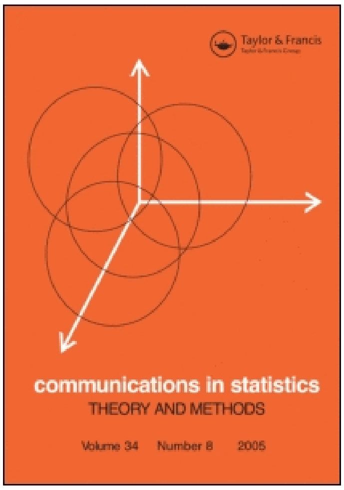

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On Functional Central Limit Theorems for Semi-Markov and Related Processes

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ABSTRACT

The semi-Markov process (SMP) has long been used as a model for the underlying process of a discrete-event stochastic system. Important refinements of this model include the continuous-time Markov chain (CTMC) and important extensions include the generalized semi-Markov process (GSMP). Functional central limit theorems (FCLTS) give basic conditions under which these various processes exhibit stable long-run behavior, as well as providing approximations for cumulative-reward distributions and confidence intervals for statistical estimators. We give FCLTS for finite-state CTMCS, SMPS, and GSMPS under minimal conditions that involve irreducibility and finite second moments on the "holding time" distributions. We consider both

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continuous and lump-sum rewards; our emphasis is on the use of martingale theory and on the explicit computation, when possible, of the variance constant in the FCLT.

Key Words: Semi-Markov processes; Markov chains; Central limit theorem; Martingales; Discrete-event systems.

I. INTRODUCTION

This paper concerns the long-run behavior of complex discrete-event stochastic systems. Such systems evolve over continuous time and make stochastic state transitions when events associated with the occupied state occur; the state transitions occur only at an increasing sequence of random times.

The underlying stochastic process of a discrete-event system records the state as it evolves over continuous time and has piecewise-constant sample paths. One model for this underlying process is the *semi-Markov process* (SMP); see, for example, Çinlar (1975). In a SMP, the sequence of states evolves according to a discrete-time Markov chain. Conditional on this sequence, the holding times in the successive states are mutually independent, and each holding time has a distribution function that depends only on the current state. The *continuous-time Markov chain* (CTMC) is an important refinement of the SMP in which the holding times are exponentially distributed; the intensity is a function of the current state (Asmussen, 1987; Çinlar, 1975; Karlin and Taylor, 1975; Ross, 1983). Each of these models can be viewed as associating exactly one event with each state.

The *generalized semi-Markov process* (GSMP) extends the SMP model to capture more complicated system behavior by associating a set of events with each state. The events compete to trigger the next state transition and each set of trigger events has its own probability distribution for determining the new state; see Glynn (1989), Glynn and Haas (2004), Haas and Shedler (1987), Konig et al. (1974), Schassberger (1978), Shedler (1993), and Whitt (1980). At each state transition, new events may be scheduled. For each of these new events, a clock indicating the time until the event is scheduled to occur is set according to an arbitrary distribution function that depends on the current state, previous state, and set of events that trigger the state transition. These clocks determine when the next state transition occurs and which of the scheduled events actually trigger this state transition. A GSMP is formally defined in terms of a general state space Markov chain that records the state of the system, together with the clock readings, at successive state transitions.

In applications, a reward structure is often associated with the underlying process $\{X(t)\colon t \geq 0\}$ of the discrete-event system under study. Specifically, denote by $\mathscr{S}$ the state space of the underlying process, by $N(t)$ the number of state transitions of the process during the interval $(0, t]$, and by S_n the state just after the nth state transition. [Set $S_0 = X(0)$.] Then a general model for the reward earned over the interval $[0, t]$ is

$$R(t) = \sum_{n=1}^{N(t)} r(S_{n-1}, S_n) + \int_0^t f(X(u))du, \tag{1}$$

where $r\colon \mathscr{S} \times \mathscr{S} \mapsto \Re$ and $f\colon \mathscr{S} \mapsto \Re$ are specified functions. Under this model, the system accrues a continuous reward at rate $f(s)$ whenever the system is in state $s \in \mathscr{S}$, as well as a lump-sum reward of $r(s, s')$ whenever the system makes a transition from s to s'.

One way to characterize the long-run stability of the underlying stochastic process $\{X(t)\colon t \geq 0\}$ is to determine whether or not time-average limits of the form $\alpha = \lim_{t\to\infty} R(t)/t$ exist, that is, whether or not the reward process $\{R(t)\colon t \geq 0\}$ obeys a *strong law of large numbers* (SLLN). If such a SLLN holds and we can compute the value of the limit α, then we can estimate $R(t)$ simply as $R(t) \approx \alpha t$ whenever t is "large." Note that, in general, the distribution and moments of $R(t)$ are difficult to compute exactly, even when $\{X(t)\colon t \geq 0\}$ is a CTMC. When $\{X(t)\colon t \geq 0\}$ is a GSMP, the limit α often cannot be computed analytically or numerically, and is typically estimated using simulation; if a SLLN holds, then the estimator $\hat{\alpha}(t) = R(t)/t$ is strongly consistent for α.

Central limit theorems (CLTs) illuminate the rate of convergence in the SLLN. Moreover, if we can compute both α and the variance constant σ^2 that appears in the CLT, then we can approximately compute various probabilities concerning the quantity $R(t)$; for example, $\{R(t) \leq x\}$ can be approximated by $P\{\alpha t + \sigma t^{1/2} N(0,1) \leq x\}$, where $N(0,1)$ is a standard normal random variable. The ordinary form of the CLT asserts that under appropriate regularity conditions, the quantity $\hat{\alpha}(t)$—suitably normalized—converges in distribution to a standard normal random variable. An ordinary CLT can often be strengthened to a *functional central limit theorem* (FCLT); see, for example, Billingsley (1999) and Ethier and Kurtz (1986). Roughly speaking, a stochastic process with time-average limit α obeys a FCLT if the associated cumulative (i.e., time-integrated) process—centered about the deterministic function $g(t) = \alpha t$ and suitably compressed in space and time—converges in distribution to a standard Brownian motion as the degree of compression increases. FCLTs can be used to approximate pathwise properties of the reward process over finite time intervals via those of Brownian motion (Billingsley, 1999; Ethier and Kurtz, 1986). As mentioned previously, α is typically estimated using simulation when the underlying process is a GSMP. In this setting, a variety of estimation methods such as the method of batch means (with a fixed number of batches) are known to yield asymptotically valid confidence intervals for α provided that a FCLT holds (Glynn and Iglehart, 1990).

