**MAT205A, FALL 2019 MIDTERM**

OCTOBER 24, 9:00-10:20 AM

**Problem 1.** *Describe a construction of a Lebesgue measurable subset $E$ of $[0,1]$ which has positive Lebesgue measure but does not contain any non-empty open interval. Explain the details.*

*Solution 0* Let $E = [0,1] \setminus \mathbb{Q}$, then $E$ is measurable, $m(E) = 1$ and $E$ does not contain any interval.

*Solution 1* Consider all rational numbers on the interval $[0,1]$ and enumerate them $\{r_n\}_{n=1}^{\infty}$. For each $r_n$ let $I_n$ be the interval with the center at $r_n$ and length $3^{-n}$. Now we define $E = [0,1] \setminus \cup_n I_n$. Then $E$ is a measurable set, it is closed as the difference of a closed and an open sets. We have $m(E) = m([0,1]) - m(\cup_n I_n) \geq 1 - \sum_n 3^{-n} \geq 1/2$ and $E \cap \mathbb{Q} = \emptyset$. Then $E$ does not contain a non-empty open interval, because each open interval contains a rational point.

*Solution 2* We repeat the construction of the Cantor set, but vary the portion of the set which is taken away each step. Let $C_0 = [0,1]$, $C_1 = [0,3/8] \cup [5/8,1]$ it is obtained from $C_0$ by deleting the middle quarter of the interval. In general if $C_n$ is the union of $2^n$ closed intervals, $C_n = \cup J_k$, we define $C_{n+1}$ to the union of $2^{n+1}$ intervals $C_n = \cup L_k$, where $L_{2k-1}, L_{2k} \subset I_k$ and they were obtained by deleting the open middle $I_k$ part of $J_k$, where $|I_k| = |J_k|/(n+2)^2$. We see that

$$|C_{n+1} = \left(1 - \frac{1}{(n+2)^2}\right)|C_n| = \frac{(n+1)(n+3)}{(n+2)^2}|C_n| = ... = \frac{n+3}{2n+4}|C_0|.$$

Now let $C = \cap_n C_n$ it is a closed set and it does not contain any interval, since $C_n$ is a union disjoint closed intervals of length less that $2^{-n}$ and it does not contain any interval of length larger than $2^{-n}$. On the other hand, $|C| = \lim_{n\to\infty} |C_n| = 1/2$.

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**Problem 2.** *Let $(X, \mathcal{M}, \mu)$ be a measure space and $\{f_n\}$ be a sequence of measurable functions such that $f_n \in L^1(\mu)$. We consider the following statements:*
(i) *there exists a function $f \in L^1(\mu)$ such that $f_n \to f$ in $L^1(\mu)$,*
(ii) *there exists a measurable function $f$ such that $f_n \to f$ in $\mu$-measure.*
*Does (i) imply (ii)? Does (ii) imply (i)? Do any of these implications hold under an additional assumption? Justify your answers.*

*Solution* First we show that (i) implies (ii). Assume that $f_n \to f$ in $L^1(\mu)$, we know that $\mu(\{|f - f_n| > \varepsilon\}) \leq \varepsilon^{-1}\|f - f_n\|_1$ and since $\|f_n - f\|_1 \to 0$ as $n \to \infty$, we conclude that $\mu(\{|f - f_n| > \varepsilon\}$ goes to zero as $n \to \infty$ for any positive $\varepsilon$. Hence $f_n$ converges to $f$ in $\mu$-measure.

To show that (ii) does not imply (i) we give an example of a sequence of functions in $L^1(\mu)$ which converges to zero in measure but does not converge in $L^1$. Let $(X, \mathcal{M}, \mu) = (\mathbb{R}, \mathcal{B}, m)$, we define $f_n = n^2\chi_{(0,n^{-1})}$, clearly $m(\{|f_n| > \varepsilon\}) \leq n^{-1}$ for any positive $\varepsilon$ and $f_n$ converges to 0 in measure, but $\|f_n\|_1 = n$ and the sequence $\{f_n\}$ does not converge in $L^1$ since it is not bounded in $L^1$.

