Bayesian Persuasion
Web Appendix

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1 Persuasion mechanisms

In this paper we study a particular game where Sender chooses a signal π whose realization is observed by Receiver who then takes her action. In Subsection 2.3 we made an observation that Sender's gain from persuasion is weakly greater in this game than in any other communication game. In this section of the Online Appendix we provide a formal statement and proof of this claim. To do so, we introduce the notion of a *persuasion mechanism*.

As before, Receiver has a continuous utility function $u(a, \omega)$ that depends on her action $a \in A$ and the state of the world $\omega \in \Omega$. Sender has a continuous utility function $v(a, \omega)$ that depends on Receiver's action and the state of the world. Sender and Receiver share a prior $\mu_0 \in int(\Delta(\Omega))$. Let $a^*(\mu)$ denote the set of actions that maximize Receiver's expected utility given her belief is μ. We assume that there are at least two actions in A and that for any action a there exists a μ s.t. $a^*(\mu) = \{a\}$. The action space A is compact and the state space Ω is finite. We will relax the latter assumption in Section 3.

A persuasion mechanism (π, c) is a combination of a signal and a message technology. Sender's private *signal* π consists of a finite realization space S and a family of distributions $\{\pi(\cdot|\omega)\}_{\omega\in\Omega}$ over S. A *message technology* c consists of a finite message space M and a family of functions $c(\cdot|s)$:

$M \to \overline{\mathbb{R}}_+$; $c(m|s)$ denotes the cost to Sender of sending message m after receiving signal realization s.[1] The assumptions that S and M are finite are without loss of generality (cf. Proposition 4) and are used solely for notational convenience.

A persuasion mechanism defines a game. The timing is as follows. First, nature selects ω from Ω according to μ_0. Neither Sender nor Receiver observe nature's move. Then, Sender privately observes a realization $s \in S$ from $\pi(\cdot|\omega)$ and chooses a message $m \in M$. Finally, Receiver observes m and chooses an action $a \in A$. Sender's payoff is $v(a,\omega) - c(m|s)$ and Receiver's payoff is $u(a,\omega)$. We represent the Sender's and Receiver's (possibly stochastic) strategies by σ and ρ, respectively. We use $\mu(\omega|m)$ to denote Receiver's posterior belief that the state is ω after observing m.

A few examples help clarify the varieties of games that are captured by the definition of a persuasion mechanism. If π is perfectly informative, $M = \mathbb{R}_+$, and $c(m|s) = m/s$, the mechanism is Michael A. Spence's (1973) education signalling game. If π is perfectly informative and c is constant, the mechanism is a cheap talk game as in Vincent Crawford and Joel Sobel (1982). If π is arbitrary and c is constant, the mechanism coincides with the information-transmission game of Jerry R. Green and Nancy L. M. Stokey (2007). If π is perfectly informative and $c(m|s) = -(m-s)^2$, the mechanism is a communication game with lying costs developed in Navin Kartik (2009). If π is perfectly informative, $M = \mathcal{P}(\Omega)$, and $c(m|s) = \begin{cases} 0 & \text{if } s \in m \\ \infty & \text{if } s \notin m \end{cases}$, the mechanism is a persuasion game as in Sanford J. Grossman (1981) and Paul Milgrom (1981).[2]

A perfect Bayesian equilibrium of a persuasion mechanism is a triplet $(\sigma^*, \rho^*, \mu^*)$ satisfying the usual conditions. We also apply an additional equilibrium selection criterion: we focus on Sender-preferred equilibria, i.e., equilibria where the expectation of $v(a,\omega) - c(m|s)$ is the greatest. For the remainder of this appendix, we use the term "equilibrium" to mean a Sender-preferred perfect

[1] $\overline{\mathbb{R}}_+$ denotes the affinely extended non-negative real numbers: $\overline{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{\infty\}$. Allowing c to take on the value of ∞ is useful for characterizing the cases where Sender cannot lie and cases where he must reveal all his information.

[2] $\mathcal{P}(X)$ denotes the set of all subsets of X.