In this paper, we provide FCLTs for finite-state CTMCs, SMPs, and GSMPs under the general reward structure in (1). In the case of SMPs and CTMCs, our emphasis is on the use of martingale theory to obtain the desired limit theorems, both because this approach appears to be somewhat less well known than the classical approach based on regenerative structure, and because use of martingale techniques often leads to algorithms for explicit computation of the variance constant in the FCLT. A major contribution of this paper is the development of minimal (and easily checkable) conditions under which FCLTs for general rewards are valid. In particular, we show that FCLTs typically hold for both SMPs and their GSMP generalizations when the "holding time" distributions have finite second moment.

II. PRELIMINARIES

Before stating the main results, we briefly review some pertinent aspects of martingale theory—see Ethier and Kurtz (1986), Bremaud (1981) and Hall and Heyde (1980) for detailed discussions—as well as a result concerning random changes of time.

Definition 1. *Let $\{\mathscr{F}_n : n \geq 0\}$ be an increasing sequence of σ-fields. The discrete-time process $\{M_n : n \geq 0\}$ is a martingale adapted to $\{\mathscr{F}_n : n \geq 0\}$ if each M_n is $\mathscr{F}_n$-measurable with $E[|M_n|] < \infty$ and $E[M_{n+1} \mid \mathscr{F}_n] = M_n$* a.s.

The FCLT for both CTMCs and SMPs can be established using the following FCLT for discrete-time martingales. Denote by $C[0,\infty)$ the space of continuous real-valued functions on $[0,\infty)$, by $W = \{W(t) : t \geq 0\}$ a standard Brownian motion, and by "$\Rightarrow$" weak convergence; see Billingsley (1999), and Ethier and Kurtz (1986) for definitions. Given a martingale $\{M_n : n \geq 0\}$, define a collection $\{U_\eta : \eta \geq 0\}$ of $C[0,\infty)$-valued stochastic processes by setting

$$U_\eta(t) = \eta^{-1/2}(M_{\lfloor \eta t \rfloor} + (\eta t - \lfloor \eta t \rfloor)(M_{\lfloor \eta t \rfloor + 1} - M_{\lfloor \eta t \rfloor}))$$

for $\eta, t \geq 0$.

Proposition 1. *Let $\{M_n : n \geq 0\}$ be a martingale adapted to an increasing sequence of σ-fields $\{\mathscr{F}_n : n \geq 0\}$. Let $D_k = M_{k+1} - M_k$ for $k \geq 0$ and suppose that*

(i) $E[M_n^2] < \infty$ *for* $n \geq 0$*;*
(ii) $\lim_{n\to\infty}(1/n)\sum_{k=0}^{n-1} E[D_k^2 \mid \mathscr{F}_k] = \sigma^2$ *a.s. for some constant* $\sigma^2 \geq 0$*; and*
(iii) $\lim_{n\to\infty}(1/n)\sum_{k=0}^{n-1} E[D_k^2 I(D_k^2 \geq \epsilon n) \mid \mathscr{F}_k] = 0$ *for all* $\epsilon > 0$*,*

where $I(A)$ is the indicator of event A. Then $U_\eta \Rightarrow \sigma W$ on $C[0,\infty)$ as $\eta \to \infty$.

This result follows from Theorem 4.1 in Hall and Heyde (1980).

Our next result gives conditions under which a FCLT in discrete time implies a corresponding FCLT in continuous time; see Serfozo (1975) for a general discussion of results of this type. Consider a reward process $\{R(t): t \geq 0\}$ defined as in (1) in terms of the underlying process $\{X(t): t \geq 0\}$ of a discrete-event stochastic system with state space $\mathscr{S}$. As before, denote by S_n ($n \geq 0$) the nth state visited by the latter process. Also denote by ζ_n the time of the nth state transition and by $\Delta_n = \zeta_{n+1} - \zeta_n$ the holding time in state S_n. Suppose that $\lim_{t\to\infty} R(t)/t = \alpha$ a.s. for some real-valued constant α, and set $f_\alpha(s) = f(s) - \alpha$ for $s \in \mathscr{S}$. Also set $R_n = R(\zeta_n) - \alpha\zeta_n$ for $n \geq 0$. Finally, set

$$\widetilde{U}_\eta(t) = \eta^{-1/2}(R_{\lfloor \eta t \rfloor} + (\eta t - \lfloor \eta t \rfloor)(R_{\lfloor \eta t \rfloor + 1} - R_{\lfloor \eta t \rfloor}))$$

and

$$U_\eta(t) = \eta^{-1/2}(R(\eta t) - \alpha \eta t)$$

for $\eta, t \geq 0$. Observe that the sample paths of U_η are elements of $D[0,\infty)$, the space of functions on $[0,\infty)$ that are right continuous and have limits from the left (Billingsley, 1999; Ethier and Kurtz, 1986).

Proposition 2. *Suppose that*

(i) $\widetilde{U}_\eta \Rightarrow \tilde{\sigma} W$ *as* $\eta \to \infty$ *on* $C[0,\infty)$ *for some* $\tilde{\sigma}^2 \geq 0$*.*
(ii) $\lim_{n\to\infty}(1/n)\sum_{k=0}^{n-1} \Delta_k = \delta$ *a.s. for some* $\delta > 0$*.*
(iii) $\lim_{n\to\infty}\max_{0\leq k\leq n}|f_\alpha(S_k)|\Delta_k/\sqrt{n} = 0$ *a.s.*

Then $U_\eta \Rightarrow \sigma W$ on $D[0,\infty)$ as $\eta \to \infty$, where $\sigma^2 = \tilde{\sigma}^2/\delta$.

