MAT205A, FALL 2019

ADDITIONAL PROBLEMS

Problem 1. Let μ be a non-negative function defined on a σ-algebra $\mathcal{M}$, such that μ is finitely additive. Show that if for any increasing sequence of sets $\{E_n\}$, $E_n \in \mathcal{M}$ and $E_n \subset E_{n+1}$ we have $\mu(\cup_n E_n) = \lim_{n\to\infty} \mu(E_n)$, then μ is countably additive.

Problem 2. Let μ be a measure on $(X, \mathcal{M})$, we define

$$\mu_0(E) = \sup\{\mu(F) : F \subset E, \mu(F) < \infty\}.$$

Show that μ_0 is a semifinite measure on $(X, \mathcal{M})$, i.e., show that μ_0 is a measure and for any $E \in \mathcal{M}$ with $\mu_0(E) = \infty$ there exists $F \subset E$ such that $0 < \mu_0(F) < \mu_0(E)$.

Problem 3. Let μ be a Radon measure on $\mathbb{R}^n$. Prove that

$$\limsup_{y\to x} \mu(\overline{B}(y,r)) \leq \mu(\overline{B}(x,r)) \quad \text{and} \quad \liminf_{y\to x} \mu(B(y,r)) \geq \mu(B(x,r)),$$

where $B(x,r) = \{p : |x-p| < r\}$ and $\overline{B}(x,r) = \{p : |x-p| \leq r\}$. Are there Radon measures μ such that these inequalities are strict?

Problem 4. Suppose that $E_1, ..., E_n$ are Lebesgue measurable sets on $[0,1]$ such that each $x \in [0,1]$ belongs to at most k of the sets. Prove that $\min_j m(E_j) \leq \frac{k}{n}$.

Problem 5. Suppose that f_n is a sequence of positive measurable functions such that $f_n \to f$ pointwise and $\int f \, d\mu = \lim_{n\to\infty} \int f_n d\mu < \infty$. Prove that $\int_E f_n \, d\mu \to \int_E f \, d\mu$ for any measurable $E \in \mathcal{M}$.
Give an example when $\int f \, d\mu = \lim_{n\to\infty} \int f_n d\mu = \infty$ and $\int_E f_n \, d\mu \to \int_E f \, d\mu$ does not hold for some $E \in \mathcal{M}$.

Problem 6. Suppose that $\mu(X) < \infty$ and $f : X \times [0,1] \to \mathbb{C}$ is a function such that $f(\cdot, y)$ is measurable for each $y \in [0,1]$ and $f(x, \cdot)$ is continuous for each $x \in X$.
(i) Let $\delta \in (0,1)$ and $\varepsilon > 0$, show that $E_{\varepsilon,\delta} = \{x : |f(x,y) - f(x,0)| \leq \varepsilon, \text{ when } y < \delta\}$ is measurable.
(ii) Prove that for any $\varepsilon > 0$ there exists a set $E \subset X$ such that $\mu(X \setminus E) < \varepsilon$ and $f(\cdot, y) \to f(\cdot, 0)$ uniformly on E.

Problem 7. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous and $A \subset \mathbb{R}$.
(a) Show that if U is open then $f(U)$ is a F_σ set, i.e., $f(U)$ is a countable union of closed sets.
(b) Suppose that f is injective and B is a Borel set, show that $f(B)$ is a Borel set. (Hint: Consider the family $\mathcal{A} = \{A \subset \mathbb{R} : f(A) \text{ is Borel }\}$.)
Remark: In general the image of a Borel set under a continuous map is not a Borel set, so be sure you use the fact that f is injective.

Problem 8. Suppose that $(X, \mathcal{M}, \mu)$ is a measure space with finite measure. Prove that if $f_n \to f$ is measure and $g_n \to g$ in measure then $f_n g_n \to fg$ in measure. Give an example when this does not hold in a space with infinite measure.

Problem 9. Deduce the monotone convergence theorem form Fatou's lemma.

Problem 10. Suppose that f is a Lebesgue measurable function on $[0, 1]$ and $0 < a < 1$ are such that $\int_E f\,dm = 0$ for any Lebesgue measurable set $E \subset [0, 1]$ with $m(E) = a$. Does this imply that $f = 0$ a.e.?