Bayesian equilibrium of a persuasion mechanism.[3] We denote Sender's equilibrium action when her belief is μ by $\hat{a}(\mu)$.[4]

We define the *value* of a mechanism to be the equilibrium expectation of $v(a, \omega) - c(m|s)$. The *gain* from a mechanism is the difference between its value and the equilibrium expectation of $v(a, \omega)$ when Receiver obtains no information. *Sender benefits from persuasion* if there is a mechanism with a strictly positive gain. A mechanism is *optimal* if no other mechanism has higher value.

The game we study in the paper is closely related to persuasion mechanism where $M = S$ and $c(m|s) = \begin{cases} k & \text{if } s \in m \\ \infty & \text{if } s \notin m \end{cases}$ for some $k \in \mathbb{R}_+$. We call such mechanisms *honest*. Specifically, the game we study in the paper is one where Sender chooses among the set of all honest mechanisms. With these definitions in hand, we can formally state and prove the aforementioned observation from Subsection 2.3.

Proposition 1. *For any $v^* \in \mathbb{R}$, there exists a persuasion mechanism with value v^* if and only if there exists an honest mechanism with value v^*.*

Proof. The "if" part of the claim is immediate. To see the "only if" part, consider an equilibrium $\varepsilon^o = (\sigma^o, \rho^o, \mu^o)$ of some mechanism (π^o, c^o) with value v^*. Let A^o be the set of actions induced with positive probability in ε^o, and for any $a \in A^o$, let M^a be the set of messages which induce a in ε^o. Define an honest mechanism by $M = S = A^o$, $\pi(a|\omega) = \sum_{m \in M^a} \sum_{s \in S} \sigma^o(m|s) \pi^o(s|\omega)$, and $c(m|s)$ equal to the ex ante expected messaging costs under ε^o if $m = s$ and $-\infty$ otherwise. Reporting truthfully is clearly an equilibrium strategy for Sender. Because a was an optimal response for Receiver to each $m \in M^a$ in ε^o, it must also be an optimal response to the message

[3] For some mechanisms it might seem more suitable to focus on sequential, rather than merely perfect Bayesian, equilibria. Given that we examine Sender-preferred equilibria, however, this distinction will not be particularly relevant.

[4] If both Sender and Receiver are indifferent between two actions at a given belief, we focus on equilibria where Receiver takes a deterministic action.

a in the proposed honest mechanism, so the distribution of Receiver's actions conditional on the state is the same as in ε^o. Since the expected messaging costs are also the same as in ε^o, the value of the honest mechanism is exactly v^*. □

2 Verifiable messages

In Subsection 2.3 we also made an observation that it is not important to assume that Sender must truthfully report realization s from π.

Specifically, if we instead assume that Sender observes s privately and then sends a verifiable message about s, the same information is revealed in equilibrium and we obtain the same outcome. As before, Receiver has a continuous utility function $u(a,\omega)$ that depends on her action $a \in A$ and the state of the world $\omega \in \Omega$. Sender has a continuous utility function $v(a,\omega)$ that depends on Receiver's action and the state of the world. Sender and Receiver share a prior $\mu_0 \in int\,(\Delta(\Omega))$. The action space A is compact and the state space Ω is finite. In this section, we assume that Receiver has a unique optimal action $a^*(\mu)$ at every belief. When this assumption is not satisfied, the equivalence of the outcomes with verifiable messages can be guaranteed by introducing a small amount of private information for Receiver, so that the distribution of Receiver's optimal actions is uniquely determined conditional on μ.

Given A, Ω, μ_0, u, and v, we define two games. Let Π be the set of all signals on Ω with a finite realization space S. The *baseline game* is the game we study in the paper. Sender chooses a signal $\pi \in \Pi$. Receiver observes Sender's choice of the signal and the signal realization $s \in S$, forms her beliefs, and then takes her action. Denote Sender's strategy by $\rho \in \Delta(\Pi)$. A strategy ρ is an equilibrium if and only if each π in its support induces a $\tau \in \arg\max_\tau E_\tau(\hat{v}(\mu))$.