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The idea behind the proof of the proposition is to fix $T > 0$ and set $\Lambda_\eta(t) = N(\eta t)/\eta$ and $\Lambda(t) = t/\delta$ for $t \geq 0$, where $N(t)$ is the number of state transitions in $(0, t]$. As discussed, for example, in Haas (1999, p. 78), it follows from the assumption in (ii) that $\lim_{\eta\to\infty}\sup_{0\leq t\leq T}|\Lambda_\eta(t)-\Lambda(t)|=0$ a.s., that is, $\Lambda_\eta \to \Lambda$ a.s. on $C[0,T]$. Arguing as in Sec. 14 in Billingsley (1999), we then have $\widetilde{U}_\eta \circ \Lambda_\eta \Rightarrow \tilde{\sigma} W \circ \Lambda = \sigma W$ as $\eta \to \infty$, where $(f \circ g)(t) = f\big(g(t)\big)$. (The equality follows from standard properties of Brownian motion.) Moreover, it is not hard to show that

$$\begin{aligned}\sup_{0\leq t\leq T}|U_\eta(t)-\widetilde{U}_\eta\circ\Lambda_\eta(t)| &\leq \left(\frac{N(\eta T)}{\eta}\right)^{1/2}\sup_{0\leq t\leq T}\frac{|f_\alpha(S_{N(\eta t)})|\Delta_{N(\eta t)}}{N^{1/2}(\eta T)}\\ &= \left(\frac{N(\eta T)}{\eta}\right)^{1/2}\max_{0\leq k\leq N(\eta T)}\frac{|f_\alpha(S_k)|\Delta_k}{N^{1/2}(\eta T)}.\end{aligned}$$

The conclusion of the proposition now follows by standard converging-together arguments (Billingsley, 1999), provided that the rightmost term converges to 0 in probability as $\eta \to \infty$. We can in fact establish a.s. convergence to 0 by noting that the assumption in (ii) implies that $N(\eta T) \to \infty$ a.s. and $N(\eta T)/\eta \to T/\delta$ a.s. as $\eta \to \infty$ (see Haas, 2002, p. 79), so that the desired result follows from the assumption in (iii).

III. THE FCLT FOR SMPs AND CTMCs

For a SMP $\{X(t): t \geq 0\}$ with finite state space $\mathscr{S}$ and initial distribution μ, denote by ζ_n the time of the nth state transition ($n \geq 0$), by $S_n = X(\zeta_n)$ the state of the process just after this state transition, and by $\Delta_n = \zeta_{n+1} - \zeta_n$ the holding time in state S_n. As is well known, the sequence of successive states $\{S_n: n \geq 0\}$ is a discrete-time Markov chain (DTMC) and, given $\{S_n: n \geq 0\}$, the random variables $\{\Delta_n: n \geq 0\}$ are mutually independent. We denote by R the Markov transition matrix of the DTMC. The distribution function of each holding time Δ_n depends only on the current state S_n, and we write $F(t; s) = P\{\Delta_n \leq t \mid S_n = s\}$. We formally define a SMP by first defining the *Markov renewal process* $\{(S_n, \zeta_n): n \geq 0\}$ as the unique stochastic process such that $\{S_0 = s\} = \mu(s)$ and

$$\begin{aligned}&P\{S_{n+1} = s', \zeta_{n+1} - \zeta_n \leq t \mid S_n, \zeta_n, \ldots, S_0, \zeta_0\}\\ &\quad = R(S_n, s')F(t; S_n) \quad \text{a.s.}\end{aligned}$$

for $n \geq 0$, $s' \in \mathscr{S}$, and $t \geq 0$. The SMP $\{X(t): t \geq 0\}$ is then defined by setting

$$X(t) = S_{N(t)}, \tag{2}$$

where $N(t) = \sup\{n \geq 0: \zeta_n \leq t\}$. By construction, the SMP has piecewise constant, right-continuous sample paths. When specifying probabilities and expectations, we often use the notation P_μ (P_s) and E_μ (E_s) to emphasize dependence on the initial distribution μ (or initial state $s \in \mathscr{S}$).

As usual, the embedded jump chain $\{S_n: n \geq 0\}$ is called *irreducible* if for each $s, s' \in \mathscr{S}$ there exists a finite positive integer $n = n(s, s')$ such that $R^n(s, s') > 0$, where R^n is the nth power of the transition matrix R. It is well known that a finite-state irreducible embedded jump chain admits a unique invariant distribution π, that is, a probability distribution on $\mathscr{S}$ that satisfies the system of linear equations

$$\pi^t R = \pi. \tag{3}$$

Moreover,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(S_n) = \pi(f) \quad \text{a.s.} \tag{4}$$

for any real-valued function f defined on $\mathscr{S}$, where $\pi(f) = E_\pi[f(S_0)] = \sum_{s\in\mathscr{S}} \pi(s)f(s)$. We call a SMP irreducible if the embedded jump chain is irreducible.

Theorem 1 below gives a FCLT for irreducible finite-state SMPs. To prepare for this result, define a reward function $R(t)$ as in (1). Whenever the embedded chain $\{S_n: n \geq 0\}$ admits an invariant distribution π, set

$$\delta = \sum_{s\in\mathscr{S}} \pi(s)m(s) \tag{5}$$

and

$$\alpha = \delta^{-1} \sum_{s,s'\in\mathscr{S}} \pi(s)R(s,s')(r(s,s') + f(s)m(s)), \tag{6}$$

where $m(s) = \int_{[0,\infty)} tF(dt; s)$. Using classical regenerative arguments (Asmussen, 1987; Shedler, 1993), it can be shown that an irreducible finite-state SMP satisfying $\max_{s\in\mathscr{S}} m(s) < \infty$ obeys a SLLN with limiting constant α: $R(t)/t \to \alpha$ a.s. Set $b(s) = \tilde{r}(s) + f_\alpha(s)m(s)$ for $s \in \mathscr{S}$, where $\tilde{r}(s) = \sum_{s'\in\mathscr{S}} R(s,s')r(s,s')$ and $f_\alpha(s) = f(s) - \alpha$, and let g be a solution of the linear system of equations

