MAT205a, Fall 2019 Part V:

Lecture 16, Following Folland, ch 7.1

1. Positive functionals on $C_0(\mathbb{R}^n)$

In this part of the lecture notes (in contract to chapter 7 of Folland's book) we consider only the case $X = \mathbb{R}^n$.

1.1. **Preliminary lemmas.** First if f is a continuous function on $\mathbb{R}^n$ we consider the largest open set O such that $f = 0$ on O and define $supp(f) = O^c$, $supp(f)$ is a closed set that is called the support of f. We denote by $C_0(\mathbb{R}^n)$ the space of continuous functions on $\mathbb{R}^n$ with compact support.

The following lemma is relatively simple on $\mathbb{R}^n$ and holds on a class of topological spaces (so called locally convex Hausdorff spaces). It is crucial in our study of the dual to $C_0(\mathbb{R}^n)$.

Lemma 1.1 (Uryson). *Let $K \subset U \subset \mathbb{R}^n$, where K is compact and U is open. There exist $f \in C(\mathbb{R}^n)$ such that $0 \le f \le 1$, $f(x) = 1$ when $x \in K$ and $supp(f) \subset U$.*

Proof. Since K is compact and U is open and $K \in U$ there exists d such that $|x - y| < d$ when $x \in K$ and $y \notin U$. Define $d_K(y) = \inf_{x\in K} |x - y|$, we have $C = \{y \in \mathbb{R}^n : d_K(y) \le d\} \subset U$ and C is compact. Let

$$f(y) = \begin{cases} 1 - d_K(y)d^{-1}, \text{ if } d_K(y) < d \\ 0, \text{ otherwise} \end{cases}.$$

Clearly $|f(x) - f(y)| \le |x - y|$ so f is continuous and $0 \le f \le 1$, $f = 1$ on K and $supp(f) \subset C \subset U$. □

A stronger statement holds in $\mathbb{R}^n$, there is such function $f \in C^\infty(\mathbb{R}^n)$. To construct it first note that there exists a function $g \in C^\infty(\mathbb{R})$ such that $g(x) = 0$ when $|x| > 1$, define

$$g(x) = \begin{cases} \exp(-(1 - x^2)^{-1}), & -1 < x < 1, \\ 0, & |x| \ge 1 \end{cases}.$$

Now define the function g_d on $\mathbb{R}^n$ by $g_d(x) = g(2d^{-1}|x|)$, we get $g(x) = 0$ when $|x| > d/2$. Let also $V = \{y \in \mathbb{R}^n : d_K(y) < d/2\}$ Finally, define

$$f(x) = \chi_V * g_d(x) = \int_V g_d(x - y)\, dm(y).$$

This convolution is a C^∞ function since $g_d \in C^\infty(\mathbb{R}^n)$, moreover $0 \le f \le 1$, $f = c = \int g_d\, dm$ on K and $supp(f) \subset C$.

Lemma 1.2 (Partition of unity)**.** *Suppose that $K \subset \mathbb{R}^n$ is a compact set and U_j are open set such that $K \subset \bigcup_1^m U_j$. Then there exist functions $g_j \in C_0(\mathbb{R}^n)$ such that $supp(g_j) \subset U_j$, $0 \le g_j \le 1$, and $\sum_1^m g_j(x) = 1$ for any $x \in K$.*

Proof. For each $x \in K$ there is a ball B_x centered at x such that $\overline{B}_x \subset U_j$ for some $j \in \{1, 2, ..., m\}$. We take a finite cover $K \subset \bigcup_l B_{x_l}$. Now we let $F_j = \bigcup \overline{B}_{x_l}$ where the union is taken over balls that are subsets of U_j. Such that F_j is a compact set, $F_j \subset U_j$ and $\bigcup_1^m F_j \supset K$.

