).$$

Thus $|F(t+is)|\le M_0^{1-t}M_1^t\exp(\varepsilon(t(1-t)+s^2))$. The last inequality holds for any $\varepsilon>0$ then the conclusion of the theorem follows. $\square$

5.2. **Two auxiliary results.** We prove two lemmas before formulating the first interpolation theorem.

Lemma 5.2. *If $1\le p<\infty$ then the set $S_0(\mu)$ of simple functions ϕ such that $\mu(\{\phi\ne 0\})<\infty$ is dense in $L^p(\mu)$.*

Proof. We want to show that for any $f\in L^p(\mu)$ there exists a sequence $\{\phi_n\}$ such that $\phi_n\in S_0(\mu)$ and $\lim_{n\to\infty}\|f-\phi_n\|_p=0$. Let ϕ_n be simple functions such that $\phi_n\to f$ a.e. and $|\phi_n|\le|\phi_{n+1}|$, then $\|\phi_n\|_p\le\|f\|_p$. Since $|\phi_n|$ takes only finitely many values, we have $\inf\{|\phi_n(x)|:\phi_n(x)\ne 0\}=c>0$. Then $\mu(\{\phi_n\ne 0\})\le c^{-p}\|\phi_n\|^p<\infty$, so

$\phi_n \in \mathcal{S}_0(\mu)$. Moreover, $|f-\phi_n|^p \le 2^p|f|^p$ point-wise. The dominated convergence theorem implies that $\|f-\phi_n\|_p \to 0$. $\square$

Lemma 5.3. *Suppose that a linear operator T is defined on $L^{p_1}(\mu)+L^{p_2}(\mu)$ and Tf is a measurable function on $(Y,\mathcal{N},\nu)$ such that $\|Tf\|_{q_0} \le M_0\|f\|_{p_0}$ and $\|Tf\|_{q_1} \le M_1\|f\|_{p_1}$. Suppose also that for some $p \in [p_0,p_1]$, $\|T\phi\|_q \le C\|\phi\|_p$, when $\phi \in \mathcal{S}_0(\mu)$. Then for $f \in L^p(\mu)$ we have $Tf \in L^q(\nu)$ and $\|Tf\|_q \le C\|f\|_p$.*

Proof. Let $f \in L^p(\mu)$ and $E=\{|f|>1\}$. There is a sequence $\phi_n \to f$ such that $\phi_n \in \mathcal{S}_0(\mu)$ and $|\phi_n| \le |\phi_{n+1}$. We define $\psi_n = \phi_n\chi_E$ and $\omega_n = \phi_n\chi_{E^c}$. Suppose that $p_0 \le p_1$ then $\psi_n \in L^{p_0}$ and $\omega_n \in L^{p_1}$, moreover if $g = f\chi_E$ and $h = f\chi_{E^c}$ then $\|g-\psi_n\|_{p_0} \to 0$ and $\|f-\omega_n\|_{p_1} \to 0$. Then $\{Tg - T\psi_n\}$ and $\{Th - T\omega_n\}$ converge to zero in $L^{q_0}(\nu)$ and in $L^{q_1}(\nu)$ respectively, and thus converges to zero in ν-measure, therefore we can find a subsequence $\phi_{n_k} = \psi_{n_k} + \omega_{n_k}$ such that $T\psi_{n_k} \to Tg$ a.e. and $T\omega_{n_k} \to Th$ a.e. Then $T\phi_{n_k} \to Tf$ a.e. and by the Fatou lemma

$$\|Tf\|_q \le \liminf_{k\to\infty} \|T\phi_{n_k}\|_q \le \liminf_{k\to\infty} C\|\phi_{n_k}\|_p \le C\|f\|_p.$$

$\square$

5.3. **Riesz – Thorin interpolation.** Let $(X,\mathcal{M},\mu)$ and $(Y,\mathcal{N},\nu)$ be two measure spaces. We know that if $r \in [p_1,p_2]$ then $L^r(\mu) \subset L^{p_1}(\mu)+L^{p_2}(\mu)$ (see homework assignment). We consider a linear operator T defined on both $L^{p_1}(\mu)$ and $L^{p_2}(\mu)$ and conclude that T is defined on $L^r(\mu)$ when $r \in [p_1,p_2]$.

