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# ON THE MARKOV PROPERTY OF THE $GI/G/\infty$ GAUSSIAN LIMIT

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**Abstract**

It is shown that the heavy-traffic Gaussian limit for $GI/G/\infty$ queues is Markovian if and only if the service-time distribution $H(t)$ is of the form $1\text{-}H(t)=pe^{-\alpha t}$ for $\alpha>0$ and $0<p\leqq 1$.

$GI/G/s$ QUEUE; LIMIT THEOREM: GAUSSIAN PROCESS

A recent paper by Whitt (1981) investigates the heavy-traffic limit behaviour of the $GI/G/\infty$ queue from a new perspective. The idea is to approximate a non-degenerate service-time distribution by an exponential phase-type distribution, and to consider the vector-valued continuous-time process that records the number of customers in service in each phase. Convergence to a multidimensional Ornstein–Uhlenbeck (O–U) process is then obtained by letting the arrival rate tend to $\infty$ with the service-time distribution held fixed. The well-known results of Iglehart (1965) and Borovkov (1967) are retrieved by summing the components of the process, thereby proving a limit theorem for the total number of customers in the system at time $t$.

The Gaussian limit of Borovkov arises as a natural consequence of the total number in system being a linear transform of the O–U process, which is itself Gaussian. The fact that Borovkov's limit is generally non-Markovian is associated with the fact that projections of Markov processes are usually non-Markov. The proof given by Whitt shows that by recording the additional information involving the number of customers in service for a time less than or equal to $t$ (basically equivalent to recording the number of phases completed, for an exponential phase-type distribution), the limit is Markov. Hence, the non-Markovian Borovkov limit arises from not keeping track of enough information.

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In light of the preceding discussion, it seems natural that the Borovkov limit should be Markovian precisely in the case in which only one exponential phase need be recorded—namely, where the service-time distribution $H(t)$ is of the form $1-H(t)=pe^{-\alpha t}$ for $0<p\leqq 1$ and $\alpha>0$. We now proceed to prove this.

The queue-length limit process of Borovkov is a Gaussian process $\{Y(t), t\geqq 0\}$ with $EY(t)=0$ and covariance function $K$ given by

$$K(s, s+t) = EY(s)Y(s+t) = \int_0^s G(x)\bar{H}(x+t)\,dx$$

for $s, t\geqq 0$, where $\bar{H}(x)=1-H(x)$ and $G(x)=\lambda H(x)+\sigma^2\lambda^3\bar{H}(x)(\sigma^2, \lambda>0)$. In order that a Gaussian process be Markovian it is necessary (and sufficient) that $K$ satisfy the factorization condition

$$K(s, s+t+u)K(s+t, s+t) = K(s, s+t)K(s+t, s+t+u)$$

for $s, t, u\geqq 0$ (see e.g. Doob (1953), p. 233). Hence, if $\{Y(t), t\geqq 0\}$ is to be Markov, $H(t)$ must satisfy

(1)
$$\int_0^s G(x)\bar{H}(x+t+u)\,dx \,.\int_0^{s+t} G(x)\bar{H}(x)\,dx = \int_0^s G(x)\bar{H}(x+t)\,dx\,.\int_0^{s+t} G(x)\bar{H}(x+u)\,dx.$$

Differentiating (1) with respect to $s$, we obtain the equality

(2)
$$\begin{aligned} G(s)\bar{H}(s+t+u)&\int_0^{s+t} G(x)\bar{H}(x)\,dx + \int_0^s G(x)\bar{H}(x+t+u)\,dx\,.\,G(s+t)\bar{H}(s+t) \\ &= G(s)\bar{H}(s+t)\int_0^{s+t} G(x)\bar{H}(x+u)\,dx + \int_0^s G(x)\bar{H}(x+t)\,dx\,.\,G(s+t)\bar{H}(s+t+u) \end{aligned}$$

for almost every $s$. As the integrals are clearly continuous functions of $s$, we have that both sides of (2) are right continuous and consequently equality must hold in (2) everywhere. In particular, setting $s=0$, we get

(3)
$$\bar{H}(t+u)\int_0^t G(x)\bar{H}(x)\,dx = \bar{H}(t)\int_0^t G(x)\bar{H}(x+u)\,dx.$$

Put

$$f_1(t, u) = \int_0^t G(x)\bar{H}(x+u)\, dx$$

$$f_2(t) = \int_0^t G(x)\bar{H}(x)\, dx.$$

It will simplify some later differentiation arguments if we now show that $G(t)$ and $\bar{H}(t)$ are continuous. We start by observing that since $H$ is the distribution function of a non-degenerate random variable, $G(0)\bar{H}(0)$ is positive and thus the right continuity of $G(t)\bar{H}(t)$ at $t=0$ implies that $f_2(t)>0$ for $t>0$. Also, the fact that $H(t)$ has at most countably many discontinuities means that $\bar{H}(t+u_n) \to \bar{H}(t+u)$ for almost every $t$ whenever $u_n \to u \geqq 0$. Hence, by bounded convergence, the integral on the right-hand side of (3) is continuous in $u$, and thus we may infer from (3) that $\bar{H}(t+u)$ is continuous in $u \geqq 0$ for each $t>0$. Putting this together with the right continuity of $\bar{H}(t)$ at $t=0$ yields the conclusion that $\bar{H}(t)$ and $G(t)$ are continuous on $[0, \infty)$.

Returning again to (3), we note that it is equivalent to

(4) $$f_2(t)\frac{d}{dt} f_1(t, u) = f_1(t, u) \,.\, \frac{d}{dt} f_2(t)$$

from which we obtain

$$\frac{d}{dt}(f_1(t, u)/f_2(t)) = 0.$$

Thus,

(5) $$f_1(t, u)/f_2(t) = \alpha_u$$

for all positive values of $t$. Letting $t \downarrow 0$, and employing L'Hôpital's rule, gives

(6) $$\alpha_u = \lim_{t\downarrow 0} \frac{d}{dt} f_1(t, u) \Big/ \frac{d}{dt} f_2(t) = \frac{\bar{H}(u)}{\bar{H}(0)}\,.$$

On the other hand, an obvious consequence of (3) is that

(7) $$\bar{H}(t+u) = \bar{H}(t) \,.\, f_1(t, u)/f_2(t).$$

Equations (5) through (7) together provide the functional equation

(8) $$a(t+u) = a(t)a(u)$$

where $a(t) = \bar{H}(t)/\bar{H}(0)$. Since $a(t)$ is bounded, the only possible solution (Feller (1950), p. 459) of (8) is $a(t) = e^{-\alpha t}$, proving the necessity of our claim. The sufficiency is easy to verify.

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