$$(I - R)g = b. \tag{7}$$

In the literature, (7) is known as *Poisson's equation*. If the SMP is irreducible with finite state space, then it is not hard to see that one solution of (7) is given by $g = (I - R + \Pi)^{-1}b$, where the matrix Π is defined by $\Pi(s, s') = \pi(s')$ for $s, s' \in \mathscr{S}$; this solution is unique up to an additive constant. Indeed, it follows from Theorem 4.3.1 in Kemeny and Snell (1960) that the "fundamental matrix" $A = (I - R + \Pi)^{-1}$ exists and has the representation $A = I + \sum_{n=1}^{\infty}(R^n - \Pi)$. Since (3) and (7) imply that $\pi^t b = 0$, it can be seen that $Ab = \sum_{n=0}^{\infty} R^n b$, and hence $(I - R)Ab = \sum_{n=0}^{\infty} R^n b - \sum_{n=1}^{\infty} R^n b = b$. Next, set $m_2(s) = \int_{[0,\infty)} t^2 F(dt; s)$ and $H(s, s') = r(s, s') + g(s') - g(s)$ for $s, s' \in \mathscr{S}$, and set

$$\sigma^2 = \tilde{\sigma}^2/\delta, \tag{8}$$

where

$$\tilde{\sigma}^2 = \sum_{s,s' \in \mathscr{S}} \pi(s)R(s,s')[H^2(s,s') + 2H(s,s')f_\alpha(s)m(s) + f_\alpha^2(s)m_2(s)]. \tag{9}$$

Finally, set

$$U_\eta(t) = \eta^{-1/2}(R(\eta t) - \alpha\eta t) \tag{10}$$

for $\eta, t \geq 0$.

Theorem 1. *Suppose that $\mathscr{S}$ is finite, $\{X(t): t \geq 0\}$ is irreducible, and $\max_{s \in \mathscr{S}} m(s) < \infty$, so that $\lim_{t\to\infty} R(t)/t = \alpha$ a.s.. Then $U_\eta \Rightarrow \sigma W$ on $D[0, \infty)$ as $\eta \to \infty$ for any initial distribution μ if and only if $m_2(s) < \infty$ for each $s \in \mathscr{S}$ such that $f(s) \neq \alpha$.*

Theorem 1 asserts that a FCLT holds for a finite-state SMP essentially under the assumptions of irreducibility and finite second moments on the holding-time distributions. To compute the variance constant σ^2 in Theorem 1, first determine π and g by solving (3) and (7). Next, determine δ, α, and $\tilde{\sigma}^2$ from (5), (6), and (9), and set $\sigma^2 = \tilde{\sigma}^2/\delta$.

Proof. To prove the "if" result, suppose that $m_2(s) < \infty$ for $s \in \mathscr{S}$ such that $f(s) \neq \alpha$. The basic idea is to first establish a FCLT for the sequence $\{R_n: n \geq 0\}$, where $R_n = R(\zeta_n) - \alpha\zeta_n$, and then apply a random time

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change as in Proposition 2. We establish the discrete-time FCLT by showing that for large n, the process $\{R_n : n \geq 0\}$ is equal to a martingale plus a bounded stochastic process. To this end, set

$$M_1(n) = \sum_{k=0}^{n-1} f_\alpha(S_k)(\Delta_k - m(S_k)),$$

$$M_2(n) = \sum_{k=0}^{n-1} (r(S_k, S_{k+1}) - \tilde{r}(S_k)),$$

and

$$M_3(n) = \sum_{k=0}^{n-1} b(S_k) + g(S_n) - g(S_0)$$

for $n \geq 0$. [Take $M_1(0) = M_2(0) = M_3(0) = 0$.] We claim that $\{M_i(n) : n \geq 0\}$ is a martingale adapted to $\{\mathscr{F}_n : n \geq 0\}$ for $i = 1, 2, 3$, where $\mathscr{F}_0 = \sigma(S_0)$ and $\mathscr{F}_n = \sigma(S_0, \ldots, S_n, \Delta_0, \ldots, \Delta_{n-1})$ for $n \geq 1$. To see this for the case $i = 1$, set $Y_k = f_\alpha(S_k)\big(\Delta_k - m(S_k)\big)$ for $k \geq 0$, observe that

$$\begin{aligned} E[Y_k \mid \mathscr{F}_k] = E[Y_k \mid S_k] &= f_\alpha(S_k)(E[\Delta_k \mid S_k] - m(S_k)) \\ &= f_\alpha(S_k) \cdot 0 = 0 \end{aligned}$$

for each k, which immediately implies that $E[M_1(n+1) \mid \mathscr{F}_n] = M_1(n)$ a.s. for each $n \geq 0$. The remaining conditions in Definition 1 are easy to verify. The argument for the case $i = 2$ is similar. For the case $i = 3$, the relation in (7) implies that we can write

$$M_3(n) = \sum_{k=0}^{n-1} (g(S_{k+1}) - E[g(S_{k+1}) \mid \mathscr{F}_k]),$$

from which the relation $E[M_3(n+1) \mid \mathscr{F}_n] = M_3(n)$ a.s. follows immediately.

Now observe that $R_n = M(n) + g(S_0) - g(S_n)$ for $n \geq 0$, where $M(n) = M_1(n) + M_2(n) + M_3(n)$. Set

$$\widetilde{V}_\eta(t) = \eta^{-1/2}(M(\lfloor \eta t \rfloor) + (\eta t - \lfloor \eta t \rfloor)(M(\lfloor \eta t \rfloor + 1) - M(\lfloor \eta t \rfloor))$$

and

$$\widetilde{U}_\eta(t) = \eta^{-1/2}(R_{\lfloor \eta t \rfloor} + (\eta t - \lfloor \eta t \rfloor)(R_{\lfloor \eta t \rfloor + 1} - R_{\lfloor \eta t \rfloor}))$$