No we want to know under what condition convergence in measure implies convergence in $L^1$. Assume that $f_n$ is a sequence of functions that converges in measure to a function $f$. We will assume in addition that $|f_n| \leq g$ for some $g \in L^1(\mu)$. Now if $\int |f_n - f|\, d\mu$ does not converge to zero, we can find a subsequence $\{f_{n_k}\}$ such that $\int |f - f_{n_k}|\, d\mu > \varepsilon > 0$. We know that the sequence $f_{n_k} - f$ converges to zero in measure, then there is a subsequence that converges to zero almost everywhere, to simplify the notation, we assume that it is the same subsequence. Noticing that $|f_{n_k} - f| \leq 2g \in l^1(\mu)$ and applying the dominated convergence theorem to the $\{|f_{n,k} - f|\}$, we get

$$\int |f_n - f|\, d\mu = 0,$$

which contradicts to our choice of $\{f_{n_k}\}$. Therefore under the additional assumptions that all functions are dominated by one integrable function, we see that convergence in measure implies convergence in $L^1$.

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**Problem 3.** Egorov's theorem says that if $(X, \mathcal{M}, \mu)$ is a measure space with finite measure, and a sequence of measurable functions $\{f_n\}$ converges to a measurable function $f$ a.e., then for any $\varepsilon > 0$ there is a subset $E \subset X$ such that $f_n \to f$ converges uniformly on $E$ and $\mu(X \setminus E) < \varepsilon$.
(a) Give an example to show that the statement of the theorem may be false when $\mu(X) = \infty$.
(b) Prove the theorem for the case when $\mu(X) = \infty$ under the additional assumption that there exists $g \in L^1(\mu)$ such that $|f_n| \leq g$ for all $n$.

*Solution* (a) Consider the real line with the Lebesgue measure and let $f_n = \chi_{[n,n+1]}$ clearly $f_n$ converges to zero everywhere. Assume that there is a set $E$ such that $f_n$ converges to zero uniformly on $E$. Then there exists $n$ such that $|f_m| < 1$ on $E$ for any $m \geq n$. It implies that $[n, +\infty) \cap E = \emptyset$ and $m(\mathbb{R} \setminus E) = \infty$.

*Solution 1* (b) We repeat the proof of the Egorov's theorem. First, let $Y$ be the set where $\{f_n\}$ converges to $f$, $\mu(X \setminus Y) = 0$. For each $n$ and $k$ we define

$$E_{n,k} = \cup_{m \geq n} \{|f - f_m| > 1/k\}.$$

Then $E_{1,k} \supset E_{2,k} \supset ..$ for each $k$ and $\cap_n E_{n,k} \subset Y^c$. Now, since $|f_n| \leq g$ then $|f| \geq g$ on $Y^c$ and for any $m$ we have $\{|f - f_m| > 1/k\} \subset \{2|g| > 1/k\}$ and then $E_{1,k} \subset \{g > 1/(2k)\}$ and it has finite measure, as $\mu(\{g > 1/(2k)\} \leq 2k \int g \, d\mu$. We have $\lim_{n\to\infty} \mu(E_{n,k}) = 0$. We choose $n(k)$ such that $\mu(E_{n(k),k}) < \varepsilon 2^{-k}$. And let $E = \cup_k E_{n(k),k}$, so that $\mu(E) \leq \varepsilon$. On the complement of $E$ we have $|f_n - f| < 1/k$ when $n \geq n(k)$. Thus $f_n$ converges to $f$ uniformly on $E^c$.

*Solution 2* (b) First, we have $|f| \geq g$ a.e., since $f_n$ converges to $f$ a.e. Let $\varepsilon > 0$, for each $n \geq 1$ let $X_n = \{g \geq 1/n\}$, then $\mu(X_n) \leq \varepsilon^{-1} \|g\|_1 < \infty$. By Egorov's theorem there exists $F_n \subset X_n$ such that $\mu(F_n) \leq \varepsilon 2^{-n}$ and $f_n$ converges to $f$ uniformly on $X_n \setminus F_n$. Denote $F_0 = \{|f| > g\}$, such that $\mu(F_0) = 0$ and let $F = \cup_0^\infty F_n$. We have $\mu(F) \leq \varepsilon$ and we want to show that $f_n$ converges to $f$ uniformly on $X \setminus F$.

Let $\delta > 0$, we choose $N > 2\delta^{-1}$; since $f_n$ converges to $f$ uniformly on $X_N \setminus F_N$ there exists $m$ such that $|f - f_n| < \delta$ on $X_N \setminus F_N$ for $n \geq m$. We have

$$X \setminus F \subset (X_N^c \setminus F_0) \cup (X_N \setminus F_N).$$

Therefore for $n \geq m$ and $x \in X \setminus F$, we obtain $|f(x) - f_n(x)| \leq \max\{N^{-1}, \delta\} = \delta$, as $|f - f_n| \leq 2|g| \leq 2N{-}1$ on $X_N^c \setminus F_0$. We showed that $f_n$ converges to $f$ uniformly on $F^c$.