By Uryson's lemma there exist functions $f_j \in C_0(\mathbb{R}^n)$ such that $supp(f_j) \subset U_j$, $0 \le f_j \le 1$ and $f_j = 1$ on F_j. Then $O = \bigcup_j \{f_j > 0\}$ is an open set and $K \subset O$. We apply the Uryson lemma once again to find a function $f \in C_0(\mathbb{R}^n)$ such that $f = 1$ on K and $supp(f) \subset O$. Now let $h = (1 - f) + \sum_1^m f_j$. Then $h > 0$ on $\mathbb{R}^n$. Finally we define $g_j = f_j h^{-1}$. Then $g_j \in C_0(\mathbb{R}^n)$ and $supp(g_j) \subset U_j$, and $\sum_1^m g_j(x) = h^{-1}(x)(h(x) + f(x) - 1)$ and when $x \in K$ we have $f(x) = 1$ and $\sum_1^m g_j(x) = 1$. □

1.2. **Positive functional is bounded.** The aim of this part of the course is to understand the dual of the space $C_0(\mathbb{R}^n)$. We will start but looking at positive functionals on this space.

Definition 1.1. *A linear functional I on $C_0(\mathbb{R})$ is called* **positive** *if $I(f) \ge 0$ for any function $f \in C_0(\mathbb{R}^n)$ such that $f \ge 0$.*

Proposition 1.1. *If I is a positive functional on $C_0(\mathbb{R}^n)$ then for any compact subset K of $\mathbb{R}^n$ there exists C_K such that $|If| \le C_K \|f\|_\infty$ when $supp(f) \subset K$.*

Proof. Let $g_K \in C_0(\mathbb{R}^n)$ be such that $0 \le g \le 1$ and $g = 1$ on K. Such function exists by the Uryson lemma. Now let f be a real-valued function with $supp(f) \in K$. Then $\|f\|_\infty g - f \ge 0$. Thay $I(\|f\|_\infty g - f) \ge 0$ and by the linearity of I we have $I(f) \le I(g)\|f\|_\infty$. □

Remind that a Borel measure μ on $\mathbb{R}^n$ is called Borel if $\mu(K) < \infty$ for any compact set and μ. satisfies the following regularity properties

$$\mu(E) = \inf\{\mu(U) : U \supset E, U \text{ is open}\}$$

for any $E \in \mathcal{B}$ and

$$\mu(G) = \inf\{\mu(K) : K \subset G, G \text{ is compact}$$

for any open $G \subset \mathbb{R}^n$. If μ is a Borel measure then $I_\mu(f) = \int_{\mathbb{R}^n} f \, d\mu$ is a positive linear functional on $C_0(\mathbb{R}^n)$. Our aim is to prove that these are all possible positive functionals on $C_0(\mathbb{R}^n)$.

1.3. **Riesz representation theorem for positive functionals.** First we will show that two distinct Radon measures define different functionals.

Lemma 1.3. *Suppose that* μ_1 *and* μ_2 *are Radon measures on* $\mathbb{R}^n$ *and* $\int f\,d\mu_1 = \int f\,d\mu_2$ *for any* $f \in C_0(\mathbb{R}^n)$*. Then* $\mu_1(E) = \mu_2(E)$*for any* $E \in \mathcal{B}$ *.*

Proof. Let $U \subset \mathbb{R}^n$ be an open set, we define

$$\nu(U) = \sup\left\{ \int f\,d\mu_1 : f \in C_0(\mathbb{R}^n), 0 \le f \le 1, supp(f) \subset U \right\}.$$

Clearly, $\nu(U) \le \mu_1(U)$. On the other hand for any $K \subset U$ compact by the Uryson lemma there is a function f with $supp(f) \subset U$ such that $0 \le f \le 1$ and $f = 1$ on K. Thus $\mu_1(K) \le \nu(U)$. Since μ_1 is a regular measure, we conclude that $\mu_1(U) \le \nu(U)$. Thus for open sets $\nu(U) = \mu_1(U)$. the second regularity condition guarantees that $\nu(E) = \mu_1(E)$ for any Borel set E. But the definition of ν is the same if μ_1 is replaced by μ_2. Thus $\mu_1(E) = \mu_2(E)$. □

Now we are ready to prove the first Riesz representation theorem.