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for $\eta, t \geq 0$. We claim that $\widetilde{V}_\eta \Rightarrow \tilde{\sigma} W$ on $C[0,\infty)$ as $\eta \to \infty$ by the FCLT for martingales. Since, for $T > 0$,

$$\sup_{0 \leq t \leq T} |\widetilde{U}_\eta(t) - \tilde{V}_\eta(t)| \leq 2 \max_{s \in \mathscr{S}} |g(s)|/\eta^{1/2} \to 0$$

as $\eta \to \infty$, a converging-together argument will then imply that $\widetilde{U}_\eta \Rightarrow \tilde{\sigma} W$. To establish the FCLT for $\{\widetilde{V}_\eta : \eta \geq 0\}$ we need to show that the conditions of Proposition 1 hold. The first condition of the proposition follows easily from the finiteness assumption on $\mathscr{S}$ and the moment condition on the holding-time distributions. To verify the second condition, set $D_k = M(k+1) - M(k)$ for $k \geq 0$ and observe that, using (7), we can write $D_k = f_\alpha(S_k)\Delta_k + r(S_k, S_{k+1}) + g(S_{k+1}) - g(S_k)$. Define a bounded real-valued function h on $\mathscr{S}$ by setting $h(s) = E[D_0^2 \mid S_0 = s]$. Appealing to (4), we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} E[D_k^2 \mid \mathscr{F}_k] = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} h(S_k) = E_\pi[h(S_0)] = \tilde{\sigma}^2,$$

and the second condition of Proposition 1 holds. To establish the final condition of Proposition 1, fix $\epsilon > 0$ and set $\pi_{\min} = \min_{s \in \mathscr{S}} \pi(s)$; observe that $\pi_{\min} > 0$ since the embedded DTMC $\{S_n : n \geq 0\}$ is irreducible with finite state space. Also set $h_n(s) = E[D_k^2 I(D_k^2 \geq \epsilon n) \mid S_k = s]$ for $n \geq 0$ and $s \in \mathscr{S}$. Observe that, since each h_n is nonnegative,

$$h_n(s) \leq \sum_{s' \in \mathscr{S}} \frac{\pi(s')}{\pi(s)} h_n(s') \leq \frac{1}{\pi_{\min}} E_\pi[h_n(S_0)] \stackrel{\text{def}}{=} \gamma_n,$$

so that

$$\begin{aligned} \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} E[D_k^2 I(D_k^2 \geq \epsilon n) | \mathscr{F}_k] &= \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} h_n(S_k) \\ &\leq \lim_{n\to\infty} \gamma_n = 0, \end{aligned}$$

where the last equality follows from the monotone convergence theorem.

We now complete the first part of the proof by applying Proposition 2. To this end, fix a state $\bar{s} \in \mathscr{S}$ and define a sequence of random indices $\{\theta(k) : k \geq 0\}$ by setting $\theta(-1) = -1$ and $\theta(k) = \inf\{n > \theta(k-1) : S_n = \bar{s}\}$ for $k \geq 0$. Observe that the sequence $\{\theta(k) : k \geq 0\}$ decomposes the process $\{\Delta_n : n \geq 0\}$ into i.i.d. cycles. It follows from standard results for DTMCs that each $\theta(k)$ is a.s. finite and the cycle length $\tau_1 = \theta(1) - \theta(0)$ has finite moments of all orders. Moreover, we have

$$E\left[\sum_{n=\theta(0)}^{\theta(1)-1} \Delta_k\right] = E\left[E\left[\sum_{n=\theta(0)}^{\theta(1)-1} \Delta_k \,\middle|\, S_0, S_1, \ldots\right]\right]$$
$$= E\left[\sum_{n=\theta(0)}^{\theta(1)-1} m(S_k)\right] \le E[\tau_1] \max_{s\in\mathscr{S}} m(s) < \infty.$$

It then follows from the SLLN for regenerative processes (Asmussen, 1987) that $\lim_{n\to\infty}(1/n)\sum_{k=0}^{n-1}\Delta_k = \delta$ a.s. A similar argument shows that $\lim_{n\to\infty}(1/n)\sum_{k=0}^{n-1} f_\alpha^2(S_k)\Delta_k^2 = \beta$ a.s. for some constant $\beta < \infty$, which in turn implies (Haas, 2002, p. 79) that $\lim_{n\to\infty}|f_\alpha(S_n)|\Delta_n/n^{1/2} = 0$ a.s. A simple argument as on p. 78 of Haas (1999) now shows that

$$\lim_{n\to\infty} \max_{0\le k\le n} |f_\alpha(S_k)|\Delta_k/n^{1/2} = 0 \quad \text{a.s.},$$

so that the conditions of Proposition 2 hold and the desired result follows.

It remains to prove the "only if" result. Suppose therefore that $U_\eta \Rightarrow \sigma W$ as $\eta \to \infty$. It follows from a minor generalization of the results in Glynn and Whitt (1987) that the quantity $E\left[Z_1^2\right]$ must be finite, where $Z_1 = \sum_{n=\theta(0)}^{\theta(1)-1}\left(f_\alpha(S_n)\Delta_n + r(S_n, S_{n+1})\right)$ and the sequence of regeneration points $\{\theta(k)\colon k \ge 0\}$ is defined as before. Since each $m(s)$ is finite, it follows that $E[f_\alpha^2(S_{\theta(0)})\Delta_{\theta(0)}^2] < \infty$, so that $m_2(\bar{s})$ must be finite if $f_\alpha(\bar{s}) \ne 0$. Since $\bar{s}$ is arbitrary, the desired result follows. □

Remark. With minor changes to the proof, we can extend Theorem 1 to the general case in which the holding time in a state s can depend on both s and the next state s'. Set $F(t; s, s') = P\{\Delta_n \le t \mid S_n = s, S_{n+1} = s'\}$, $m(s, s') = \int_{[0,\infty)} tF(dt; s, s')$, and $m_2(s, s') = \int_{[0,\infty)} t^2 F(dt; s, s')$. Then we take $\delta = \sum_{s,s'\in\mathscr{S}} \pi(s)R(s, s')m(s, s')$.

$$\alpha = \delta^{-1} \sum_{s,s'\in\mathscr{S}} \pi(s)R(s, s')(r(s, s') + f(s)m(s, s')),$$

and

$$\tilde{\sigma}^2 = \sum_{s,s'\in\mathscr{S}} \pi(s)R(s, s')$$
$$\times \left[H^2(s, s') + 2H(s, s')f_\alpha(s)m(s, s') + f_\alpha^2(s)m_2(s, s')\right],$$

where, as before, $H(s,s') = r(s,s') + g(s') - g(s)$ and g solves (7). Now, however, the function b that appears in (7) is given by

$$b(s) = \tilde{r}(s) + f_\alpha(s) \sum_{s' \in \mathscr{S}} R(s,s')m(s,s')$$

for $s \in \mathscr{S}$. The necessary and sufficient condition becomes: $m_2(s,s') < \infty$ for all $s, s' \in \mathscr{S}$ such that $f(s) \neq \alpha$ and $R(s,s') > 0$.