Theorem 5.1. *Suppose that $1 \le p_0,p_1,q_0,q_1 \le \infty$ and that if $q_0 = q_1 = \infty$ then ν is σ-finite. Suppose that T is a linear operator from $L^{p_1}(\mu)+l^{p_2}(\mu)$ to $L^{q_0}(\nu)+L^{q-1}(\nu)$ such that $\|Tf\|_{q_0} \le M_0\|f\|_{p_0}$ and $\|Tf\|_{q_1} \le M_1\|f\|_{p_1}$. Then for p_t and q_t defined by*

$$\frac{1}{p_t} = \frac{1-t}{p_0} + \frac{t}{p_1}, \quad \frac{1}{q_t} = \frac{1-t}{q_0} + \frac{t}{q-1},$$

T is a bounded operator from $L^{p_t}(\mu)$ to $L^{q_t}(\mu)$ and $\|Tf\|_{q_t} \le M_0^{1-t}M_1^t\|f\|_{p_t}$..

Proof. First if $p_0 = p_1$ we estimate $\|Tf\|_{q_t}$ applying the Hölder inequality to $|Tf|^{q_t} = |Tf|^{(1-t)q_t}|Tf|^{tq_t}$ with exponents $r = q_0q_t^{-1}(1-t)^{-1}$ and $r' = q_1q_t^{-1}t^{-1}$. We get $\|Tf\|_{q_t} \le \|Tf\|_{q_0}^{1-t}\|Tf\|_{q_1}^t$.

Now assume that $p_1 \ne p_2$. We want to show that $\|Tf\|_{q_t} \le M_0^{1-t}M_1^t\|f\|_{p_t}$. By Lemma 5.3 it suffices to prove the inequality for the case $f \in \mathcal{S}_0(\mu)$. Furthermore, we note that by Theorem 2.1, $\|Tf\|_{q_t} = \sup\{\left|\int(Tf)g\,d\nu\right| : g \in L^{q_t'}(\nu), \|g\|_{q_t'} = 1\}$. Applying Lemma 5.2 we may take the supremum over the functions $g \in S_0(\nu)$ only. Let $f(x) = \sum_j c_j\chi_{E_j}(x)$ and $g \in S_0(\nu)$ be $g(y) = \sum_k d_k\chi_{F_k}(y)$, where both sums are finite and $c_j, d_k \in \mathbb{C}$, $c_j = |c_j|e^{i\gamma_j}$ and $d_j = |d_j e^{i\delta_j}$. We also assume that $\|f\|_{p_t} = \|g\|_{q_t'} = 1$. We have

$$\int_Y (Tf)g\,d\nu = \sum_{j,k} c_j d_k \int_{F_k} T\chi_{E_j}\,d\nu = \sum_{j,k} c_j d_k C_{jk}.$$

Our aim is to construct a holomorphic function in the strip P and apply the three line inequality. We define

$$a(z) = \frac{1-z}{p_0} + \frac{z}{p_1}, \quad b(z) = \frac{1-z}{q_0'} + \frac{z}{q_1'}.$$

Such that $a(0) = p_0^{-1}, a(t) = p_t^{-1}, a(1) = p_1^{-1}$ and $b(0) = q_0'^{-1}, b(t) = q_t'^{-1}, b(1) = q_1'^{-1}$. Now we fix t and let

$$f_z(x) = \sum_j |c_j|^{a(z)p_t} e^{i\gamma_j} \chi_{E_j}(x), \quad \text{and} \quad g_z(y) = \sum_k |d_k|^{b(z)q_t'} e^{i\delta_k} \chi_{F_k}(y).$$

Clearly $f_t = f$ and $g_t = g$. Furthermore, let

$$F(z) = \int_Y (Tf_z)g_z\,d\nu = \sum_{j,k} |c_j|^{a(z)p_t} |d_k|^{b(z)q_t'} e^{i(\gamma_j+\delta_k)} C_{jk}.$$

