MAT205A, FALL 2019 MIDTERM

OCTOBER 24, 9:00-10:20 AM

Problem 1. Describe a construction of a Lebesgue measurable subset E of $[0,1]$ which has positive Lebesgue measure but does not contain any non-empty open interval. Explain the details.

Problem 2. Let $(X, \mathcal{M}, \mu)$ be a measure space and $\{f_n\}$ be a sequence of measurable functions such that $f_n \in L^1(\mu)$. We consider the following statements:
(i) there exists a function $f \in L^1(\mu)$ such that $f_n \to f$ in $L^1(\mu)$,
(ii) there exists a measurable function f such that $f_n \to f$ in μ-measure.
Does (i) imply (ii)? Does (ii) imply (i)? Do any of these implications hold under an additional assumption? Justify your answers.

Problem 3. Egorov's theorem says that if $(X, \mathcal{M}, \mu)$ is a measure space with finite measure, and a sequence of measurable functions $\{f_n\}$ converges to a measurable function f a.e., then for any $\varepsilon > 0$ there is a subset $E \subset X$ such that $f_n \to f$ converges uniformly on E and $\mu(X \setminus E) < \varepsilon$.
(a) Give an example to show that the statement of the theorem may be false when $\mu(X) = \infty$.
(b) Prove the theorem for the case when $\mu(X) = \infty$ under the additional assumption that there exists $g \in L^1(\mu)$ such that $|f_n| \leq g$ for all n. (You may use Egorov's theorem.)

Problem 4. (a) Suppose that $f : X \to \mathbb{R}$ is a non-negative integrable function on a measure space $(X, \mathcal{M}, \mu)$ with a σ-finite measure μ. Show that

$$\int f d\mu = \int_0^\infty \mu\{f \geq t\} dt.$$

(b) Give an example of a measure space $(X, \mathcal{M}, \mu)$ and a non-negative measurable function $f : X \to [0, +\infty)$ such that $f \notin L^1(\mu)$ but there exists a constant C such that $\mu(\{f > t\}) \leq C t^{-1}$ for any $t > 0$.