The other game we refer to as the *verifiable message game*. Sender chooses a signal π. Receiver observes Sender's choice of the signal. Sender privately observes a signal realization s from the signal. He then sends a message $m \in \mathcal{P}(S)$ s.t. $s \in m$. Receiver observes the message, forms her belief, and then takes an action. Let $\sigma = \left(\rho, (\gamma^{\pi})_{\pi \in \Pi}\right)$ denote a strategy for Sender. This consists of a possibly stochastic choice of signal $\rho \in \Delta(\Pi)$ and a messaging policy $\gamma^{\pi} : S \to \Delta(\mathcal{P}(S))$ following each signal π. Let γ^* denote the truthful messaging policy, i.e., $\gamma^*(s)$ puts probability one on $\{s\}$ for all s. Let $\tilde{\mu}(m) \in \Delta(S)$ denote Receiver's belief about the signal realization observed by Sender when she sees message m. The support of $\tilde{\mu}(m)$ must be m. Each belief $\tilde{\mu}(m)$ implies a unique belief μ about ω. A pair $(\sigma, \tilde{\mu})$ is a (perfect Bayesian) equilibrium if and only if $\tilde{\mu}$ obeys Bayes' rule on the equilibrium path and σ is a best response to $\tilde{\mu}$ at every information set.

In both games, we refer to the equilibrium distribution of Receiver's beliefs conditional on the state as the equilibrium *outcome*. We can now formally state our claim from Subsection 2.3:

Proposition 2. *The set of equilibrium outcomes of the baseline game and the set of equilibrium outcomes of the verifiable message game coincide.*

Let VG_{π} denote the subgame of the verifiable message game following Sender's choice of π. A (perfect Bayesian) equilibrium of the subgame VG_{π} is a pair $(\gamma^{\pi}, \tilde{\mu}^{\pi})$. Standard arguments ensure that for any π, such an equilibrium of VG_{π} exists.

Lemma 1. *Suppose ρ is an equilibrium of the baseline game. Then for each π in the support of ρ there exists an equilibrium of VG_{π} in which $\gamma = \gamma^*$.*

Proof. Consider some equilibrium ρ of the baseline game and some π in the support of ρ. There exists an equilibrium $(\gamma^{\pi}, \tilde{\mu}^{\pi})$ of VG_{π}. It will suffice to show that given $\tilde{\mu}^{\pi}$, Sender's payoff from γ^* must be the same as his payoff from γ^{π} following any s. Define these payoffs to be $y^*(s)$ and $y(s)$ respectively.

Since $(\gamma^{\pi}, \tilde{\mu}^{\pi})$ is an equilibrium, we must have $y(s) \geq y^*(s) \, \forall s$. Moreover, because Sender can choose a signal π' in the baseline game to induce any Bayes-plausible distribution of Receiver's beliefs, he can in particular choose a π' that produces the same distribution of Receiver's beliefs as $(\gamma^{\pi}, \tilde{\mu}^{\pi})$. Sender's expected payoff in the baseline game following π' must be the same as his payoff in VG_{π}, which is $E_{\pi} y(s)$. Since π is played in an equilibrium of the baseline game, we must have $E_{\pi} y^*(s) \geq E_{\pi} y(s)$. Given $y(s) \geq y^*(s) \, \forall s$, this implies $y(s) = y^*(s) \, \forall s$. □

With Lemma 1, it is straightforward to establish one direction of Proposition 2.