Problem 11. Let $f \in L^1(\mathbb{R}^n)$ and Mf be the Hardy-Littlewood maximal function of f, $Mf(x) = \sup_{r>0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y)|\,dm(y)$. Prove that $\lim_{t\to\infty} tm(\{Mf > t\}) = 0$.

Problem 12. Suppose that f is a non-negative Lebesgue measurable function on $\mathbb{R}$, define

$$G = \{(x, y) : 0 \le y \le f(x)\}.$$

Prove that G is a Lebesgue measurable set in $\mathbb{R}^2$ and $m_2(G) = \int_{\mathbb{R}} f(x)\,dm(x)$.

Problem 13. Define a function f on $(0, 1) \times (0, 1)$ by

$$f(x, y) = \begin{cases} x^{-2}, & x > y \\ 0, & x = y \\ -y^{-2}, & x < y \end{cases}.$$

Compute the integrals $\int_{(0,1)} \int_{(0,1)} f(x, y)\,dm(x)\,dm(y)$ and $\int_{(0,1)} \int_{(0,1)} f(x, y)\,dm(y)\,dm(x)$.

Problem 14. Define the function f by

$$f(x) = \int_0^\infty \frac{te^{-t}}{(x + t)^2}\,dt, \quad x > 0.$$

Find the value of $\int_0^\infty f(x)dx$.

Problem 15. Let f be a measurable function on $\mathbb{R}$. Show that f satisfies the inequality $|f(x) - f(y)| \le M|x - y|$ for all $x, y \in \mathbb{R}$ if and only if F is absolutely continuous on $\mathbb{R}$ and $|F'| \le M$ a.e.

Problem 16. Suppose that μ is a Radon measure on $\mathbb{R}^n$ and $f \in L^1_{loc}(\mu)$. Prove that for μ-a.e. $x \in \mathbb{R}^n$

$$\lim_{r\to 0} \frac{1}{\mu(B(x,r)} \int_{B_{x,r}} f(y)\, d\mu = f(x).$$

Problem 17. Let μ be a non-zero Radon measure on $\mathbb{R}^n$ that satisfies the following doubling property: there exits a constant C such that $\mu(2B) \leq C\mu(B)$ for any ball $B \subset \mathbb{R}^n$. For $f \in L^1_{loc}(\mu)$, define the maximal function

$$M_\mu f(x) = \sup_{r>0} \frac{1}{\mu(B_r)} \int_{B(x,r)} |f(y)|\, d\mu(y).$$

Prove that $\|M_\mu f_- p \leq C_p \|f\|_p$ when $1 < p \leq \infty$. (Hint: Apply the Marcinkiewicz interpolation theorem.)

Problem 18. Suppose that f, g are two absolutely continuous functions on a bounded interval $[a, b]$. Show that fg is also absolutely continuous and

$$\int_a^b (f'g + fg')\, dm = f(b)g(b) - f(a)f(a).$$

Problem 19. Let $(X, \mathcal{M}, \mu)$ is a measure space and $\mathcal{N}$ is a sub-σ algebra of $\mathcal{M}$, the measure ν is defined on $\mathcal{N}$ by $\nu(E) = \mu(E)$. Suppose that $f \in L^1(\mu)$. Show that there exists $g \in L1(\nu)$ such that $\int_E f d\mu = \int_E g d\nu$ for any $E \in \mathcal{N}$. (Hint: Apply the Radon-Nikodym theorem.)

Problem 20. Suppose that $(X, \mathcal{M}, \mu)$ is a measure space and $1 \leq p < r < \infty$, we define the norm $\|f\|_* = \|f\|_p + \|f\|_r$ on the space $L = L^p(\mu) + L^r(\mu)$.
(a) Show that $(L, \|\cdot\|_*)$ is a Banach space.
(b) For $q, p < q < r$ prove that $L \subset L^q$ and $\|f\|_q \leq C\|f$
$_*$ for any $f \in L$.

Problem 21. Let f be a measurable function on $(X, \mathcal{M}, \mu)$ and $\lambda_f(t) = \mu\{|f| > t\}$ be its distribution function. Prove that $f \in L^1(\mu)$ if and only if $\sum_{k=-\infty}^{\infty} 2^k \lambda_f(2^k) < \infty$.