Then $F(z)$ is a holomorphic function of z. When $\mathrm{Re}(z) \in [0,1]$, we know that $\mathrm{Re}(a(z))$ and $\mathrm{Re}(b(z))$ are bounded and therefore $F(z)$ is bounded. When $\mathrm{Re}(z) = 0$ we have $a(is) = p_0^{-1} + is(p_1^{-1} - p_0^{-1})$ and $b(s) = (q_0')^{-1} + is((q_1')^{-1} - (q_0')^{-1})$. Then

$$|f_{is}| = |f|^{p_t/p_0}, \quad |g_{is}| = |g|^{q_t'/q_0'}.$$

Then we can estimate $|F(is)| = \left|\int (Tf_{is} g_{is}\,d\nu\right|$ applying the Hölder inequality

$$|F(is)| \le \|Tf_{is}\|_{q_0} \|g_{is}\|_{q_0'} \le M_0 \|f_{is}\|_{p_0} \|g_{is}\|_{q_0'} = M_0 \|f\|_{p_t} \|g\|_{q_t'} = M_0.$$

Similarly, when $z = 1 + is$ we have

$$|f_{1+is}| = |f|^{p_t/p_1}, \quad |g_{1+is}| = |g|^{q_t'/q_1'}.$$

Then $|F(1+is)| \le M_1$. Then

$$\left|\int_Y (Tf)g\,d\nu\right| = |F(t)| \le M_0^{1-t} M_1^t.$$

□

Using the Riesz–Thorin interpolation theorem we can simplify the prove of the boundedness of an integral operator in Theorem 3.1.

5.4. **An application.** Let f, g be measurable functions on $(\mathbb{R}, \mathcal{L}, m)$. Suppose that for some $y \in \mathbb{R}$ we have $\int_{\mathbb{R}} |f(x)g(y-x)| dm(x) < \infty$ then we define

$$(f * g)(y) = \int_{\mathbb{R}} f(x)g(y-x)dm(x).$$

The function $f * g$ is called the convolution of f and g. A simple change of variables $x - y \to y$ shows that $f * g = g * f$. The Hölder inequality implies that if $f \in L^p$ and $g \in L^{p'}$ then $f * g$ is defined everywhere and $\|f * g\|_\infty \leq \|f\|_p \|g\|_{p'}$. We use the interpolation theorem to give a generalization of this inequality.

Proposition 5.1 (Young's inequality)**.** *Suppose that* $1 \leq p, q, r \leq \infty$ *and* $p^{-1} + q^{-1} = 1 + r^{-1}$. *If* $f \in L^p$ *and* $g \in L^q$ *then* $f * g$ *is defined and* $f * g \in L^r$. *Moreover*

$$\|f * g\|_r \leq \|f\|_p \|g\|_q.$$

Proof. We fix $g \in L^q$ and consider a linear operator $Tf = f * g$, we know that it is defined on $L^{q'}$ and $\|Tf\|_\infty \leq \|f\|_{q'} \|g\|_q$. Now suppose that $f \in L^1$ then by Minkowski's inequality

$$\left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(x)||g(x-y)| dm(x) \right)^q dm(y) \right)^{1/q} \leq \int_{\mathbb{R}} |f(x)| \|g\|_q \, dm(x) = \|f\|_1 \|g\|_q.$$

Thus $T : L^{q'} \to L^\infty$ and $T : L^1 \to L^q$ with estimates $\|Tf\|_\infty \leq \|f\|_{q'}$ and $\|Tf\|_q \leq \|f\|_1$. We apply the Riesz–Thorin theorem to T and obtain that $\|Tf\|_r \leq \|f\|_p$ when $1 \leq p \leq q'$ and $1/r = (1-t)/\infty + t/q$ while $1/p = (1-t)/q' + t$. We see that $1/p = t/q + 1/q' = 1/r + 1/q'$ and $1/p + 1/q = 1 + 1/r$. Moreover we can obtain the inequality for any $p \in [1, q']$ by choosing t appropriately. $\square$

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Lecture 15, Following Folland, ch 6.5

6. Marcinkiewicz Interpolation theorem

6.1. **Weak L^p spaces.** The Chebyshev inequality implies that when $f \in L^p(\mu)$, we have $\lambda_f(t) \le t^{-p}\|f\|_p^p$.