Lemma 2. *Any equilibrium outcome of the baseline game is an equilibrium outcome of the verifiable message game.*

Proof. Consider any equilibrium ρ of the baseline game with outcome τ. Let v denote Sender's payoff in this equilibrium. We construct an equilibrium $(\hat{\sigma}, \hat{\mu})$ of the verifiable message game as follows. First, specify that $\hat{\sigma}$ chooses signals according to ρ at the initial node, and set $\gamma^{\pi} = \gamma^*$ for all π in the support of ρ. Note that the outcome of the verifiable message game under this strategy is τ. Impose further that $\hat{\mu}$ is consistent with Bayes rule on the equilibrium path given these strategies. To construct off-equilibrium messaging policies $(\gamma^{\pi})_{\pi \notin Supp(\rho)}$, consider for each such π some equilibrium $\left(\gamma^{\pi}_{eq}, \tilde{\mu}^{\pi}_{eq}\right)$ of VG_{π}. Set $\gamma^{\pi} = \gamma^{\pi}_{eq}$ and set $\hat{\mu} = \tilde{\mu}^{\pi}_{eq}$ at nodes following π.

Because each γ^{π}_{eq} is an equilibrium of VG_{π}, we know that $\hat{\sigma}$ is a best response to $\hat{\mu}$ at all information sets following each π. Consider, then, the first node at which Sender chooses π. For any π', let $v_{\pi'}$ be Sender's payoff if she chooses π' followed by the γ^{π} prescribes by $\hat{\sigma}$. By Proposition 1 in the paper, we know that Sender could have chosen a signal in the baseline game that would have generated the same distribution of posteriors and thus also yielded payoff $v_{\pi'}$. Because ρ is an equilibrium of the baseline game, any $\pi \in Supp(\rho)$ followed by γ^* must yield a payoff weakly higher than $v_{\pi'}$ for all π'. This implies that $\hat{\sigma}$ is a best response to $\hat{\mu}$ at the initial information set of the verifiable message game. □

It remains to establish the following Lemma:

Lemma 3. *Any equilibrium outcome of the verifiable message game is an equilibrium outcome of the baseline game.*

Proof. Consider some equilibrium of the verifiable message game $(\sigma, \tilde{\mu})$, where $\sigma = \left(\rho, (\gamma^{\pi})_{\pi \in \Pi}\right)$, with outcome τ. Let v be Sender's payoff in this equilibrium. Let $v_{\pi'}$ denote Sender's payoff in if he deviates to π' followed by $\gamma^{\pi'}$. Let $v^*_{\pi'}$ denote Sender's payoff if he deviates to π' followed by γ^*. We know (i) $v \geq v_{\pi'} \forall \pi'$ (by the fact that σ is an equilibrium); (ii) $v_{\pi'} \geq v^*_{\pi'}$ (by the fact that $\gamma^{\pi'}$ was a best response at the information sets following π'). By Proposition 1 in the paper, we know that there exists a $\hat{\rho}$ such that playing $\hat{\rho}$ in the baseline game yields sender payoff v and produces outcome τ. Since for any π' Sender's payoff in the baseline game from choosing π' is $v^*_{\pi'}$, inequalities (i) and (ii) imply that $\hat{\rho}$ is an equilibrium of the baseline game. □

Lemmas 2 and 3 jointly establish Proposition 2.

3 Relaxing the assumption that Ω is finite

In the paper, we assumed that Ω is finite. We also claimed this assumption was made primarily for expositional convenience. In this section, we show that the approach used in the paper extends to the case when Ω is a compact metric space.[5]

As before, Receiver has a continuous utility function $u(a, \omega)$ that depends on her action $a \in A$ and the state of the world $\omega \in \Omega$. Sender has a continuous utility function $v(a, \omega)$ that depends on Receiver's action and the state of the world. The action space A is assumed to be compact and the state space Ω is assumed to be a compact metric space. Let $\Delta(\Omega)$ denote the set of Borel probabilities on Ω, a compact metric space in the weak* topology. Sender and Receiver share a prior $\mu_0 \in \Delta(\Omega)$.

[5]We are very grateful to Max Stinchcombe for help with this extension.

A *persuasion mechanism* is a combination of a signal and a message technology. A *signal* (π, S) consists of a compact metric realization space S and a measurable function $\pi : [0,1] \to \Omega \times S$, $x \mapsto (\pi_1(x), \pi_2(x))$. Note that we define π to be a measurable function whose second component (i.e., the signal realization) is correlated with ω. We assume that x is uniformly distributed on $[0,1]$ and that Sender observes $\pi_2(x)$.