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**Problem 4.** (a) *Suppose that $f : X \to \mathbb{R}$ is a non-negative integrable function on a measure space $(X, \mathcal{M}, \mu)$ with a $\sigma$-finite measure $\mu$. Show that*

$$\int f \, d\mu = \int_0^\infty \mu\{f \geq t\} \, dt.$$

(b) *Give an example of a measure space $(X, \mathcal{M}, \mu)$ and a non-negative measurable function $f : X \to [0, +\infty)$ such that $f \notin L^1(\mu)$ but there exists a constant $C$ such that $\mu(\{f > t\}) \leq Ct^{-1}$ for any $t > 0$.*

*Soluion 1* (a) Consider the function $F : \mathbb{R}\times X \to [0,\infty)$ defined by $F(t,x) = \chi_{f\geq t}(x)$, in other words

$$F(t,x) = \begin{cases} 1, \text{ when } f(x) \geq t > 0 \\ 0, \text{ when } f(x) < t, \\ 0, \text{ when } t \leq 0. \end{cases}$$

Since $\mu$ and the Lebesgue measure are $\sigma$-finite, we can apply the Tonelli theorem to $F$. Note that

$$\int_{\mathbb{R}} F(x,t)dm(t) = \int_{(0,f(x)} 1dm(t) = f(x).$$

Therefore the double integral is equal to

$$\int_X \int_{\mathbb{R}} F(t,x)\, dm(t)\, d\mu(x) = \int_X f(x)\, d\mu(x).$$

On the other hand, if we change the order of integration, then for $t > 0$ the inner integral becomes

$$\int_X F(x,t)d\mu(x) = \int_X \chi_{\{f\geq t\}}\, d\mu = \mu(\{f \geq t\}),$$

and it is zero when $t \leq 0$. Thus the double integral is equal to

$$\int_0^\infty \mu(\{f \geq t\})dt.$$

We have shown that the two integrals are equal as required.

*Solution 2* (a) Assume first that $f$ is a non-negative simple function, $f = \sum a_k \chi_{A_k}$, where $A_k$ are disjoint measurable sets. Then $\mu(\{f > t\}) = \sum_{k:\ a_k>t} \mu(A_k)$. Therefore

$$\int_0^\infty \mu(\{f > t\}\, dt = \sum_k \int_{0<t<a_k} \mu(A_k)dt = \sum_k a_k\mu(A_k) = \int f\, d\mu.$$

Now let $\psi_n$ be an increasing sequence of simple functions that converges to $f$, by the monotone convergence theorem $\int f\, d\mu = \lim \int \psi_n\, d\mu$. Let $h(t) = \mu(\{f > t\})$ and $h_n(t) = \mu(\{\psi_n > t\})$, since $\{\psi_n\}$ is an increasing sequence of function, we get

$$\{\psi_{n+1} > t\} \supset \{\psi_n > t\} \quad \text{and} \quad \bigcup_n \{\psi_n > t\} = \{f > t\}.$$

Then $h_{n+1}(t) \geq h_n(t)$ and $\lim_{n\to\infty} h_n(t) = h(t)$. Once again we apply the monotone convergence theorem and conclude that

$$\int_0^\infty \mu(\{f > t\})dt = \int_0^\infty h\, dt = \lim_{n\to\infty} \int_0^\infty h_n dt =$$

$$\lim_{n\to\infty} \int_0^\infty \mu(\{\psi_n > t\})dt = \lim_{n\to\infty} \int \psi_n d\mu = \int f d\mu.$$

Finally, we remark that since $\mu$ is $\sigma$-finite $\mu(\{f > t\}) = \mu(\{f \geq t\}$ for almost every $t$. Indeed let $\mu = \sum_n \mu_n$, where $\mu_n(X) < \infty$. Suppose that for $t \in T_n$ we have $\mu_n(\{f = t\}) > 0$, then $\sum_{t\in T} \mu_n(\{t\} < +\infty$ and $T_n$ is countable. Therefore $T = \cup_n T_n$ is countable and has Lebesgue measure zero.

*Solution* (b) Let $(X, \mathcal{M}, \mu) = (\mathbb{R}, \mathcal{B}, m)$ and consider the function $f(x) = x^{-1}$ when $x > 0$ and $f(x) = 0$ when $x \leq 0$. Then $\int f(x)\, dx = \infty$. On the other hand $\{f > t\} = \{x : x < 1/t\}$ and $m(\{f > t\} = t^{-1}$.