Theorem 1.1. *If* I *is a positive linear functional on* $C_0(\mathbb{R}^n)$ *then there exists a Radon measure* μ *such that* $I(f) = \int f\,d\mu$*.*

Proof. For an open set U we define

$$\mu(U) = \sup\left\{ I(f) : f \in C_0(\mathbb{R}^n), 0 \le f \le 1, supp(f) \subset U \right\}.$$

Then for open sets $U \subset V$ by the definition of μ we have $\mu(U) \le \mu(V)$. For any $E \subset \mathbb{R}^n$ let

$$\mu^*(E) = \inf\{\mu(U) : U \supset E, U \text{ is open}\}.$$

We have also that $\mu^*(U) = \mu(U)$ for any open set U.

The proof consists of several steps.

Step 1 μ^* *is an outer measure on* $\mathbb{R}^n$*.* Clearly $\mu(\emptyset) = 0$, the general construction of an outer measure from a non-negative function defined on a family of sets states that

$$\tilde{\mu}^*(E) = \inf\{\sum_1^\infty \mu(U_j) : E \subset \bigcup_j U_j, U_j \text{ are open}\}$$

is an outer measure. We want to show that $\mu^*(E) = \tilde{\mu}^*(E)$, clearly $\tilde{\mu}^*(E) \le \mu^*(E)$. It is enough to show that if $U = \bigcup_1^\infty U_j$ and U_j are open then $\mu(U) \le \sum_j \mu(U_j)$. Let $0 \le f \le 1$, $f \in C_0(\mathbb{R}^n)$ and $K = supp(f) \subset U$, then $K \subset \bigcup_1^k U_j$. We can find a partition of unity $\{g_j\}_1^k$ such that $\sum_j g_j = 1$ on K and $supp(g_j) \subset U_j$, moreover $0 \le g_j \le 1$. Then for each j we have $0 \le fg_j \le 1$ and $supp(fg_j) \subset U_j$. This implies

$$I(f) = I(f\sum_j g_j) = \sum_j I(fg_j)\,d\mu \le \sum_j \mu(U_j).$$

Taking the supremum over all f with $supp(f) \in U$ we get $\mu(U) \leq \sum_j \mu(U_j)$.
Step 2 *Open sets are μ^*-measurable.* Let U be an open set. We want to show that for any $E \subset \mathbb{R}^n$ we have

$$\mu^*(E) = \mu^*(E \cap U) + \mu^*(E \cap U^c).$$

We know from the fact that μ^* is an outer measure that $\mu^*(E) \leq \mu^*(E \cap U) + \mu^*(E \cap U^c)$. To prove the opposite inequality first let V be an open set and assume that $\mu^*(V) = \mu(V) < \infty$. Then $V \cap U$ is open and $\mu(U \cap V) < \infty$. For any $\varepsilon > 0$ here is a function f such that $0 \leq f \leq 1$, $K = supp(f) \subset V \cap U$ and $I(f) > \mu(U \cap V) - \varepsilon$. We have $V \cap U^c \supset V \setminus K$ and $V \setminus K$ is an open set. We can also find $g \in C_0(\mathbb{R}^n)$ with $supp(g) \subset V \setminus K$ such that $0 \leq g \leq 1$ and $I(g) > \mu(V \setminus K) - \varepsilon$. Then $f + g \in C_0(\mathbb{R}^n)$, $f + g \leq 1$ and $supp(f + g) \subset V$. We have

$$\mu(V) \geq I(f+g) = I(f)+i(g) > \mu(U\cap V)+\mu(V\setminus K)-2\varepsilon > \mu^*(V\cap U)+\mu^*(V\cap U^c)-2\varepsilon.$$

We let $\varepsilon \to 0$ and conclude that $\mu^*(V) = \mu^*(V \cap U0 + \mu^*(V \cap U^c)$. Now for general $E \subset \mathbb{R}^n$ and $\varepsilon > 0$ there is an open set V such that $E \subset V$ and $\mu^*(E) \geq \mu(V) - \varepsilon$. Then

$$\mu^*(E) \geq \mu(V) - \varepsilon = \mu^*(V \cap U) + \mu^*(V \cap U^C) - \varepsilon \geq \mu^*(E \cap U) + \mu^*(E \cap U^c) - \varepsilon.$$

Since it holds for any $\varepsilon > 0$ we see that $\mu^*(E) = \mu^*(E \cap U) + \mu^*(E \cap U^c)$ and U is μ^*-measurable.