Definition 6.1. *Let f be a measurable function on $(X, \mathcal{M}, \mu)$ and let $\lambda_f(t)$ be its distribution function, $\lambda_f(t) = \mu(\{|f| > t\})$. We say that f belongs to* **weak $L^p(\mu)$-space** *if*

$$[f]_p = \left(\sup_{t>0} t^p \lambda_f(t)\right)^{1/p}$$

is finite.

Clearly if $f \in L^p(\mu)$ then f in weak L^p and $[f]_p \le \|f\|_p$. Moreover if f and g are in weak $L^p(\mu)$ then $f+g$ is also in weak $L^p(\mu)$. However $[f+g]_p$ is not a norm. A standard example of a function in weak $L^p(\mathbb{R}, m)$ that is not in $L^p(\mathbb{R}, m)$ is $f(x) = x^{-1/p}$.

We will not distinguish between L^∞ and weak L^∞. In terms of the distribution function, $f \in L^\infty$ if $\lambda_f(t) = 0$ for $t > t_0$ and the smallest such t_0 is the norm $\|f\|_\infty$.

Definition 6.2. *We say that a map T which sends measurable functions on $(X, \mathcal{M}, \mu)$ to measurable functions on $(Y, \mathcal{N}, \nu)$ is of* **weak type** *(p, q) if for any $f \in L^p(\mu)$ the image Tf is in weak $L^q(\nu)$ and $[Tf]_q \le C\|f\|_p$.*

6.2. **Marcinkiewicz interpolation.** We will now prove our second interpolation theorem. It can be applied to a large class of maps and not only to linear operators, however there are additional restrictions on the order of p and q in this result.

Definition 6.3. *Let $\mathcal{F}$ be a linear subset of measurable functions on $(X, \mathcal{M}, \mu)$ such that $\mathcal{F}$ contains all finite linear combinations of characteristic functions of sets of finite measure and also if $f \in \mathcal{F}$ and $C > 0$ then $\min\{f, C\}$ is also in $\mathcal{F}$. We say that a map T from $\mathcal{F}$ to measurable functions on $(Y, \mathcal{N}, \nu)$ is* **sublinear** *if*

$$(i)\ |T(af)(y)| = a|Tf(y)|, \quad (ii)|T(f_1 + f_2)(y)| \le |Tf_1(y)| + |Tf_2(y)|.$$

Theorem 6.1. *Suppose that T is a sublinear map such that*

$$[Tf]_{q_j} \le C_j \|f\|_{p_j}$$

for $f \in L^{p_j}(X) \cap \mathcal{F}$ and $j = 0, 1$, where $q_0 \neq q_1$ and $p_j \leq q_j$. Then $\|Tf\|_{q_t} \leq C_t \|f\|_{p_t}$, for any $f \in \mathcal{F} \cap l^{p_t}(\mu)$, where $0 < t < 1$ and

$$\frac{1}{p_t} = \frac{1-t}{p_0} + \frac{t}{p_1}, \quad \frac{1}{q_t} = \frac{1-t}{q_0} + \frac{t}{q_1}.$$

We will prove it for the case $p_0 = q_0$ and $p_1 = q_1$. In the proof we work with distribution functions on two spaces $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ we continue to denote the first distribution function of f on X (corresponding to the measure μ) by λ_f and denote the second one (of a function g on Y corresponding to the measure ν) by κ_g.

Proof. Let $f \in \mathcal{F} \cap L^{p_t}$, we want to estimate the distribution function $\kappa_{Tf}(t)$. Assume that $p_0 < p_1$ (for the case we consider $p_0 \neq p_1$) and let first $p_1 < \infty$.