Remark. The martingale-based approach has been used to obtain FCLTS in the context of discrete-time Markov chains on a general state space (Duflo 1990; Glynn and Meyn, 1996; Maigret, 1978; Meyn and Tweedie, 1993) as well as continuous time Markov processes on a general state space (Glynn and Meyn, 1996; Bhattacharya, 1982). In related work, martingale methods have been used to obtain CLTS and FCLTS for the process $\{\Delta_n : n \geq 0\}$; see Durrett and Resnick (1978) and references therein. For examples of the classical regenerative approach to obtaining CLTS and FCLTS for Markov chains and SMPS, see, for example, Asmussen (1987), Chung (1967), Glynn and Whitt (1987, 2002), Haas (2002), Meyn and Tweedie (1993), Pyke and Schaufele (1964) and Shedler (1993). To our knowledge, the current paper contains the first presentation of a martingale-based FCLT for SMPS with a general reward structure as in (1).

The foregoing FCLT for SMPS immediately leads to a FCLT for finite state irreducible CTMCS. As is well known, a (time homogeneous) CTMC $\{X(t): t \geq 0\}$ with state space $\mathscr{S}$, initial distribution μ, and infinitesimal generator matrix (or rate matrix) Q can be treated as a special case of a SMP. In particular, $F(t;s) = 1 - \exp(-q(s)t)$ and $R(s,s') = I(s \neq s')$ $Q(s,s')/q(s)$, where $q(s) = -Q(s,s) = \sum_{s' \neq s} Q(s,s')$. Since each $F(\cdot;s)$ has finite moments of all orders, we obtain Theorem 2 below. In the theorem, $R(t)$ is a reward function defined as in (1); we take $r(s,s) = 0$ for $s \in \mathscr{S}$ since a CTMC never makes a transition from a state s back to itself.

Theorem 2. *Suppose that $\mathscr{S}$ is finite and $\{X(t): t \geq 0\}$ is irreducible, so that $\lim_{t\to\infty} R(t)/t = \alpha$ a.s. with α defined by* (6). *Let $\{U_\eta : \eta \geq 0\}$ be a collection of random functions defined as in* (10). *Then $U_\eta \Rightarrow \sigma W$ on $D[0,\infty)$ as $\eta \to \infty$ for any initial distribution μ, where σ^2 is defined as in* (8).

Remark. The parameters α and σ^2 in Theorem 2 can be expressed in terms of the infinitesimal generator matrix Q and the invariant distribution ν of the CTMC. (The latter distribution always exists for an irreducible CTMC with finite state space). The idea is to use the well known fact that

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$m(s) = q^{-1}(s)$ and

$$\nu(s) = \frac{\pi(s)q^{-1}(s)}{\sum_{s' \in S} \pi(s')q^{-1}(s')} = \pi(s)q^{-1}(s)/\delta \tag{11}$$

for $s \in \mathscr{S}$; here π is, as before, the invariant distribution of the embedded chain $\{X(\zeta_n): n \geq 0\}$. Specifically, set $\check{r}(s) = \sum_{s' \in \mathscr{S}} Q(s, s')r(s, s')$ and $\check{b}(s) = \check{r}(s) + f(s)$ for $s \in \mathscr{S}$. Then (11) implies that $\alpha = \nu(\check{b})$, where we use the notation $\nu(h) = \sum_{s \in \mathscr{S}} \nu(s)h(s)$. Moreover, it follows from (7) and (11) that

$$\sigma^2 = \sum_{s,s' \in \mathscr{S}} \nu(s)Q(s, s')H^2(s, s'),$$

where, as before, $H(s, s') = r(s, s') + g(s') - g(s)$ and g solves (7). At first sight this reformulation may appear incomplete because the definition of the function g involves R rather than Q. Using (11), however, it can be shown that (7) is equivalent to the linear system

$$-Qg = \check{b} - \nu(\check{b}). \tag{12}$$

Remark. Theorem 2 can also be established by directly exploiting the theory of martingales in continuous time. The idea is to write $R(t) - \nu(\check{b})t = M(t) - g\big(X(t)\big)$, where $M(t) = M_1(t) + M_2(t)$ with

$$M_1(t) = \sum_{n=1}^{N(t)} r(S_{n-1}, S_n) - \int_0^t \check{r}(X(u))du$$

and

$$M_2(t) = g(X(t)) + \int_0^t \check{b}(X(u))du - \nu(\check{b})t.$$

It follows from Lévy's formula (Bremaud, 1981, p. 294) that $\{M_1(t): t \geq 0\}$ is a martingale. Moreover, using (12) (which is the continuous-time version of Poisson's equation), we have

$$M_2(t) = g(X(t)) - \int_0^t Qg(X(u))du,$$

where $Qg(s) = \sum_{s' \in \mathscr{S}} Q(s, s')g(s')$ for $s \in \mathscr{S}$. Lévy's formula then implies that $\{M_2(t): t \geq 0\}$, and hence $\{M(t): t \geq 0\}$ is a martingale. The proof now proceeds analogously to the proof of Theorem 1.

IV. THE FCLT FOR GSMPs

As discussed previously, a generalized semi-Markov process (GSMP) is a continuous time stochastic process that makes a state transition when one or more "events" associated with the occupied state occur. In general, more than one event can occur simultaneously; see Chapter 6 in Shedler (1993). A GSMP is defined in terms of a general state space Markov chain that describes the process at successive state-transition times. After recalling the definition of a GSMP, we give a FCLT. Unlike in the CTMC or SMP setting, it is usually not possible to compute either α or σ^2 explicitly; these quantities are typically estimated using simulation. Also, as one might expect, the FCLT for GSMPs entails more complicated conditions than the FCLT for either CTMCs or SMPs.