We denote a realization of $\pi_2(x)$ by s. Note that since S is a compact metric space (hence, complete and separable), there exists a regular conditional probability (i.e., a posterior probability) obtained by conditioning on $\pi_2(x) = s$ (A. N. Shiryaev 1996, p.230). A *message technology* c consists of a message space M and a family of functions $\left\{c(\cdot|s) : M \to \overline{\mathbb{R}}_+\right\}_{s \in S}$. As before, a mechanism is honest if $M = S$ and $c(m|s) = \begin{cases} k & \text{if } s \in m \\ \infty & \text{if } s \notin m \end{cases}$ for some $k \in \mathbb{R}_+$. A persuasion mechanism defines a game just as before. Perfect Bayesian equilibrium is still the solution concept and we still select Sender-preferred equilibria. Definitions of value and gain are same as before.

Let $a^*(\mu)$ denote the set of actions optimal for Receiver given her beliefs are $\mu \in \Delta(\Omega)$:

$$a^*(\mu) \equiv \arg\max_a \int u(a, \omega)\, d\mu(\omega).$$

Note that $a^*(\cdot)$ is an upper hemicontinuous, non-empty valued, compact valued, correspondence from $\Delta(\Omega)$ to A.

Let $\hat{v}(\mu)$ denote the maximum expected value of v if Receiver takes an action in $a^*(\mu)$:

$$\hat{v}(\mu) \equiv \max_{a \in a^*(\mu)} \int v(a, \omega)\, d\mu(\omega).$$

Since $a^*(\mu)$ is non-empty and compact and $\int v(a, \omega)\, d\mu(\omega)$ is continuous in a, $\hat{v}$ is well defined. We first show that the main ingredient for the existence of an optimal mechanism, namely the upper semicontinuity of $\hat{v}$, remains true in this setting.

Proof. $\hat{v}$ is upper semicontinuous.

Given any a, the random variable $v(a,\omega)$ is dominated by the constant random variable $\max_\omega v(a,\omega)$ (since v is continuous in ω and Ω is compact, the maximum is attained). Hence, by the Lebesgue's Dominated Convergence Theorem, $\int v(a,\omega)\,d\mu(\omega)$ is continuous in μ for any given a. Now, suppose that $\hat{v}$ is discontinuous at some μ. Since u is continuous, by Berge's Maximum Theorem this means that Receiver must be indifferent between a set of actions at μ, i.e., $a^*(\mu)$ is not a singleton. By definition, however, $\hat{v}(\mu) \equiv \max_{a\in a^*(\mu)} \int v(a,\omega)\,d\mu(\omega)$.

Hence, $\hat{v}$ is upper semicontinuous. □

Now, *a distribution of posteriors*, denoted by τ, is an element of the set $\Delta(\Delta(\Omega))$, the set of Borel probabilities on the compact metric space $\Delta(\Omega)$. We say a distribution of posteriors τ is *Bayes-plausible* if $\int_{\Delta(\Omega)} \mu d\tau(\mu) = \mu_0$. We say that π *induces* τ if conditioning on $\pi_2(x) = s$ gives posterior κ_s and the distribution of $\kappa_{\pi_2(x)}$ is τ given that x is uniformly distributed. Since Ω is a compact metric space, for any Bayes-plausible τ there exists a π that induces it.[6] Hence, the problem of finding an optimal mechanism is equivalent to solving

$$\max_{\tau\in\Delta(\Delta(\Omega))} \int_{\Delta(\Omega)} \hat{v}(\mu)\,d\tau(\mu)$$
$$\text{s.t.} \int_{\Delta(\Omega)} \mu d\tau(\mu) = \mu_0.$$