We fix $t > 0$ and decompose f into sum of two functions $f = f_0 + f_1$, where

$$f_0 = \begin{cases} 0, \ |f| \leq At \\ f, \ |f| > At \end{cases}, \quad f_1 = \begin{cases} f, \ |f| \leq At \\ 0, \ |f| > At \end{cases}.$$

By our assumption $f_1, f_2 \in \mathcal{F}$ and $|Tf| \leq |Tf_1| + |Tf_2|$. Then

$$\kappa_{Tf}(t) \leq \kappa_{Tf_0}(t/2) + \kappa_{Tf_1}(t/2).$$

We note that $f_1 \in L^{p_0} \cap \mathcal{F}$ and $f_2 \in L^{p_1} \cap \mathcal{F}$ since $p_0 \leq p_t \leq p_1$. Further,

$$\lambda_{f_0}(s) = \begin{cases} \lambda_f(At), \ s < At \\ \lambda_f(s), \quad s > At \end{cases}, \quad \lambda_{f_1}(s) = \begin{cases} \lambda_f(s) - \lambda_f(At), \ s < At \\ 0, \quad s > At \end{cases}.$$

Applying the weak estimate for T in L^{p_0} we get

$$\kappa_{Tf_0}(t/2) \leq C_0^{p_0} 2^{p_0} t^{-p_0} \|f_0\|_{p_0}^{p_0} = (2C_0)^{p_0} t^{-p_0} \int_0^\infty p_0 s^{p_0 - 1} \lambda_{f_0}(s) ds.$$

Using the formula for λ_{f_0}, we get

$$\kappa_{Tf_0}(t/2) \leq (2C_0)^{p_0} t^{-p_0} \left((At)^{p_0} \lambda_f(At) + \int_{At}^\infty p_0 s^{p_0 - 1} \lambda_f(s) ds \right).$$

On the other hand for $f_1 \in L^{p_1}$ we get

$$\kappa_{Tf_1}(t/2) \leq (2C_1)^{p_1} t^{-p_1} \int_0^{At} p_1 s^{p_1 - 1} \lambda_f(s) ds.$$

Thus for any $t > 0$ we obtain

$$\kappa_{Tf}(t) \leq (2C_0)^{p_0} t^{-p_0} \left((At)^{p_0} \lambda_f(At) + \int_{At}^\infty p_0 s^{p_0 - 1} \lambda_f(s) ds \right) \\ + (2C_1)^{p_1} t^{-p_1} \int_0^{At} p_1 s^{p_1 - 1} \lambda_f(s) ds.$$

We forget about our decomposition $f = f_0 + f_1$ after we obtained this inequality and start to vary t.

Now we mulpily the last inequality by pt^{p-1} and integrate it,

$$\int_0^\infty pt^{p-1}\kappa_{Tf}(t)dt \leq (2C_0A)^{p_0}A^{-p}\int_0^\infty ps^{p-1}\lambda_f(s)ds$$
$$+\frac{(2C_0)^{p_0}A^{p_0-p}}{p-p_0}\int_0^\infty ps^{p-1}\lambda_f(s)ds + \frac{(2C_1)^{p_1}A^{p_1-p}}{p_1-p}\int_0^\infty ps^{p-1}\lambda_f(s)ds.$$

This implies $\|Tf\|_{L^p(\nu)} \leq C\|f\|_{L^p(\mu)}$. To minimize the constant we should choose A in an appropriate way. We see that C blows up when p approaches p_0 or p_1, this is natural as we assumed only weak inequalities at the end points.

Let us now consider the case $p_1 = \infty$ then we choose $A < 1/2$ and conclude that with decomposition $f = f_1 + f_2$ as above, we have $\kappa_{Tf_1}(t/2) = 0$. Thus we still get the estimate $\|Tf\|_p \leq C\|f\|_p$ for some C. □

6.3. **An application: Maximal function.** We remind that for $f \in L^1_{loc}(\mathbb{R}^n)$ we defined the maximal function Mf by

$$Mf(x) = \sup_{r>0}\frac{1}{m(B(x,r))}\int_{B(x,r)}|f(y)|\,dm(y).$$

We claim that $f \mapsto Mf$ is a sublinear map. Clearly $M(cf) = |c|M(f)$, also

$$M(f+g)(x) \leq \sup_{r>0}\frac{1}{m(B(x,r))}\int_{B(x,r)}|f(y)| + |g(y)|\,dm(y) \leq Mf(x) + Mg(x).$$