A. Definition

Following (Shedler, 1993), let $E = \{e_1, e_2, \ldots, e_M\}$ be a finite set of *events* and $\mathscr{S}$ be a finite set of *states*. For $s \in \mathscr{S}$, let $s \mapsto E(s)$ be a mapping from $\mathscr{S}$ to the nonempty subsets of E; here $E(s)$ denotes the set of all events that can occur when the process is in state s. An event $e \in E(s)$ is said to be *active* in state s. When the process is in state s, the occurrence of one or more active events triggers a state transition. Denote by $p(s'; s, E^*)$ the probability that the new state is s' given that the events in the set E^* [$\subseteq E(s)$] occur simultaneously in state s. A "clock" is associated with each event. The clock reading for an active event indicates the remaining time until the event is scheduled to occur. These clocks, along with the speeds at which the clocks run down, determine which of the active events actually trigger the next state transition. Denote by $r(s, e)$ (≥ 0) the *speed* (finite, deterministic rate) at which the clock associated with event e runs down when the state is s; we assume that, for each $s \in \mathscr{S}$, we have $r(s, e) > 0$ for some $e \in E(s)$. Typically in applications, all speeds for active events are equal to 1; zero speeds can be used to model "preemptive-resume" behavior. Let $C(s)$ be the set of possible *clock-reading vectors* when the state is s:

$$C(s) = \{c = (c_1, \ldots, c_M) : c_i \in [0, \infty) \text{ and } c_i > 0 \text{ if and only if } e_i \in E(s)\}.$$

[The ith component of a clock-reading vector $c = (c_1, \ldots, c_M)$ is the clock reading associated with event e_i.] Beginning in state s with clock-reading vector $c = (c_1, \ldots, c_M) \in C(s)$, the time $t^*(s, c)$ to the next state transition is given by

$$t^*(s,c) = \min_{\{i: e_i \in E(s)\}} c_i / r(s,e_i), \tag{13}$$

where $c_i/r(s,e_i)$ is taken to be $+\infty$ when $r(s,e_i)=0$. The set of events $E^*(s,c)$ that trigger the next state transition is given by

$$E^*(s,c) = \{e_i \in E(s): c_i - t^*(s,c)r(s,e_i) = 0\}. \tag{14}$$

At a transition from state s to state s' triggered by the simultaneous occurrence of the events in the set E^*, a finite clock reading is generated for each *new event* $e' \in N(s';s,E^*) = E(s') - (E(s) - E^*)$. Denote the *clock-setting distribution function* (that is, the distribution function of such a new clock reading) by $F(\cdot;s',e',s,E^*)$. We assume that $F(0;s',e',s,E^*)=0$, so that new clock readings are a.s. positive, and that $\lim_{x\to\infty} F(x;s',e',s,E^*)=1$, so that each new clock reading is a.s. finite. For each *old event* $e' \in O(s';s,E^*) = E(s') \cap \big(E(s) - E^*\big)$, the old clock reading is kept after the state transition. For $e' \in (E(s) - E^*) - E(s')$, event e' is cancelled and the clock reading is discarded. When E^* is a singleton set of the form $E^* = \{e^*\}$, we write $p(s';s,e^*) = p(s';s,\{e^*\})$, $O(s';s,e^*) = O(s';s,\{e^*\})$, and so forth. The GSMP is a continuous-time stochastic process $\{X(t): t \geq 0\}$ that records the state of the system at time t.

Formal definition of the process $\{X(t): t \geq 0\}$ is in terms of a general state space Markov chain $\{(S_n, C_n): n \geq 0\}$ that describes the process at successive state-transition times. Heuristically, S_n represents the state and $C_n = (C_{n,1}, \ldots, C_{n,M})$ represents the clock-reading vector just after the nth state transition; see (Shedler, 1993) for a formal definition of the chain. The chain takes values in the set $\Sigma = \bigcup_{s\in\mathscr{S}}\big(\{s\} \times C(s)\big)$. Denote by μ the initial distribution of the chain. As with SMPs and CTMCs, we use the notations P_μ, $P_{(s,c)}$, E_μ, and $E_{(s,c)}$ when specifying probabilities and expected values associated with the chain in order to emphasize the dependence on the initial distribution μ or the initial state $(s,c) \in \Sigma$.

We construct the continuous time process $\{X(t): t \geq 0\}$ from the chain $\{(S_n, C_n): n \geq 0\}$ in a manner almost identical to the construction of a SMP. Let ζ_n $(n \geq 0)$ be the (nonnegative, real-valued) time of the nth state transition: $\zeta_0 = 0$ and

$$\zeta_n = \sum_{j=0}^{n-1} t^*(S_j, C_j)$$

for $n \geq 1$. The GSMP $\{X(t): t \geq 0\}$ is now defined exactly as in (2). As with a SMP or CTMC, the GSMP has piecewise constant, right-continuous sample paths.

B. The Limit Theorem

For a GSMP with state space $\mathscr{S}$ and event set E and for $s, s' \in \mathscr{S}$ and $e \in E$, write $s \xrightarrow{e} s'$ if $p(s'; s, e) r(s, e) > 0$ and write $s \rightarrow s'$ if $s \xrightarrow{e} s'$ for some $e \in E(s)$. Also write $s \rightsquigarrow s'$ if either $s \rightarrow s'$ or there exist states $s_1, s_2, \ldots, s_n \in \mathscr{S}$ $(n \geq 1)$ such that $s \rightarrow s_1 \rightarrow \cdots \rightarrow s_n \rightarrow s'$.

Definition 2. *A GSMP is irreducible if $s \rightsquigarrow s'$ for each $s, s' \in \mathscr{S}$.*

Recall that a nonnegative function G is a *component* of a distribution function F if G is not identically equal to 0 and $G \leq F$. If G is a component of F and G is absolutely continuous, so that G has a density function g, then we say that g is a *density component* of F.