Step 2 implies that all Borel sets are μ^*-measurable, thus we have a measure μ on $(\mathbb{R}^n, \mathcal{B})$ and on open sets this measure is given by

$$\mu(U) = \sup \left\{ I(f) : f \in C_0(\mathbb{R}^n), 0 \leq f \leq 1, supp(f) \subset U \right\}.$$

We now want to show that on compact sets it satisfies

$$\mu(K) = \inf\{I(f) : f \in C_0(\mathbb{R}^n), f \geq \chi_K\}.$$

Step 3 *The formula above holds for all compact sets K.* Let $f \in C_0(\mathbb{R}^n)$ and $f \geq \chi_K$. For every $\varepsilon \in (0, 1)$ we define $U_\varepsilon = \{f > 1 - \varepsilon\}$. Then $K \subset U_\varepsilon$. If now $g \in C_0(\mathbb{R}^n)$, $0 \leq g \leq 1$ and $supp(g) \subset U_\varepsilon$ then $(1 - \varepsilon)g < f$ and since I is a positive linear functional we see that $(1 - \varepsilon)I(g) \leq I(f)$. Taking the supremum over all g we get $(1 - \varepsilon)\mu(U_\varepsilon) \leq I(f)$ and $(1 - \varepsilon)\mu(K) \leq I(f)$. Thus we have one inequality $\mu(K) \leq \inf\{I(f), f \in C_0(\mathbb{R}^n), f \geq \chi_K\}$. To prove the opposite inequality we take U open and such that $K \subset U$ and by Uryson lemma find a function $f \in C_0(\mathbb{R}^n)$ such that $f \geq \chi_K$ and $0 \leq f \leq 1$, $supp(f) \subset U$. Then $I(f) \leq \mu(U)$ and then the opposite inequality holds: $\mu(K) \geq \inf\{I(f), f \in C_0(\mathbb{R}^n), f \geq \chi_K\}$.
Step 4 $I(f) = I_\mu(f)$. Finally we want to show that I is defined by integration with

respect to μ. Let $f \in C_0(\mathbb{R}^n)$ and $0 \leq f \leq 1$, consider $K_a = \{f \geq a\}$. Fix N and let define

$$f_j(x) = \begin{cases} 0 & \text{if} \quad f(x) < (j-1)/N \\ f(x) - (j-1)/N & \text{if} \quad (j-1)/N \leq f(x) < j/N \\ N^{-1} & \text{if} \quad f(x) \geq j/N \end{cases} .$$

Then $f(x) = \sum_{j=1}^{N} f_j$, $\chi_{K_{(j/N)}} \leq N f_j \leq \chi_{K_{(j-1)/N}}$. Then we have

$$\mu(K_{j/N}) \leq N \int f_j \, d\mu \leq \mu(K_{(j-1)/N}).$$

On the other hand, $NI(f_j) \geq \mu(K_{j/N})$ by Step 3. Moreover if U is open and $K_{(j-1)/N} \subset U$ then $NI(f_j) \leq \mu(U)$ and therefore $NI(f_j) \leq \mu(K_{(j-1)/N}$. Summing up the inequalities we get

$$\left| I(f) - \int f \, d\mu \right| \leq N^{-1} \mu(K_0).$$

Since $f \in C_0(\mathbb{R}^n)$ and μ is finite on compact sets, we conclude that $I(f) = I_\mu(f)$. $\square$

MAT205a, Fall 2019 Part V:

Lecture 17, Following Folland, ch 7.2,7.3

2. Bounded functionals $C_0(\mathbb{R}^n)$

2.1. **The closure of $C_0(\mathbb{R}^n)$.** Let $\bar{C}_0(\mathbb{R}^n)$ denote the closure of the space of continuous functions with compact support in the uniform norm $\|f\|_\infty$. It is easy to see that $\bar{C}_0(\mathbb{R}^n)$ is the space of continuous functions that tend to zero at infinity. If I is a bounded linear functional on $C_0(\mathbb{R}^n)$, $|If| \le C\|f\|_\infty$, then I uniquely extends to a linear functional on $\bar{C}_0(\mathbb{R}^n)$.