If $f \in L^\infty(\mathbb{R}^n, m)$ then $Mf(x) \leq \|f\|_\infty$ and as we proved earlier for $f \in L^1(m)$ we have

$$m(\{Mf > t\}) \leq Ct^{-1}\|f\|_1.$$

Then the Marcinkiewicz interpolation theorem implies that for $f \in L^p(\mathbb{R}^n)$ with $1 < p \leq \infty$ we have $Mf \in L^p$ and $\|Mf\|_p \leq C_p\|f\|_p$. Examining the proof of the interpolation ytheorem above, we see that $C_p \leq C(n)p/(p-1)$.

6.4. **Fractional integration.** Let f on $\mathbb{R}^m$, we define the convolution of f and $|x|^{\alpha-m}$ with $0 < \alpha < m$ as

$$I_\alpha f(x) = \int |x-y|^{\alpha-m}f(y)dy,$$

when the integral of the absolute value is finite.

Such operators appear naturally. For example, when $\alpha = 2$, $m \geq 3$ and f is a bounded function with compact support, we have $\Delta(I_2 f) = c_m f$.

Theorem 6.2. *Let $p > 1$ and $\alpha < m/p$, then the following inequality holds*

$$\|I_\alpha f\|_q \le C\|f\|_p, \quad when \quad \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{m}.$$

Proof. By the Marcinkiewicz interpolation theorem, to prove the inequality for all $p > 1$ and q such that $1/q = 1/p - \alpha/m$, it is enough to prove the weak type inequality for such p and q for two different p_1 and p_2 and then interpolate. Fix $\lambda > 0$ and let $k_1 = |x|^{\alpha-m}$ when $|x| < R = R(\lambda)$ and zero otherwise and $k_2(x) = |x|^{\alpha-m}$ when $|x| \ge R$ and zero otherwise. We have $I_\alpha f = k_1 * f + k_2 * f$. First we estimate $\|k_2 * f\|_\infty$, applying the Hölder inequality.

$$|k_2 * f(x)| \le \|f\|_p \|k_2\|_{p'}.$$

We have

$$\int |k_2|^{p'} dx = \int_R^\infty r^{m-1+(\alpha-m)p'} dr = cR^{m+(\alpha-m)p'} = cR^{m+(\alpha-m)p'},$$

if $m < (m-\alpha)p'$, that follows from the condition $\alpha < m/p$. We choose $R = R(\lambda)$ such that

$$cR^{m/p'+\alpha-m}\|f\|_p = cR^{\alpha-m/p}\|f\|_p = \lambda/2.$$

Then $\|f * k_2\|_\infty \le \lambda/2$. Now we look at the set $\{|f * k_1| > \lambda/2\}$. If we show that $f * k_1$ is in L^p we would estimate the measure of this set by $(2/\lambda)^p \|f * k\|_p^p$. Indeed,

$$\|k_1 * f\|_p \le \|k_1\|_1 \|f\|_p = \int_{|x|<R} |x|^{\alpha-m} dx \|f\|_p = c\int_0^R r^{\alpha-1} dr \|f\|_p = cR^\alpha \|f\|_p.$$

Thus $|\{|I_\alpha f| > \lambda\}| \le C\lambda^{-p} R^{p\alpha} \|f\|_p^p$. We have

$$R^{p\alpha} = (R^{\alpha-m/p})^{p^2\alpha/(p\alpha-m)} = c\lambda^{p^2\alpha/(p\alpha-m)} \|f\|_p^{-p^2\alpha/(p\alpha-m)}.$$

Finally, we obtain

$$|\{|I_\alpha * f| > \lambda\}| \le c\lambda^{-pm/(m-\alpha p)} \|f\|_p^{pm/(m-p\alpha)}$$

and recall that $q = (1/p - \alpha/m)^{-1} = pm/(m-\alpha p)$. Therefore $[I_\alpha f]_q \le C\|f\|_p$. $\square$