Assumption PD(q), defined below, encapsulates the key conditions that we impose on the building blocks of a GSMP to obtain limit theorems. Denote by $\mathscr{H}$ the subset of the clock-setting distribution functions such that $F(\cdot; s', e', s, E^*) \in \mathscr{H}$ if and only if $E(s') = \{e'\}$. (Observe that every clock-setting distribution function is an element of $\mathscr{H}$ if the GSMP is in fact a SMP.)

Definition 3. *Assumption PD(q) holds for a specified GSMP and real number $q \geq 0$ if*

(i) *the state space $\mathscr{S}$ of the GSMP is finite;*
(ii) *the GSMP is irreducible;*
(iii) *all speeds of the GSMP are positive;*
(iv) *each clock-setting distribution function $F(\cdot; s', e', s, E^*)$ has finite qth moment; and*
(v) *there exists $\bar{x} \in (0, \infty)$ such that each clock-setting distribution function $F(\cdot; s', e', s, E^*) \notin \mathscr{H}$ has a density component that is positive and continuous on $(0, \bar{x})$.*

If Assumption PD(q) holds for some $q \geq 0$, then obviously Assumption PD(r) holds for $0 \leq r \leq q$. Also observe that, whenever Assumption PD(q) holds, there can be at most a finite number of state transitions at which two or more events occur simultaneously.

Recall that a probability distribution π is *invariant* with respect to a Markov chain $\{Z_n : n \geq 0\}$ with transition kernel P and (possibly uncountable) state space Γ if and only if $\int P(z, A) \pi(dz) = \pi(A)$ for each measurable set $A \subseteq \Gamma$. The following result is a consequence of Proposition 3.13 and Theorem 4.5 in Haas (1999).

Proposition 3. *If Assumption PD*(1) *holds, then there exists a unique invariant distribution* π *for the underlying chain* $\{(S_n, C_n): n \geq 0\}$.

We often write $\pi(\tilde{f}) = E_\pi[\tilde{f}(S_0, C_0)]$ for a function $\tilde{f}: \Sigma \mapsto \Re$. In contrast to CTMCs and SMPs, the invariant distribution π typically cannot be computed simply by solving a system of linear equations—one generally resorts to computer simulation.

Define a reward function $R(t)$ as in (1). Recall the definition of the functions t^* and E^* from (13) and (14), and set

$$\tilde{h}(s,c) = f(s)t^*(s,c) + \sum_{s' \in \mathscr{S}} r(s,s')p(s'; s, E^*(s,c))$$

for $(s,c) \in \Sigma$. Also set $\alpha = \pi(\tilde{h})/\pi(t^*)$, and observe that α is well defined and finite if and only if $\pi(|\tilde{h}|) < \infty$. Straightforward modifications of results in Glynn and Haas (2004) establish the following SLLN.

Proposition 4. *If Assumption PD*(1) *holds, then* $\pi(|\tilde{h}|) < \infty$ *and* $\lim_{t\to\infty} R(t)/t = \alpha$ *a.s. for any initial distribution* μ.

Other direct modifications of the results in Glynn and Haas (2004) yield an FCLT. As before, assuming that $\lim_{t\to\infty} R(t)/t = \alpha$ a.s., set $U_\eta(t) = \eta^{-1/2}\big(R(\eta t) - \alpha\eta t\big)$ for $\eta, t \geq 0$.

Theorem 3. *If Assumption PD*(2) *holds, then there exists a finite deterministic constant* $\sigma^2 \geq 0$ *such that* $U_\eta \Rightarrow \sigma W$ *on* $D[0,\infty)$ *as* $\eta \to \infty$ *for any initial distribution* μ.

The moment condition in Theorem 3 is natural in light of the previously discussed conditions for SMPs and CTMCs. Indeed, the necessity of the second moment condition appearing in Theorem 1 proves that the moment condition in Theorem 3 is the weakest general condition possible. In the GSMP setting, we also impose a positive-density condition on the clock-setting distributions. Although this particular condition is by no means necessary, some such condition is needed in the face of the additional complexity caused by the presence of multiple clocks. Indeed, the following example shows that even the SLLN can fail in the absence of any such structural assumption; this example is taken from Glynn and Haas (2004).

Example 1 (An irreducible GSMP with no unique time-average limit)**.** Consider a GSMP with unit speeds, state space $\mathscr{S} = \{1, 2, 3, 4\}$,

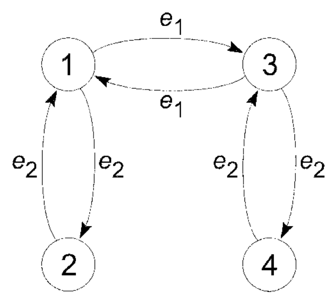

Figure 1. State transition diagram for GSMP of Example 1.

event set $E = \{e_1, e_2\}$ and active event sets given by $E(1) = E(3) = \{e_1, e_2\}$ and $E(2) = E(4) = \{e_2\}$. The state-transition probabilities are $p(1; 3, e_1) = p(3; 1, e_1) = 1$ and $p(1; 2, e_2) = p(2; 1, e_2) = p(3; 4, e_2) = p(4; 3, e_2) = 1$; see Fig. 1. Observe that this GSMP is irreducible in the sense of Definition 2. Suppose that each successive new clock reading for event e_i $(i = 1, 2)$ is uniformly distributed on a specified interval $[a_i, b_i]$, and that $0 \leq a_2 < b_2 < a_1 < b_1$. Then with probability 1 event e_2 always occurs before event e_1 whenever both events simultaneously become active. It follows that if the initial state is equal to 1 or 2, then the GSMP never visits state 3 or 4; if the initial state is equal to 3 or 4, then the GSMP never visits state 1 or 2. Thus, in general, the value of a limit of the form $\lim_{t\to\infty}(1/t)\int_0^t f\big(X(u)\big)du$ depends on the initial distribution. Similar observations hold for the underlying chain. Of course, this GSMP does not satisfy Assumption PD(q) for any $q \geq 0$ since the clock-setting distribution function for event e_1 does not have a density component that is positive on an interval of the form $(0, \bar{x})$.

In general, obtaining structural conditions that are weaker than those used in Theorem 3 would involve analysis of the detailed structure of the GSMP under consideration.

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