Now, let

$$V \equiv \sup\{z|\,(\mu,z)\in co\,(hyp\,(\hat{v}))\},$$

where $co(\cdot)$ denotes the convex hull and $hyp(\cdot)$ denotes the hypograph. Recall that given a subset K of an arbitrary vector space, $co(K)$ is defined as $\cap\{C|K\subset C,\, C \text{ convex}\}$. Let $g(\mu_0)$ denote

[6] Personal communication with Max Stinchcombe. Detailed proof available upon request.

the subset of $\Delta(\Delta(\Omega))$ that generate the point $(\mu_0, V(\mu_0))$, i.e.,

$$g(\mu_0) \equiv \left\{ \tau \in \Delta(\Delta(\Omega)) \,|\, \int_{\Delta(\Omega)} \mu d\tau(\mu) = \mu_0, \int_{\Delta(\Omega)} \hat{v}(\mu) d\tau(\mu) = V(\mu_0) \right\}.$$

Note that we still have not established that $g(\mu_0)$ is non-empty. That is the primary task of the proof of our main proposition.

Proposition 3. *An optimal mechanism exists. The value of an optimal mechanism is $V(\mu_0)$. Sender benefits from persuasion iff $V(\mu_0) > \hat{v}(\mu_0)$. An honest mechanism with a signal that induces an element of $g(\mu_0)$ is optimal.*

Proof. By construction of V, there can be no mechanism with value strictly greater than $V(\mu_0)$. We need to show there exists a mechanism with value equal to $V(\mu_0)$, or equivalently, that $g(\mu_0)$ is not empty. Without loss of generality, suppose the range of v is $[0,1]$. Consider the set $H = \{(\mu, z) \in hyp(V) \,|\, z \geq 0\}$. Since $\hat{v}$ is upper semicontinuous, H is compact. By construction of V, H is convex. Therefore, by Choquet's Theorem (e.g., Robert R. Phelps 2001), for any $(\mu', z') \in H$, there exists a probability measure η s.t. $(\mu', z') = \int_H (\mu, z)\, d\eta(\mu, z)$ with η supported by extreme points of H. In particular, there exists η s.t. $(\mu_0, V(\mu_0)) = \int_H (\mu, z)\, d\eta(\mu, z)$ with η supported by extreme points of H. Now, note that if (μ, z) is an extreme point of H, then $V(\mu) = \hat{v}(\mu)$; moreover, if $z > 0$, $z = V(\mu) = \hat{v}(\mu)$. Hence, we can find an η s.t. $(\mu_0, V(\mu_0)) = \int_H (\mu, z)\, d\eta(\mu, z)$ with support of η entirely within $\{(\mu, \hat{v}(\mu)) \,|\, \mu \in \Delta(\Omega)\}$. Therefore, there exists a $\tau \in g(\mu_0)$. $\square$

4 Bound on the size of S

In footnote 7, we claimed that $|S|$ need not exceed $\min\{|A|, |\Omega|\}$. Proposition 1 implies that there exists an optimal signal with $|S| \leq |A|$. Hence, we only need to establish the following:

Proposition 4. *There exists an optimal mechanism with $|S| \leq |\Omega|$.*

Proof. Since $\hat{v}$ is bounded, $hyp\left(\hat{v}\right)$ is path-connected. Therefore, it is connected. The Fenchel-Bunt Theorem (Hiriart-Urruty and Lemaréchal 2004, Thm 1.3.7) states that if $S \subset \mathbb{R}^n$ has no more than n connected components (in particular, if S is connected), then any $x \in co\left(S\right)$ can be expressed as a convex combination of n elements of S. Hence, since $hyp\left(\hat{v}\right) \subset \mathbb{R}^{|\Omega|}$, any element of $co\left(hyp\left(\hat{v}\right)\right)$ can be expressed as a convex combination of $|\Omega|$ elements of $hyp\left(\hat{v}\right)$. In particular, $\left(\mu_0, V\left(\mu_0\right)\right) \in co\left(hyp\left(\hat{v}\right)\right)$ can be expressed a convex combination of $|\Omega|$ elements of $hyp\left(\hat{v}\right)$. This further implies that $\left(\mu_0, V\left(\mu_0\right)\right)$ can be expressed as a convex combination of $|\Omega|$ elements of the graph of $\hat{v}$. Hence, there exists an optimal straightforward mechanism which induces a distribution of posteriors whose support has no more than $|\Omega|$ elements. □

5 Monotonicity of $\hat{v}$ and optimality of worst beliefs

In Section 5.1, we observed that when Sender's payoffs are monotonic in Receiver's beliefs, there is a sense in which Sender always induces the worst belief consistent with a given action. Here we formalize that claim.