Problem 22. Let $f \in L^p(\mathbb{R})$, $1 < p < \infty$, prove that $\lim_{h\to 0} \|f(x) - f(x+h)\|_p \to 0$. (Hint: f can be approximated by a continuous function in L^p.)

Problem 23. Let $(X, \mathcal{M}, \mu)$ be a measure space and $f : X \to \mathbb{C}$ be a measurable function. Define $R_f = \{z \in \mathbb{C} : \mu(\{|f - z| > \varepsilon\}) > 0 \text{ for any} \varepsilon > 0\}$. Prove that if $f \in L^\infty(\mu)$ then R_f is compact and $\|f\|_\infty = max\{|z| : z \in R_f\}$.

Problem 24. Let $f \in L^1(\mathbb{R})$, define $g(x) = \int_{\mathbb{R}} f(t)e^{itx}\, dt$ for $x \in \mathbb{R}$. Show that g is continuous and $\lim_{x\to\pm\infty} g(x) = 0$.

Problem 25. Suppose that $K \in L^p([0,1] \times [0,1])$, $1 < p < \infty$ and $1/p + 1/q = 1$.For $f \in L^q([0,1])$, let $(Tf)(x) = \int_0 1^1 K(x,y) f(y) dy$. Show that $(Tf)(x)$ indeed exists for almost every $x \in [0,1]$ and $\|Tf\|_p \le C\|f\|_q$.

Problem 26. Suppose that w is a measurable function on $\mathbb{R}^n$ which is finite and strictly positive almost everywhere. Suppose also that K is a measurable function on $\mathbb{R}^{2n}$ such that

$$\int_{\mathbb{R}^n} |K(x,y)| w(y) dy \le A w(x) \qquad \int_{\mathbb{R}^n} |K(x,y)| w(x) dx \le A w(y)$$

for almost every x, and for almost every y, respectively. Prove that the integral operator $Tf(x) = \int_{\mathbb{R}^n} K(x,y) f(y)\, dy$ is bounded on $L^2(\mathbb{R}^n)$ and $\|Tf\|_2 \le A\|f\|_2$.

Problem 27. Consider functions on $[0,\infty)$. Let $\alpha > 0$, we define

$$J_\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t) dt,$$

where $\Gamma(\alpha)$ is the Gamma function, $\Gamma(n) = (n-1)!$. Let also $T_\alpha f(x) = x^{-\alpha} J_\alpha f(x)$. Show that $\|T_\alpha f\|_p \le C_p \|f\|_p$ when $1 < p \le \infty$.

Problem 28. Let I_a be an operator defined on function in $\mathbb{R}^m$ by $I_\alpha f(x) = \int |x - y|^{\alpha-m} f(y)\, dy$, where $0 < a < m$. Show that if $f \in L^1$ then Tf is in week L^q space with $q = m(m-\alpha)^{-1}$. (The case $p > 1$, $1/q = 1/p - \alpha/m$ and strong norm $\|Tf\|_q \le C\|f\|_p$ can be found in the lecture notes.

Problem 29. Let $(X, \mathcal{M}, \mu)$ be a measure space with finite measure μ and let $\phi : X \to \mathbb{R}$ be a measurable function. Show that there exists a Radon measure on $\mathbb{R}$ such that $\int f\, d\mu = \int f(\phi(x))\, d\mu(x)$ for any $f \in C_0(\mathbb{R})$.

Problem 30. Let μ_n be a sequence of Radon measures on $[0,1]$ with $\sup_n \mu_n([0,1]) < \infty$.
(a) Suppose that $f \in C[0,1]$, show that there exists a subsequence $\{n_k\}$ such that $\int_{[0,1]} f\, d\mu_{n_k}$ converges.
(b) There exists a countable dense subset A of $C[0,1]$. Show that there exists a subsequence $\{n_k\}$ such that $\int_{[0,1]} f\, d\mu_n$ converges for any $f \in A$.
(c) Prove that there exist a Radon measure μ on $[0,1]$ such that $\int f\, d\mu_{n_k} \to \int f\, d\mu$ for every $f \in C([0,1])$.