Note that for example integration with respect to the Lebesgue measure is a positive functional on $C_0(\mathbb{R}^n)$ but it is not bounded and it can not be extended to a functional on $\bar{C}_0(\mathbb{R}^n)$.In this section we show that bounded functionals on $\bar{C}_0(\mathbb{R}^n)$ correspond to finite signed measures.

We remind that if μ is a signed Borel measure on $\mathbb{R}^n$ then $\mu = \mu^+ - \mu^-$ where μ^+, μ^- are (positive) measures and $\mu^+ \perp \mu^-$. We write $|\mu| = \mu^+ + \mu^-$ and say that μ is finite if $\|\mu\| = |\mu|(\mathbb{R}^n) < \infty$. We say that μ is a radon signed measure if both μ^+ and μ^- are radon measures. It is equivalent to the statement that $|\mu|$ is a Radon measure. Let $M(\mathbb{R}^n)$ denote the space of all finite signed radon measures on $\mathbb{R}^n$. Clearly it is a linear space. Moreover $\|\mu\|$ is a norm on this space.

2.2. **Bounded functional as the difference of two positive ones.** We already know that positive functional correspond to measures. Now we decompose any bounded functional into the sum of two positive ones.

Lemma 2.1. *Suppose that I is a linear bounded functional on $\bar{C}_0(\mathbb{R}^n)$, then there exist positive functionals I^+ and I^- such that $I = I^+ - I^-$.*

Proof. For a non-negative function $f \in \bar{C}_0(\mathbb{R}^n)$ we define

$$I^+(f) = \sup\{I(g) : 0 \le g \le f, g \in \bar{C}_0(\mathbb{R}^n)\}.$$

Then since $I(0) = 0$ and $|I(g)| \le C\|g\|_\infty$ we conclude that $0 \le I^+(f) \le C\|f\|_\infty$. Moreover, since I is linear, we see that $I^+(cf) = cI^+(f)$ when $c > 0$ and if $f = f_1 + f_2$ where $f_1, f_2 \in \bar{C}_0(\mathbb{R}^n)$ and $f_1, f_2 \ge 0$ then for any g_1, g_2 such that $0 \le g_1 \le f_1$, $0 \le g_2 \le f_2$ we have $0 \le g_1 + g_2 \le f$ and therefore $I^+(f) \ge I^+(f_1) + I^+(f_2)$. On the other hand for $g \in \bar{C}_0(\mathbb{R}^n)$ such that $0 \le g \le f$ we can define $g_1 = \min\{g, f_1\}$ and $g_2 = \min\{g, f_2\}$. Then $0 \le g_1 \le f_1$, $0 \le g_2 \le f_2$ and $g_1 + g_2 \ge g$. It implies that $I(g) \le I(g_1) + I(g_2)$ and then we get $I^=(f) \le I^+(f_1) + I^+(f_2)$. So I^+ is linear on non-negative functions. For general $f \in \bar{C}_0(\mathbb{R}^n)$ we have $f = f^+ - f^-$ where

$f^+, f^- \geq 0$ and define $I^+(f) = I^+(f^+) - I^+(f^-)$. Clearly I^+ is a positive functional, it is linear and bounded since

$$|I^+(f)| \leq \max\{I^+(f^+), I^+(f^-)\} \leq C\|f\|_\infty.$$

Finally let $I^-(f) = I^+(f) - I(f)$, then for $f \geq 0$ we have $I^-(f) \geq 0$ so I^- is also a linear positive and bounded functional. □

2.3. **Riesz representation theorem.** Now we can describe all bounded functionals on $\bar{C}_0(\mathbb{R}^n)$.