Say that $\hat{v}$ is monotonic if for any μ, μ', $\hat{v}\left(\alpha\mu + (1-\alpha)\mu'\right)$ is monotonic in α. When $\hat{v}$ is monotonic in μ, it is meaningful to think about beliefs that are better or worse from Sender's perspective. The simplest definition would be that μ is worse than μ' if $\hat{v}\left(\mu\right) \leq \hat{v}\left(\mu'\right)$. Note, however, that because $v\left(a, \omega\right)$ depends on ω directly, whether μ is worse in this sense depends both on how Receiver's action changes at μ and how μ affects Sender's expected utility directly. It turns out that for our result we need a definition of worse that isolates the way beliefs affect Receiver's actions.

When $\hat{v}$ is monotonic, there is a rational relation on A defined by $a \succsim a'$ if $\hat{v}\left(\mu\right) \geq \hat{v}\left(\mu'\right)$ whenever $a = \hat{a}\left(\mu\right)$ and $a' = \hat{a}\left(\mu'\right)$. This relation on A implies a partial order on $\Delta\left(\Omega\right)$: say that

$\mu \rhd \mu'$ if

$$E_\mu u(a,\omega) - E_\mu u\left(a',\omega\right) > E_{\mu'} u\left(a,\omega\right) - E_{\mu'} u\left(a',\omega\right)$$

for any $a \succsim a'$. In other words, a belief is higher in this partial order if it makes better actions (from Sender's perspective) more desirable for Receiver. The order is partial since a belief might make both a better and a worse action more desirable for Receiver. We say that μ is *a worst belief inducing* $\hat{a}(\mu)$ if there is no $\mu' \lhd \mu$ s.t. $\hat{a}(\mu) = \hat{a}(\mu')$. We then have the following:

Proposition 5. *Suppose Assumption 1 holds. If $\hat{v}$ is monotonic, A is finite, and Sender benefits from persuasion, then for any interior belief μ induced by an optimal mechanism either: (i) μ is a worst belief inducing $\hat{a}(\mu)$, or (ii) both Sender and Receiver are indifferent between two actions at μ.*

Proof. Suppose Assumption 1 holds, $\hat{v}$ is monotonic, A is finite, and Sender benefits from persuasion. Now, suppose μ is an interior belief induced by an optimal mechanism. Since Assumption 1 holds and Sender benefits from persuasion, Receiver's preference at μ is not discrete by Proposition 6. Therefore, Lemma 1 tells us $\exists a$ such that $E_\mu u(\hat{a}(\mu), \omega) = E_\mu u(a,\omega)$. If (ii) does not hold, we know $E_\mu v(\hat{a}(\mu), \omega) > E_\mu v(a,\omega)$. Therefore, $\hat{a}(\mu) \succsim a$. Hence, given any $\mu' \lhd \mu$,

$$0 = E_\mu u(\hat{a}(\mu), \omega) - E_\mu u(a,\omega) > E_{\mu'} u(\hat{a}(\mu), \omega) - E_{\mu'} u(a,\omega).$$

Since $E_{\mu'} u(a,\omega) > E_{\mu'} u(\hat{a}(\mu), \omega)$, we know that $\hat{a}(\mu)$ is not Receiver's optimal action when her beliefs are μ.

Hence, for any $\mu' \lhd \mu$, $\hat{a}(\mu') \neq \hat{a}(\mu)$, which means that μ is a worst belief inducing $\hat{a}(\mu)$. $\square$

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