Theorem 2.1. *For any $\mu \in M(\mathbb{R}^n)$ $I_\mu(f) = \int f\, d\mu$ is a bounded linear functional on $\bar{C}_0(\mathbb{R}^n)$ and for any bounded linear functional functional I there is $\mu \in M(\mathbb{R}^n)$ such that $I = I_\mu$.*

Proof. First if μ is a bounded signed Radon measure then $I_\mu(f)$ is a linear functional. Moreover,

$$|I_\mu(f)| = |I_{\mu^+}(f) - I_{\mu^-}(f)| \leq |I_{\mu^+}(f)| + |I_{\mu^-}(f)| \leq \|f\|_\infty(\mu^+(\mathbb{R}^n) + \mu^-(\mathbb{R}^n)) = \|\mu\|\|f\|_\infty.$$

Thus $\|I_\mu\| \leq \|\mu\|$.

Now if I is a bounded linear functional on $\bar{C}_0(\mathbb{R}^n)$ then we have $I = I^+ - I^-$, where I^+ and I^- are positive. Then there are corresponding measures μ^+ and μ^-. We have $I = I_\mu$ with $\mu = \mu_1 - \mu_2$. □

2.4. **The norm of the functional.** We want to show that the norm of I_μ equals $\|\mu\|$. We will use a version of the Lusin theorem in $\mathbb{R}^n$.

Theorem 2.2. *Suppose that μ is finite Radon measure on $\mathbb{R}^n$. If f is a Borel measurable function then for any $\varepsilon > 0$ there exists a compact set $K \subset \mathbb{R}^n$ such that $\mu(\mathbb{R}^n \setminus K) < \varepsilon$ and f is continuous on K.*

We want to combine the Lusin theorem with the following extension theorem

Theorem 2.3 (Tietze)**.** *If $K \subset \mathbb{R}^n$ is compact and $f \in C(K)$ then there exists $F \in C_0(\mathbb{R}^n)$ such that $F = f$ on K and $\|F\|_\infty \leq \|f\|_\infty$.*

Proof. Suppose that $m = \max|f(x)|$. Consider two sets $E_1 = \{x \in K : f \geq m/3\}$ and $E_2 = \{x \in K : f \leq -m/3\}$. Then E_1 and E_2 are disjoint compact subsets of $\mathbb{R}^n$. Applying the Uryson lemma twice we can find a function $g \in C_0(\mathbb{R}^n)$ such that $g = m/3$ on E_1, $g = -m/3$ on E_3 and $\|g\|_\infty = m/3$. Then $|f - g| < m/3$ on K. Denote $f_1 = f - g$ and construct g_1 for f_1. We can assume that the supports of all g_k are in the same ball that contains K. Then we define $F = g + g - 1 + ...$, the series converges uniformly, thus F is continuous and F has compact support. Moreover $F = f$ on K and $\|F\|_\infty \leq \|g\|_\infty + \|g_1\|_\infty + .. \leq m$. □

Proposition 2.1. *If* $\mu \in M(\mathbb{R}^n)$ *then* $\|I_\mu\| = \sup\{|If| : \|f\|_\infty = 1, f \in \bar{C}_0(\mathbb{R}^n)\} = \|\mu\|$.

Proof. Let $\mathbb{R}^n = X^+ \cup X^-$ be the Hahn decomposition for μ. Define $h = \chi_{x^+} - \chi_{X^-}$, then $\|h\|_\infty = 1$ and $\int h\, d\mu = \|\mu\|$. We use the Lusin theorem to find a compact set K such that $|\mu|(\mathbb{R}^n \setminus K) < \varepsilon$ and h is continuous on K. Then by the Tietze extension theorem there is $F \in C_0(\mathbb{R}^n)$ such that $\|f||_\infty \leq \|h\|_\infty = 1$ and $f\mathcal{F} = h$ on K. We have

$$\|I_\mu\| \geq |I_\mu(F)| = \left| \int_K h\, d\mu + \int_{K^c} F\, d\mu \right| \geq |\mu|(K) - |\mu|(\mathbb{R}^n \setminus K) \geq \|\mu\| - 2\varepsilon.$$

□